11.6 Calculus of Vector-Valued Functions

We now turn to the topic of ultimate interest in this chapter: the calculus of vector-valued functions. Everything you learned about differentiating and integrating functions of the form $y = f(x)$ carries over to vector-valued functions $\mathbf{r}(t)$; you simply apply the rules of differentiation and integration to the individual components of $\mathbf{r}$.

The Derivative and Tangent Vector

Orientation of Curves

Integrals of Vector-Valued Functions

Quick Quiz

SECTION 11.6 EXERCISES

Review Questions

1. Explain how to compute the derivative of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

2. Explain the geometric meaning of $\mathbf{r}'(t)$.

3. Given a tangent vector on an oriented curve, how do you find the unit tangent vector?

4. Compute $\mathbf{r}''(t)$ when $\mathbf{r}(t) = \langle t^{10}, 8t, \cos t \rangle$.

5. How do you find the indefinite integral of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$?

6. How do you evaluate $\int_a^b \mathbf{r}(t)\,dt$?

Basic Skills

7-12. Derivatives of vector-valued functions *Differentiate the following functions.*

7. $\mathbf{r}(t) = \langle 2t^3, 6\sqrt{t}, 3/t \rangle$

8. $\mathbf{r}(t) = \langle 4, 3\cos 2t, 2\sin 3t \rangle$

9. $\mathbf{r}(t) = \langle e^t, 2e^{-t}, -4e^{2t} \rangle$

10. $\mathbf{r}(t) = \langle \tan t, \sec t, \cos^2 t \rangle$

11. $\mathbf{r}(t) = \langle t e^{-t}, t\ln t, t\cos t \rangle$

12. $\mathbf{r}(t) = \langle (t+1)^{-1}, \tan^{-1} t, \ln(t+1) \rangle$

13-16. Tangent vectors *For the following curves, find a tangent vector at the given value of t.*

13. $\mathbf{r}(t) = \langle t, \cos 2t, 2\sin t \rangle, \quad t = \pi/2$

14. $\mathbf{r}(t) = \langle 2\sin t, 3\cos t, \sin(t/2) \rangle, \quad t = \pi$

15. $\mathbf{r}(t) = \langle 2\,t^4,\, 6\,t^{3/2},\, 10/t\rangle,\quad t = 1$

16. $\mathbf{r}(t) = \langle 2\,e^t,\, e^{-2\,t},\, 4\,e^{2\,t}\rangle,\quad t = \ln 3$

17-20. Unit tangent vectors *For the following parametrized curves, find the unit tangent vector.*

17. $\mathbf{r}(t) = \langle 8,\, \cos 2\,t,\, 2\sin 2\,t\rangle$, for $0 \le t \le 2\,\pi$

18. $\mathbf{r}(t) = \langle \sin t,\, \cos t,\, \cos t\rangle$, for $0 \le t \le 2\,\pi$

19. $\mathbf{r}(t) = \langle t,\, 2,\, 2/t\rangle$, for $t \ge 1$

20. $\mathbf{r}(t) = \langle e^{2\,t},\, 2\,e^{2\,t},\, 2\,e^{-3\,t}\rangle$, for $t \ge 0$

21-24. Unit tangent vectors at a point *For the following parametrized curves, find the unit tangent vector at the given value of t.*

21. $\mathbf{r}(t) = \langle \cos 2\,t,\, 4,\, 3\sin 2\,t\rangle$, for $0 \le t \le \pi$, $t = \pi/2$

22. $\mathbf{r}(t) = \langle \sin t,\, \cos t,\, e^{-t}\rangle$, for $0 \le t \le \pi$, $t = 0$

23. $\mathbf{r}(t) = \langle 6\,t,\, 6,\, 3/t\rangle$, for $0 < t < 2$, $t = 1$

24. $\mathbf{r}(t) = \langle \sqrt{7}\; e^t,\, 3\,e^t,\, 3\,e^t\rangle$, for $0 \le t \le 1$, $t = \ln 2$

25-30. Derivative rules *Let*

$$\mathbf{u}(t) = 2\,t^3\,\mathbf{i} + \left(t^2 - 1\right)\mathbf{j} - 8\,\mathbf{k} \quad \text{and} \quad \mathbf{v}(t) = e^t\,\mathbf{i} + 2\,e^{-t}\,\mathbf{j} - e^{2\,t}\,\boldsymbol{k}.$$

Compute the derivative of the following functions.

25. $\left(t^{12} + 3\,t\right)\mathbf{u}(t)$

26. $\left(4\,t^8 - 6\,t^3\right)\mathbf{v}(t)$

27. $\mathbf{u}\left(t^4 - 2\,t\right)$

28. $\mathbf{v}\left(\sqrt{t}\,\right)$

29. $\mathbf{u}(t)\cdot\mathbf{v}(t)$

30. $\mathbf{u}(t) \times \mathbf{v}(t)$

31-34. Derivative rules *Compute the following derivatives.*

31. $\dfrac{d}{d\,t}\left[t^2(\mathbf{i} + 2\,\mathbf{i} - 2\,t\,\mathbf{k})\cdot\left(e^t\,\mathbf{i} + 2\,e^t\,\mathbf{j} - 3\,e^{-t}\,\mathbf{k}\right)\right]$

32. $\dfrac{d}{d\,t}\left[\left(t^3\,\mathbf{i} - 2\,t\,\mathbf{j} - 2\,\mathbf{k}\right) \times \left(t\,\mathbf{i} - t^2\,\mathbf{j} - t^3\,\mathbf{k}\right)\right]$

33. $\dfrac{d}{d\,t}\left[\left(3\,t^2\,\mathbf{i} + \sqrt{t}\;\mathbf{j} - 2\,t^{-1}\,\mathbf{k}\right)\cdot(\cos t\,\mathbf{i} + \sin 2\,t\,\mathbf{j} - 3\,t\,\mathbf{k})\right]$

34. $\dfrac{d}{dt}\left[\left(t^3\,\mathbf{i} + 6\,\mathbf{j} - 2\sqrt{t}\,\mathbf{k}\right) \times \left(3\,t\,\mathbf{i} - 12\,t^2\,\mathbf{j} - 6\,t^{-2}\,\mathbf{k}\right)\right]$

35-40. Higher derivatives *Compute* $\mathbf{r}''(t)$ *and* $\mathbf{r}'''(t)$ *for the following functions.*

35. $\mathbf{r}(t) = \left\langle t^2 + 1,\ t + 1,\ 1\right\rangle$

36. $\mathbf{r}(t) = \left\langle 3\,t^{12} - t^2,\ t^8 + t^3,\ t^{-4} - 2\right\rangle$

37. $\mathbf{r}(t) = \langle \cos 3\,t,\ \sin 4\,t,\ \cos 6\,t\rangle$

38. $\mathbf{r}(t) = \left\langle e^{4\,t},\ 2\,e^{-4\,t} + 1,\ 2\,e^{-t}\right\rangle$

39. $\mathbf{r}(t) = \sqrt{t + 4}\ \mathbf{i} + \dfrac{t}{t + 1}\,\mathbf{j} - e^{-t^2}\,\mathbf{k}$

40. $\mathbf{r}(t) = \tan t\,\mathbf{i} + \left(t + \dfrac{1}{t}\right)\mathbf{j} - \ln(t + 1)\,\mathbf{k}$

41-44. Indefinite integrals *Compute the indefinite integral of the following functions.*

41. $\mathbf{r}(t) = \left\langle t^4 - 3\,t,\ 2\,t - 1,\ 10\right\rangle$

42. $\mathbf{r}(t) = \left\langle 5\,t^{-4} - t^2,\ t^6 - 4\,t^3,\ 2/t\right\rangle$

43. $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin 3\,t,\ 4\cos 8\,t\rangle$

44. $\mathbf{r}(t) = t\,e^t\,\mathbf{i} + t\sin t^2\,\mathbf{j} - \dfrac{2\,t}{\sqrt{t^2 + 4}}\,\mathbf{k}$

45-48. Finding r from r' *Find the function* **r** *that satisfies the following conditions.*

45. $\mathbf{r}'(t) = \left\langle 1,\ 2\,t,\ 3\,t^2\right\rangle$; $\mathbf{r}(1) = \langle 4,\ 3,\ -5\rangle$

46. $\mathbf{r}'(t) = \left\langle \sqrt{t},\ \cos \pi\,t,\ 4/t\right\rangle$; $\mathbf{r}(1) = \langle 2,\ 3,\ 4\rangle$

47. $\mathbf{r}'(t) = \left\langle e^{2\,t},\ 1 - 2\,e^{-t},\ 1 - 2\,e^{t}\right\rangle$; $\mathbf{r}(0) = \langle 1,\ 1,\ 1\rangle$

48. $\mathbf{r}'(t) = \dfrac{t}{t^2 + 1}\,\mathbf{i} + t\,e^{-t^2}\,\mathbf{j} - \dfrac{2\,t}{\sqrt{t^2 + 4}}\,\mathbf{k}$; $\mathbf{r}(0) = \mathbf{i} + \dfrac{3}{2}\,\mathbf{j} - 3\,\mathbf{k}$

49-54. Definite integrals *Evaluate the following definite integrals.*

49. $\displaystyle\int_{-1}^{1}\left(\mathbf{i} + t\,\mathbf{j} + 3\,t^2\,\mathbf{k}\right)dt$

50. $\displaystyle\int_{0}^{4}\left(\sqrt{t}\ \mathbf{i} + t^{-3}\,\mathbf{j} - 2\,t^2\,\mathbf{k}\right)dt$

51. $\displaystyle\int_{-\pi}^{\pi}(\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2\,t\,\mathbf{k})\,dt$

52. $\int_0^{\ln 2} \left(e^{-t}\,\mathbf{i} + 2\,e^{2t}\,\mathbf{j} - 4\,e^t\,\mathbf{k}\right) dt$

53. $\int_0^2 t\,e^t(\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})\,dt$

54. $\int_0^{\pi/4} \left(\sec^2 t\,\mathbf{i} - 2\cos t\,\mathbf{j} - \mathbf{k}\right) dt$

Further Explorations

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel for all values of t in the domain.

b. The curve described by the function $\mathbf{r}(t) = \langle 1,\ t^2 - 2t,\ \cos \pi t\rangle$ is smooth, for $-\infty < t < \infty$.

c. If f, g, and h are odd integrable functions and a is a real number, then

$$\int_{-a}^{a} (f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})\,dt = \mathbf{0}.$$

56-61. Derivative rules *Let* $\mathbf{u}(t) = \langle 1,\ t,\ t^2\rangle$, $\mathbf{v}(t) = \langle t^2,\ -2t,\ 1\rangle$, *and* $g(t) = 2\sqrt{t}$. *Compute the derivatives of the following functions.*

56. $\mathbf{u}(t^3)$

57. $\mathbf{v}(e^t)$

58. $g(t)\,\mathbf{v}(t)$

59. $\mathbf{v}(g(t))$

60. $\mathbf{u}(t)\cdot\mathbf{v}(t)$

61. $\mathbf{u}(t)\times\mathbf{v}(t)$

62-67. Relationship between r and r'

62. Consider the circle $\mathbf{r}(t) = \langle a\cos t,\ a\sin t\rangle$, for $0 \le t \le 2\pi$, where a is a positive real number. Compute $\mathbf{r}'$ and show that it is orthogonal to $\mathbf{r}$ for all t.

63. Consider the parabola $\mathbf{r}(t) = \langle a\,t^2 + 1,\ t\rangle$, for $-\infty < t < \infty$, where a is a positive real number. Find all points on the parabola at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

64. Consider the curve $\mathbf{r}(t) = \langle \sqrt{t},\ 1,\ t\rangle$, for $t > 0$. Find all points on the curve at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

65. Consider the helix $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ t\rangle$, for $-\infty < t < \infty$. Find all points on the helix at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

66. Consider the ellipse $\mathbf{r}(t) = \langle 2\cos t,\ 8\sin t,\ 0\rangle$, for $0 \le t \le 2\pi$. Find all points on the ellipse at which $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

67. For what curves in $\mathbb{R}^3$ is it true that $\mathbf{r}$ and $\mathbf{r}'$ are parallel for all t in the domain?

68. Derivative rules Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions at $t = 0$ with $\mathbf{u}(0) = \langle 0,\ 1,\ 1\rangle$, $\mathbf{u}'(0) = \langle 0,\ 7,\ 1\rangle$, $\mathbf{v}(0) = \langle 0,\ 1,\ 1\rangle$, and $\mathbf{v}'(0) = \langle 1,\ 1,\ 2\rangle$. Evaluate the following expressions.

a. $\left.\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})\right|_{t=0}$

b. $\left.\frac{d}{dt}(\mathbf{u}\times\mathbf{v})\right|_{t=0}$

c. $\left.\frac{d}{dt}(\mathbf{u}(t)\cos t)\right|_{t=0}$

Additional Exercises

69. Vectors r and r' for lines

a. If $\mathbf{r}(t) = \langle at, bt, ct\rangle$ with $\langle a, b, c\rangle \neq \langle 0, 0, 0\rangle$, show that the angle between $\mathbf{r}$ and $\mathbf{r}'$ is constant for all t.

b. If $\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$, where x_0, y_0, and z_0 are not all zero, show that the angle between $\mathbf{r}$ and $\mathbf{r}'$ varies with t.

c. Explain the results of parts (a) and (b) geometrically.

70. Proof of Sum Rule By expressing $\mathbf{u}$ and $\mathbf{v}$ in terms of their components, prove that

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t).$$

71. Proof of Product Rule By expressing $\mathbf{u}$ in terms of its components, prove that

$$\frac{d}{dt}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t).$$

72. Proof of Cross Product Rule Prove that

$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t).$$

There are two ways to proceed: Either express $\mathbf{u}$ and $\mathbf{v}$ in terms of their three components or use the definition of the derivative.

T **73. Cusps and noncusps**

a. Graph the curve $\mathbf{r}(t) = \langle t^3, t^3\rangle$. Show that $\mathbf{r}'(0) = \mathbf{0}$ and the curve does not have a cusp at $t = 0$. Explain.

b. Graph the curve $\mathbf{r}(t) = \langle t^3, t^2\rangle$. Show that $\mathbf{r}'(0) = \mathbf{0}$ and the curve has a cusp at $t = 0$. Explain.

c. The functions $\mathbf{r}(t) = \langle t, t^2\rangle$ and $\mathbf{p}(t) = \langle t^2, t^4\rangle$ both satisfy $y = x^2$. Explain how the curves they parametrize are different.

d. Consider the curve $\mathbf{r}(t) = \langle t^m, t^n, t^p\rangle$, where m, n, and p are positive integers, not all equal. Is it true that the curve has a cusp at $t = 0$ if one or more of m, n, or p is even? Explain.

74. Motion on a sphere Prove that $\mathbf{r}$ describes a curve that lies on the surface of a sphere centered at the origin ($x^2 + y^2 + z^2 = a^2$ with $a \geq 0$) if and only if $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal at all points of the curve.