11.7 Motion in Space

It is a remarkable fact that given the forces acting on an object and its initial position and velocity, the motion of the object in three-dimensional space can be modeled for all future times. To be sure, the accuracy of the results depends on how well the various forces on the object are modeled. For example, it may be more difficult to predict the trajectory of a spinning soccer ball than the path of a space station orbiting Earth. Nevertheless, as shown in this section, by combining Newton's Second Law of Motion with everything we have learned about vectors, it is possible to solve a variety of moving body problems.

Position, Velocity, Speed, Acceleration

Straight-Line and Circular Motion

Two-Dimensional Motion in a Gravitational Field

Three-Dimensional Motion

Quick Quiz

SECTION 11.7 EXERCISES

Review Questions

1. Given the position function **r** of a moving object, explain how to find the velocity, speed, and acceleration of the object.

2. What is the relationship between the position and velocity vectors for motion on a circle?

3. State Newton's Second Law of Motion in vector form.

4. Write Newton's Second Law of Motion for three-dimensional motion with only the gravitational force (acting in the z-direction).

5. Given the acceleration of an object and its initial velocity, how do you find the velocity of the object for $t \geq 0$?

6. Given the velocity of an object and its initial position, how do you find the position of the object for $t \geq 0$?

Basic Skills

7-14. Velocity and acceleration from position *Consider the following position functions.*

a. *Find the velocity and speed of the object.*
b. *Find the acceleration of the object.*

7. $\mathbf{r}(t) = \langle 2 + 2t, 1 - 4t \rangle$, for $t \geq 0$

8. $\mathbf{r}(t) = \left\langle 1 - t^2, 3 + 2t^3 \right\rangle$, for $t \geq 0$

9. $\mathbf{r}(t) = \langle 8 \sin t, 8 \cos t \rangle$, for $0 \leq t \leq 2\pi$

10. $\mathbf{r}(t) = \langle 3 \cos t, 4 \sin t \rangle$, for $0 \leq t \leq 2\pi$

11. $\mathbf{r}(t) = \langle 3 + t, 2 - 4t, 1 + 6t \rangle$, for $t \geq 0$

12. $\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle$, for $0 \le t \le 2\pi$

13. $\mathbf{r}(t) = \left\langle 1, t^2, e^{-t}\right\rangle$, for $t \ge 0$

14. $\mathbf{r}(t) = \langle 13\cos 2t, 12\,\text{sint}\,2t, 5\sin 2t\rangle$, for $0 \le t \le \pi$

T **15-18. Comparing trajectories** *Consider the following position functions* **r** *and* **R** *for two objects.*

- ***a.*** *Find the interval* $[c, d]$ *over which the* **R** *trajectory is the same as the* **r** *trajectory over* $[a, b]$.
- ***b.*** *Find the velocity for both objects.*
- ***c.*** *Graph the speed of the two objects over the intervals* $[a, b]$ *and* $[c, d]$, *respectively.*

15. $\mathbf{r}(t) = \langle \cos t, 4\sin t\rangle$, $[a, b] = [0, 2\pi]$, $\mathbf{R}(t) = \langle \cos 3t, 4\sin 3t\rangle$ on $[c, d]$

16. $\mathbf{r}(t) = \left\langle 2 - e^t, 4 - e^{-t}\right\rangle$, $[a, b] = [0, \ln 10]$, $\mathbf{R}(t) = \langle 2 - t, 4 - 1/t\rangle$ on $[c, d]$

17. $\mathbf{r}(t) = \left\langle 4 + t^2, 3 - 2t^4, 1 + 3t^6\right\rangle$, $[a, b] = [0, 6]$, $\mathbf{R}(t) = \left\langle 4 + \ln t, 3 - 2\ln^2 t, 1 + 3\ln^3 t\right\rangle$ on $[c, d]$. For graphing, let $c = 1$ and $d = 20$.

18. $\mathbf{r}(t) = \left\langle 2\cos 2t, \sqrt{2}\sin 2t, \sqrt{2}\sin 2t\right\rangle$, $[a, b] = [0, \pi]$, $\mathbf{R}(t) = \left\langle 2\cos 4t, \sqrt{2}\sin 4t, \sqrt{2}\sin 4t\right\rangle$ on $[c, d]$.

19-24. Trajectories on circles and spheres *Determine whether the following trajectories lie on a circle in* $\mathbb{R}^2$ *or sphere in* $\mathbb{R}^3$ *centered at the origin. If so, find the radius of the circle or sphere and show that the position vector and the velocity vector are everywhere orthogonal.*

19. $\mathbf{r}(t) = \langle 8\cos 2t, 8\sin 2t\rangle$, for $0 \le t \le \pi$

20. $\mathbf{r}(t) = \langle 4\sin t, 2\cos t\rangle$, for $0 \le t \le 2\pi$

21. $\mathbf{r}(t) = \left\langle \sin t + \sqrt{3}\cos t, \sqrt{3}\sin t - \cos t\right\rangle$, for $0 \le t \le 2\pi$

22. $\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle$, for $0 \le t \le 2\pi$

23. $\mathbf{r}(t) = \langle \sin t, \cos t, \cos t\rangle$, for $0 \le t \le 2\pi$

24. $\mathbf{r}(t) = \left\langle \sqrt{3}\cos t + \sqrt{2}\sin t, -\sqrt{3}\cos t + \sqrt{2}\sin t, \sqrt{2}\sin t\right\rangle$, for $0 \le t \le 2\pi$

25-28. Solving equations of motion *Given an acceleration vector, initial velocity* $\langle u_0, v_0\rangle$, *and initial position* $\langle x_0, y_0\rangle$, *find the velocity and position vectors for* $t \ge 0$.

25. $\mathbf{a}(t) = \langle 0, 10\rangle$, $\langle u_0, v_0\rangle = \langle 0, 5\rangle$, $\langle x_0, y_0\rangle = \langle 1, -1\rangle$

26. $\mathbf{a}(t) = \langle 1, t\rangle$, $\langle u_0, v_0\rangle = \langle 2, -1\rangle$, $\langle x_0, y_0\rangle = \langle 0, 8\rangle$

27. $\mathbf{a}(t) = \langle \cos t, 2\sin t\rangle$, $\langle u_0, v_0\rangle = \langle 0, 1\rangle$, $\langle x_0, y_0\rangle = \langle 1, 0\rangle$

28. $\mathbf{a}(t) = \left\langle e^{-t}, 1\right\rangle$, $\langle u_0, v_0\rangle = \langle 1, 0\rangle$, $\langle x_0, y_0\rangle = \langle 0, 0\rangle$

T **29-32. Two-dimensional motion** *Consider the motion of the following objects. Assume the x-axis is horizontal, the positive y-axis is vertical (opposite g), the ground is horizontal, and only the gravitational force acts on the object.*

- ***a.*** *Find the velocity and position vectors for* $t \ge 0$.
- ***b.*** *Graph the trajectory.*

c. *Determine the time of flight and range of the object.*

d. *Determine the maximum height of the object.*

29. A soccer ball has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ when it is kicked with an initial velocity of $\langle u_0, v_0 \rangle = \langle 30, 6 \rangle$ m/s.

30. A golf ball is has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ when it is hit at an angle of 30° with an initial speed of 150 ft/s.

31. A projectile is launched from a platform 20 ft above the ground at an angle of 60° with a speed of 250 ft/s. Assume the origin is at the base of the platform.

32. A rock is thrown from the edge of a vertical cliff 40 m above the ground at an angle of 45° with a speed of $10\sqrt{2}$ m/s. Assume the origin is at the foot of the cliff.

33-36. Solving equations of motion *Given an acceleration vector, initial velocity $\langle u_0, v_0, w_0 \rangle$, and initial position $\langle x_0, y_0, z_0 \rangle$, find the velocity and position vectors for $t \ge 0$.*

33. $\mathbf{a}(t) = \langle 0, 0, 10 \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 1, 5, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 5, 0 \rangle$

34. $\mathbf{a}(t) = \langle 1, t, 4t \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 20, 0, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 0, 0 \rangle$

35. $\mathbf{a}(t) = \langle \sin t, \cos t, 1 \rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 0, 2, 0 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 0, 0, 0 \rangle$

36. $\mathbf{a}(t) = \left\langle t, e^{-t}, 1 \right\rangle$, $\langle u_0, v_0, w_0 \rangle = \langle 0, 0, 1 \rangle$, $\langle x_0, y_0, z_0 \rangle = \langle 4, 0, 0 \rangle$

T **37-40. Three-dimensional motion** *Consider the motion of the following objects. Assume the x-axis points east, the y-axis points north, the positive z-axis is vertical (opposite g), the ground is horizontal, and only the gravitational force acts on the object unless otherwise stated.*

a. *Find the velocity and position vectors for $t \ge 0$.*

b. *Make a sketch of the trajectory.*

c. *Determine the time of flight and range of the object.*

d. *Determine the maximum height of the object.*

37. A bullet is fired from a rifle 1 m above the ground in a northeast direction. The initial velocity of the bullet is $\langle 200, 200, 0 \rangle$ m/s.

38. A golf ball is hit east down a fairway with an initial velocity of $\langle 50, 0, 30 \rangle$ m/s. A crosswind blowing to the south produces an acceleration of the ball of -0.8 m/s^2.

39. A small rocket is fired from a launch pad 10 m above the ground with an initial velocity of $\langle 300, 400, 500 \rangle$ m/s. A crosswind blowing to the north produces an acceleration of the rocket of 2.5 m/s^2.

40. A soccer ball is kicked from the point $\langle 0, 0, 0 \rangle$ with an initial velocity of $\langle 0, 80, 80 \rangle$ ft/s. The spin on the ball produces an acceleration of $\langle 1.2, 0, 0 \rangle \text{ ft/s}^2$.

Further Explorations

41. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the speed of an object is constant, then its velocity components are constant.

b. The functions $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{R}(t) = \left\langle \sin t^2, \cos t^2 \right\rangle$ generate the same set of points for $t \ge 0$.

c. It is not possible for a velocity vector to have a constant direction but a variable magnitude for all $t \ge 0$.

d. If the acceleration of an object is zero for all $t \ge 0$ ($\mathbf{a}(t) = \mathbf{0}$), then the velocity of the object is constant.

e. If you double the initial speed of a projectile, its range also doubles (assume no forces other than gravity act on the projectile).

f. If you double the initial speed of a projectile, its time of flight also doubles (assume no forces other than gravity).

g. A trajectory with $\mathbf{v}(t) = \mathbf{a}(t) \neq \mathbf{0}$ for all t is possible.

42-45. Trajectory properties *Find the time of flight, range, and maximum height of the following two-dimensional trajectories, assuming no forces other than gravity. In each case the initial position is* $\langle 0, 0 \rangle$ *and the initial velocity is* $\mathbf{v}_0 = \langle u_0, v_0 \rangle$.

42. $\langle u_0, v_0 \rangle = \langle 10, 20 \rangle$ ft/s

43. Initial speed $|\mathbf{v}_0| = 150$ m/s, launch angle $\alpha = 30°$

44. $\langle u_0, v_0 \rangle = \langle 40, 80 \rangle$ m/s

45. Initial speed $|\mathbf{v}_0| = 400$ ft/s, launch angle $\alpha = 60°$

46. Motion on the moon The acceleration due to gravity on the moon is approximately $g/6$ (one-sixth its value on Earth). Compare the time of flight, range, and maximum height of a projectile on the moon with the corresponding values on Earth.

47. Firing angles A projectile is fired over horizontal ground from the origin with an initial speed of 60 m/s. What firing angles will produce a range of 300 m?

T **48. Firing strategies** Suppose you wish to fire a projectile over horizontal ground from the origin and attain a range of 1000 m.

a. Make a graph of the initial speed required for all firing angles $0 < \alpha < \pi/2$.

b. What firing angle requires the least initial speed?

c. What firing angle requires the least flight time?

49. Nonuniform straight-line motion Consider the motion of an object given by the position function

$$\mathbf{r}(t) = f(t)\langle a, b, c \rangle + \langle x_0, y_0, z_0 \rangle, \quad \text{for } t \geq 0,$$

where a, b, c, x_0, y_0, and z_0 are constants and f is a differentiable scalar function for $t \geq 0$.

a. Explain why this function describes motion along a line.

b. Find the velocity function. In general, is the velocity constant in magnitude or direction along the path?

50. A race Two people travel from $P(4, 0)$ to $Q(-4, 0)$ along the paths given by

$$\mathbf{r}(t) = \langle 4\cos(\pi t/8), 4\sin(\pi t/8) \rangle \quad \text{and}$$
$$\mathbf{R}(t) = \left\langle 4 - t, (4 - t)^2 - 16 \right\rangle.$$

a. Graph both paths between P and Q.

b. Graph the speeds of both people between P and Q.

c. Who arrives at Q first?

51. Circular motion Consider an object moving along the circular trajectory $\mathbf{r}(t) = \langle A\cos\omega t, A\sin\omega t \rangle$, where A and ω are constants.

a. Over what time interval $[0, T]$ does the object traverse the circle once?

b. Find the velocity and speed of the object. Is the velocity constant in either direction or magnitude? Is the speed constant?

c. Find the acceleration of the object.

d. How are the position and velocity related? How are the position and acceleration related?

e. Sketch the position, velocity, and acceleration vectors at four different points on the trajectory with $A = \omega = 1$.

52. A linear trajectory An object moves along a straight line from the point $P(1, 2, 4)$ to the point $Q(-6, 8, 10)$.

a. Find a position function $\mathbf{r}$ that describes the motion if it occurs with a constant speed over the time interval $[0, 5]$.

b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed e^t.

53. A circular trajectory An object moves clockwise around a circle centered at the origin with radius 5 m beginning at the point $(0, 5)$.

a. Find a position function $\mathbf{r}$ that describes the motion if the object moves with a constant speed, completing 1 lap every 12 s.

b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed e^{-t}.

54. A helical trajectory An object moves on the helix $\langle \cos t, \sin t, t \rangle$ for $t \geq 0$.

a. Find a position function $\mathbf{r}$ that describes the motion if it occurs with a constant speed of 10.

b. Find a position function $\mathbf{r}$ that describes the motion if it occurs with speed t.

T **55. Speed on an ellipse** An object moves along an ellipse given by the function $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$, for $0 \leq t \leq 2\pi$, where $a > 0$ and $b > 0$.

a. Find the velocity and speed of the object in terms of a and b, for $0 \leq t \leq 2\pi$.

b. With $a = 1$ and $b = 6$, graph the speed function for $0 \leq t \leq 2\pi$. Mark the points on the trajectory at which the speed is a minimum and a maximum.

c. Is it true that the object speeds up along the flattest (straightest) parts of the trajectory and slows down where the curves are sharpest?

d. For general a and b, find the ratio of the maximum speed to the minimum speed on the ellipse (in terms of a and b).

T **56. Travel on a cycloid** Consider an object moving on the cycloid $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ for $0 \leq t \leq 4\pi$.

a. Graph the trajectory.

b. Find the velocity and speed of the object. At what point(s) on the trajectory does the object move fastest? Slowest?

c. Find the acceleration of the object and show that $|\mathbf{a}(t)|$ is constant.

d. Explain why the trajectory has a cusp at $t = 2\pi$.

57. Analyzing a trajectory Consider the trajectory given by the position function

$$\mathbf{r}(t) = \left\langle 50\, e^{-t} \cos t,\ 50\, e^{-t} \sin t,\ 5\left(1 - e^{-t}\right) \right\rangle \text{ for } t \geq 0.$$

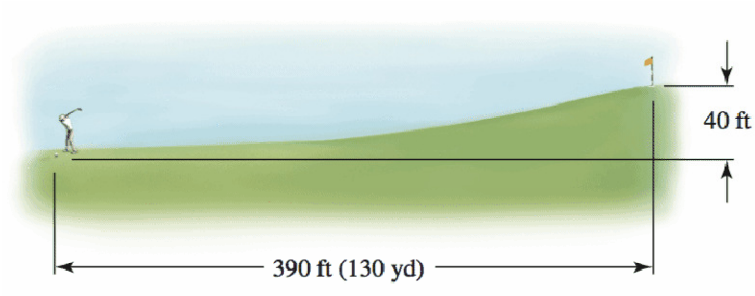
a. Find the initial point ($t = 0$) and the "terminal" point ($\lim_{t \to \infty} \mathbf{r}(t)$) of the trajectory.

b. At what point on the trajectory is the speed the greatest?

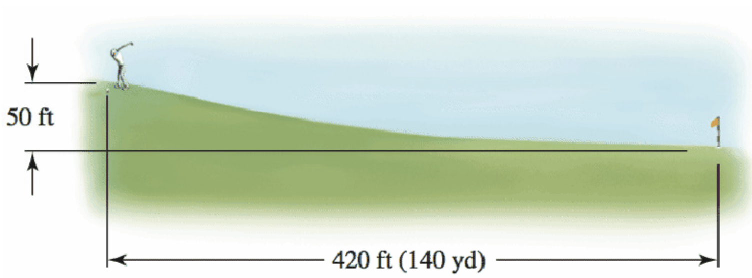
c. Graph the trajectory.

Applications

58. Golf shot A golfer stands 390 ft (130 yd) horizontally from the hole and 40 ft below the hole (see figure). Assuming the ball is hit with an initial speed of 150 ft/s, at what angle should it be hit to land in the hole? Assume the path of the ball lies in a plane.



59. **Another golf shot** A golfer stands 420 ft (140 yd) horizontally from the hole and 50 ft above the hole (see figure). Assuming the ball is hit with an initial speed of 120 ft/s, at what angle should it be hit to land in the hole? Assume the path of the ball lies in a plane.

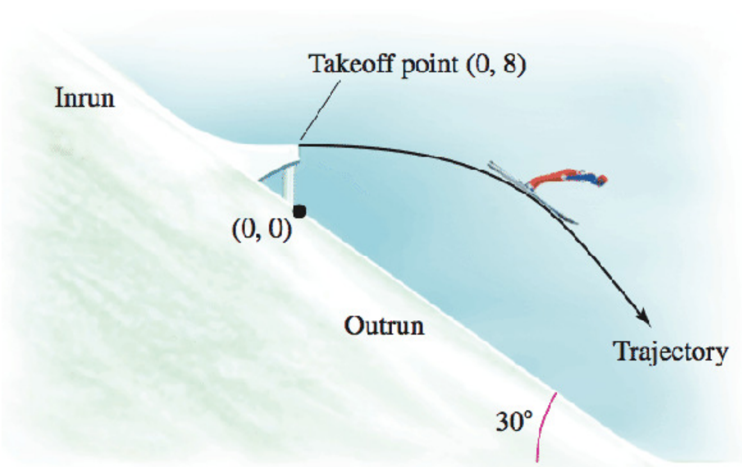


60. **Ski jump** The lip of a ski jump is 8 m above the outrun that is sloped at an angle of 30° to the horizontal (see figure).

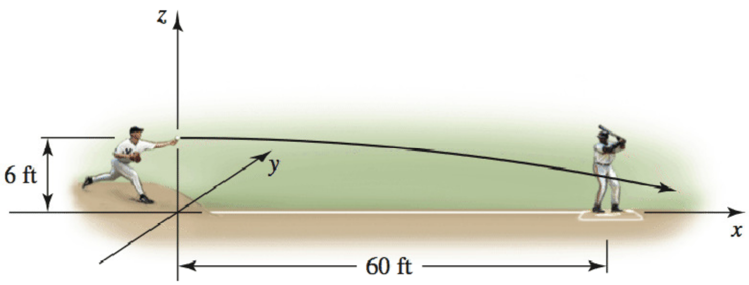
a. If the initial velocity of a ski jumper at the lip of the jump is $\langle 40, 0\rangle$ m/s, how far down the outrun does he land? Assume only gravity affects the motion.

b. Assume that air resistance produces a constant horizontal acceleration of 0.15 m/s^2 opposing the motion. How far down the outrun does the ski jumper land?

c. Suppose that the takeoff ramp is tilted upward at an angle of $\theta°$, so that the skier's initial velocity is $40\langle\cos\theta, \sin\theta\rangle$ m/s. What value of θ maximizes the length of the jump? Express your answer in degrees and neglect air resistance.



61. **Designing a baseball pitch** A baseball leaves the hand of a pitcher 6 vertical feet above home plate and 60 ft from home plate. Assume the coordinate axes are oriented as shown in the figure.



a. In the absence of all forces except gravity, assume that a pitch is thrown with an initial velocity of $\langle 130, 0, -3\rangle$ ft/s (about 90 mi/hr). How far above the ground is the ball when it crosses home plate and how long does it take for the pitch to arrive?

b. What vertical velocity component should the pitcher use so that the pitch crosses home plate exactly 3 ft above the ground?

c. A simple model to describe the curve of a baseball assumes that the spin of the ball produces a constant sideways acceleration (in the y-direction) of c ft/s^2. Assume a pitcher throws a curve ball with $c = 8$ ft/s^2 (one fourth the acceleration of gravity). How far does the ball move in the y-direction by the time it reaches home plate, assuming an initial velocity of $\langle 130, 0, -3\rangle$ ft/s?

d. In part (c), does the ball curve more in the first half of its trip to the plate or in the second half? How does this fact affect the batter?

e. Suppose the pitcher releases the ball from an initial position of $\langle 0, -3, 6\rangle$ with initial velocity $\langle 130, 0, -3\rangle$. What value of the spin parameter c is needed to put the ball over home plate passing through the point $\langle 60, 0, 3\rangle$?

62. Trajectory with a sloped landing Assume an object is launched from the origin with an initial speed $|\mathbf{v}_0|$ at an angle α to the horizontal, where $0 < \alpha < \frac{\pi}{2}$.

a. Find the time of flight, range, and maximum height (relative to launch point) of the trajectory if the ground slopes *downward* at a constant angle of θ from the launch site where $0 < \theta < \frac{\pi}{2}$.

b. Find the time of flight, range, and maximum height of the trajectory if the ground slopes *upward* at a constant angle of θ from the launch site.

63. Time of flight, range, height Derive the formulas for time of flight, range, and maximum height in the case that an object is launched from the initial position $\langle 0, y_0\rangle$ with initial velocity $|\mathbf{v}_0|\langle \cos\alpha, \sin\alpha\rangle$.

Additional Exercises

64. Parabolic trajectories Show that the two-dimensional trajectory

$$x(t) = u_0 t + x_0 \quad \text{and} \quad y(t) = -\frac{g t^2}{2} + v_0 t + y_0, \quad \text{for } 0 \le t \le T$$

of an object moving in a gravitational field is a segment of a parabola for some value of $T > 0$. Find T such that $y(T) = 0$.

65. Tilted ellipse Consider the curve $\mathbf{r}(t) = \langle \cos t, \sin t, c \sin t\rangle$ for $0 \le t \le 2\pi$, where c is a real number. It can be shown that the curve lies in a plane. Prove that the curve is an ellipse in that plane.

66. Equal area property Consider the ellipse $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$, for $0 \le t \le 2\pi$, where a and b are real numbers. Let θ be the angle between the position vector and the x-axis.

a. Show that $\tan\theta = (b/a)\tan t$.

b. Find $\theta'(t)$.

c. Recall that the area bounded by the polar curve $r = f(\theta)$ on the interval $[0, \theta]$ is $A(\theta) = \frac{1}{2}\int_0^{\theta} (f(u))^2\, du$. Letting $f(\theta(t)) = |\mathbf{r}(\theta(t))|$, show that $A'(t) = \frac{1}{2}\, a\, b$.

d. Conclude that as an object moves around the ellipse, it sweeps out equal areas in equal times.

67. Another property of constant |*r*| motion Suppose an object moves on the surface of a sphere with $|\mathbf{r}(t)|$ constant for all t. Show that $\mathbf{r}(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$ satisfy $\mathbf{r}(t) \cdot \mathbf{a}(t) = -|\mathbf{v}(t)|^2$.

68. Conditions for a circular/elliptical trajectory in the plane An object moves along a path given by

$$\mathbf{r}(t) = \langle a \cos t + b \sin t,\ c \cos t + d \sin t \rangle \text{ for } 0 \le t \le 2\pi.$$

a. What conditions on a, b, c and d guarantee that the path is a circle?

b. What conditions on a, b, c and d guarantee that the path is an ellipse?

69. Conditions for a circular/elliptical trajectory in space An object moves along a path given by

$$\mathbf{r}(t) = \langle a \cos t + b \sin t,\ c \cos t + d \sin t,\ e \cos t + f \sin t \rangle \text{ for } 0 \le t \le 2\pi.$$

a. What conditions on a, b, c, d, e, and f guarantee that the path is a circle (in a plane)?

b. What conditions on a, b, c, and d guarantee that the path is an ellipse (in a plane)?