11.8 Length of Curves

With the methods of Section 11.7, it is possible to model the trajectory of an object moving in three-dimensional space. Although we can predict the position of the object at all times, we still don't have the tools needed to answer a simple question: How far does the object travel along its flight path over a given interval of time? In this section we answer this question of *arc length*.

Arc Length

Arc Length of a Polar Curve

Quick Quiz

SECTION 11.8 EXERCISES

Review Questions

1. Find the length of the line given by $\mathbf{r}(t) = \langle t, 2t \rangle$ for $a \le t \le b$.

2. Explain how to find the length of the curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $a \le t \le b$.

3. Express the arc length of a curve in terms of the speed of an object moving along the curve.

4. Suppose an object moves in space with the position function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Write the integral that gives the distance it travels between $t = a$ and $t = b$.

5. An object moves on a trajectory given by $\mathbf{r}(t) = \langle 10\cos 2t, 10\sin 2t \rangle$ for $0 \le t \le \pi$. How far does it travel?

6. How do you find the arc length of the polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$?

Basic Skills

7-18. Arc length calculations *Find the length of the following two- and three-dimensional curves.*

7. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$, for $0 \le t \le \pi$

8. $\mathbf{r}(t) = \langle 4\cos 3t, 4\sin 3t \rangle$, for $0 \le t \le 2\pi/3$

9. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t \rangle$, for $0 \le t \le \pi/2$

10. $\mathbf{r}(t) = \langle \cos t + \sin t, \cos t - \sin t \rangle$, for $0 \le t \le 2\pi$

11. $\mathbf{r}(t) = \langle 2 + 3t, 1 - 4t, -4 + 3t \rangle$, for $1 \le t \le 6$

12. $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$, for $0 \le t \le 6\pi$

13. $\mathbf{r}(t) = \langle t, 8\sin t, 8\cos t \rangle$, for $0 \le t \le 4\pi$

14. $\mathbf{r}(t) = \left\langle t^2/2, (2t + 1)^{3/2}/3 \right\rangle$, for $0 \le t \le 2$

15. $\mathbf{r}(t) = \left\langle t^2/2, 8(t + 1)^{3/2}/3 \right\rangle$, for $0 \le t \le 2$

16. $\mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle$, for $0 \le t \le 4$

17. $\mathbf{r}(t) = \langle \cos^3 t,\ \sin^3 t \rangle$, for $0 \le t \le \pi/2$

18. $\mathbf{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t \rangle$, for $0 \le t \le 2\pi$

19-22. Speed and arc length *For the following trajectories, find the speed associated with the trajectory and then find the length of the trajectory on the given interval.*

19. $\mathbf{r}(t) = \langle 2t^3,\ -t^3,\ 5t^3 \rangle$, for $0 \le t \le 4$

20. $\mathbf{r}(t) = \langle t^2,\ 2t^2,\ t^3 \rangle$, for $1 \le t \le 2$

21. $\mathbf{r}(t) = \langle 13\sin 2t,\ 12\cos 2t,\ 5\cos 2t \rangle$, for $0 \le t \le \pi$

22. $\mathbf{r}(t) = \langle e^t \sin t,\ e^t \cos t,\ e^t \rangle$, for $0 \le t \le \ln 2$

T **23-26. Arc length approximations** *Use a calculator to approximate the length of the following curves. In each case, simplify the arc length integral as much as possible before finding an approximation.*

23. $\mathbf{r}(t) = \langle 2\cos t,\ 4\sin t \rangle$, for $0 \le t \le 2\pi$

24. $\mathbf{r}(t) = \langle 2\cos t,\ 4\sin t,\ 6\cos t \rangle$, for $0 \le t \le 2\pi$

25. $\mathbf{r}(t) = \langle t,\ 4t^2,\ 10 \rangle$, for $-2 \le t \le 2$

26. $\mathbf{r}(t) = \langle e^t,\ 2e^{-t},\ t \rangle$, for $0 \le t \le \ln 3$

27-34. Arc length of polar curves *Find the length of the following polar curves.*

27. The complete circle $r = a\sin\theta$, where $a > 0$

28. The complete cardioid $r = 2 - 2\sin\theta$

29. The complete cardioid $r = 4 + 4\sin\theta$

30. The spiral $r = 4\theta^2$, for $0 \le \theta \le 6$

31. The spiral $r = 2e^{2\theta}$, for $0 \le \theta \le \ln 8$

32. The curve $r = \sin^2(\theta/2)$, for $0 \le \theta \le \pi$

33. The curve $r = \sin^3(\theta/3)$, for $0 \le \theta \le \pi/2$

34. The parabola $r = \sqrt{2}\big/(1 + \cos\theta)$, for $0 \le \theta \le \pi/2$

Further Explorations

35. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If an object moves on a trajectory with constant speed S over a time interval $a \le t \le b$, then the length of the trajectory is $S(b - a)$.

b. The curves defined by $\mathbf{r}(t) = \langle f(t),\ g(t) \rangle$ and $\mathbf{R}(t) = \langle g(t),\ f(t) \rangle$ have the same length over the interval $[a, b]$.

c. The curve $\mathbf{r}(t) = \langle f(t),\ g(t) \rangle$ for $0 \le a \le t \le b$ and the curve $\mathbf{R}(t) = \langle f(t^2),\ g(t^2) \rangle$ for $\sqrt{a} \le t \le \sqrt{b}$ have the same length.

36. Length of a line segment Consider the line segment joining the points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$.

- **a.** Find a parametric description of the line segment PQ.
- **b.** Use the arc length formula to find the length of PQ.
- **c.** Use geometry (distance formula) to verify the result of part (b).

37. Tilted circles Let the curve C be described by $\mathbf{r}(t) = \langle a\cos t, b\sin t, c\sin t\rangle$, where a, b, and c are real positive numbers.

- **a.** Assume that C lies in a plane and show that C is a circle centered at the origin provided $a^2 = b^2 + c^2$.
- **b.** Find the arc length of the circle.
- **c.** Assuming that the curve lies in a plane, find the conditions under which $\mathbf{r}(t) = \langle a\cos t + b\sin t, c\cos t + d\sin t, e\cos t + f\sin t\rangle$ describes a circle. Then find its arc length.

38. A family of arc length integrals Find the length of the curve $\mathbf{r}(t) = \left\langle t^m, t^m, t^{3m/2}\right\rangle$ for $0 \le a \le t \le b$, where m is a real number. Express the result in terms of m, a, and b.

39. A special case Suppose a curve is described by $\mathbf{r}(t) = \langle A\,h(t), B\,h(t)\rangle$, for $a \le t \le b$, where A and B are constants and h has a continuous derivative.

- **a.** Show that the length of the curve is

$$\sqrt{A^2 + B^2}\int_a^b |h'(t)|\,dt.$$

- **b.** Use part (a) to find the length of the curve $x = 2t^3$, $y = 5t^3$, for $0 \le t \le 4$.
- **c.** Use part (a) to find the length of the curve $x = 4/t$, $y = 10/t$, for $1 \le t \le 8$.

40. Spiral arc length Consider the spiral $r = 4\theta$ for $\theta \ge 0$.

- **a.** Use a trigonometric substitution or a calculator to find the length of the spiral for $0 \le \theta \le \sqrt{8}$.
- **b.** Find $L(\theta)$, the length of the spiral on the interval $[0, \theta]$ for any $\theta \ge 0$.
- **c.** Show that $L'(\theta) > 0$. Is $L''(\theta)$ positive or negative? Interpret your answers.

41. Spiral arc length Find the length of the entire spiral $r = e^{-a\theta}$ for $\theta \ge 0$, and $a > 0$.

T **42-45. Arc length using technology** *Use a calculator to find the approximate length of the following curves.*

42. The three-leaf rose $r = 2\cos 3\theta$

43. The lemniscate $r^2 = 6\sin 2\theta$

44. The limaçon $r = 2 - 4\sin\theta$

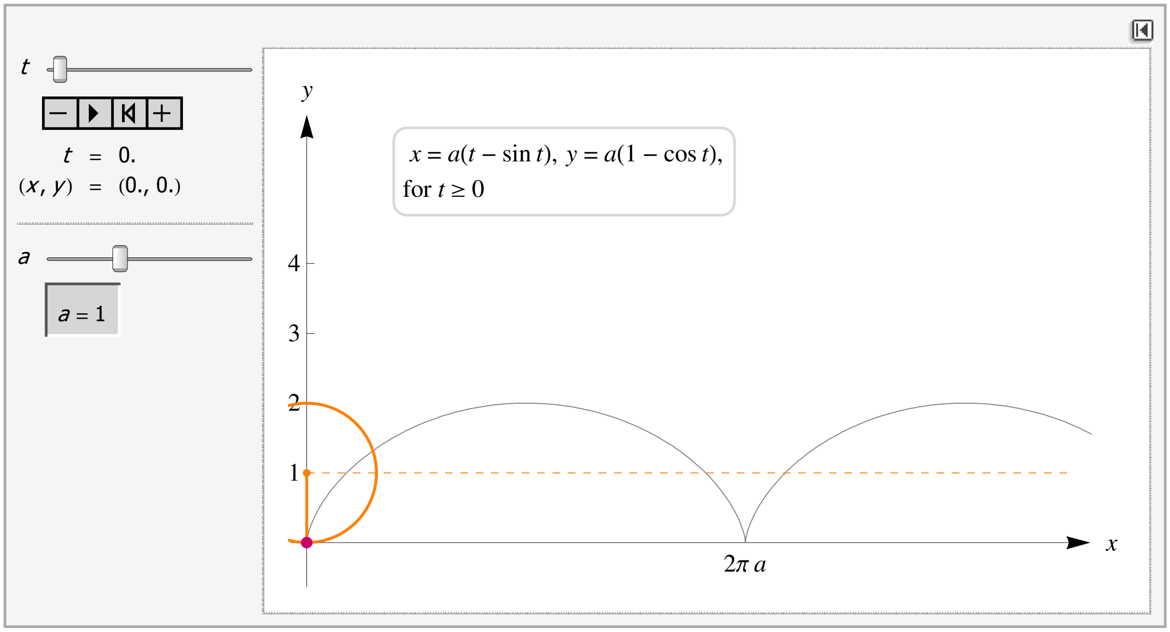
45. The limaçon $r = 4 - 2\cos\theta$

Applications

46. A cycloid A cycloid is the path traced by a point on a rolling circle (think of a light on the rim of a moving bicycle wheel). The cycloid generated by a circle of radius a is given by the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t);$$

the parameter range $0 \le t \le 2\pi$ produces one arch of the cycloid (see figure). Show that the length of one arch of a cycloid is $8a$.



47. Projectile trajectories A projectile (such as a baseball or a cannonball) launched from the origin with an initial horizontal velocity u_0 and an initial vertical velocity v_0 moves in a parabolic trajectory given by

$$x = u_0\, t, \quad y = -\left(\frac{1}{2}\right) g\, t^2 + v_0\, t, \quad \text{for } t \geq 0,$$

where air resistance is neglected and $g \approx 9.8\ \text{m}/\text{s}^2$ is the acceleration due to gravity.

a. Let $u_0 = 20\ \text{m}/\text{s}$ and $v_0 = 25\ \text{m}/\text{s}$. Assuming the projectile is launched over horizontal ground, at what time does it return to Earth?

b. Find the integral that gives the length of the trajectory from launch to landing.

c. Evaluate the integral in part (b) by first making the change of variables $u = -g\, t + v_0$. The resulting integral is evaluated either by making a second change of variables or by using a calculator. What is the length of the trajectory?

d. How far does the projectile land from its launch site?

48. Variable speed on a circle Consider a particle that moves in a plane according to the equations $x = \sin t^2$ and $y = \cos t^2$ with a starting position $(0, 1)$ at $t = 0$.

a. Describe the path of the particle, including the time required to return to the starting position.

b. What is the length of the path in part (a)?

c. Describe how the motion of this particle differs from the motion described by the equations $x = \sin t$ and $y = \cos t$.

d. Now consider the motion described by $x = \sin t^n$ and $y = \cos t^n$, where n is a positive integer. Describe the path of the particle, including the time required to return to the starting position.

e. What is the length of the path in part (d) for any positive integer n?

f. If you were watching a race on a circular path between two runners, one moving according to $x = \sin t$ and $y = \cos t$ and one according to $x = \sin t^2$ and $y = \cos t^2$, who would win and when would one runner pass the other?

Additional Exercises

49. Lengths of related curves Suppose a curve is given by $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, where f' and g' are continuous for $a \le t \le b$. Assume the curve is traversed once for $a \le t \le b$ and the length of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is L. Prove that for any nonzero constant c the length of the curve defined by $\mathbf{r}(t) = \langle c\, f(t), c\, g(t)\rangle$ for $a \le t \le b$ is $|c|\, L$.

50. Arc length for polar curves Prove that the length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\; d\theta.$$

51. Arc length for $y = f(x)$ The arc length formula for functions of the form $y = f(x)$ on $[a, b]$ found in Section 6.5 is

$$L = \int_{a}^{b} \sqrt{1 + f'(x)^2}\; dx.$$

Derive this formula from the arc length formula for vector curves. (*Hint:* Let $x = t$ be the parameter.)

52. Change of variables Consider the parametrized curves $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and $\mathbf{R}(t) = \langle f(u(t)), g(u(t)), h(u(t))\rangle$, where f, g, h, and u are continuously differentiable functions and u has an inverse on $[a, b]$.

a. Show that the curve generated by $\mathbf{r}$ on the interval $a \le t \le b$ is the same as the curve generated by $\mathbf{R}$ on $u^{-1}(a) \le t \le u^{-1}(b)$ (or $u^{-1}(b) \le t \le u^{-1}(a)$).

b. Show that the lengths of the two curves are equal. (*Hint:* Use the Chain Rule and a change of variables in the arc length integral for the curve generated by $\mathbf{R}$.)