11.9 Curvature and Normal Vectors

We know how to find tangent vectors and lengths of curves in space, but much more can be said about the shape of such curves. In this section, we introduce two new concepts: *curvature* and *normal vectors*. *Curvature* measures how *fast* a curve turns at a point and the *normal vector* describes the *direction* in which a curve turns.

Arc Length as a Parameter

Curvature

Principal Unit Normal Vector

Components of the Acceleration

Quick Quiz

SECTION 11.9 EXERCISES

Review Questions

1. Explain what it means for a curve to be parameterized by its arc length.

2. Is the curve $\mathbf{r}(t) = \langle \cos t,\ \sin t \rangle$ parametrized by its arc length? Explain.

3. Is the curve $\mathbf{r}(t) = \langle t,\ t,\ t \rangle$ parametrized by its arc length? Explain.

4. Explain in words the meaning of *the curvature of a curve*. Is it a scalar function or a vector function?

5. Give a practical formula for computing the curvature.

6. Interpret *the principal unit normal vector of a curve*. Is it a scalar function or a vector function?

7. Give a practical formula for computing the principal unit normal vector.

8. Explain how to decompose the acceleration vector of a moving object into its tangential and normal components.

Basic Skills

9-14. Arc length parametrization *Determine whether the following curves use arc length as a parameter. If not, find a description that uses arc length as a parameter.*

9. $\mathbf{r}(t) = \langle t,\ 2t \rangle$, for $0 \le t \le 3$

10. $\mathbf{r}(t) = \langle t + 1,\ 2t - 3,\ 6t \rangle$, for $0 \le t \le 10$

11. $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$, for $0 \le t \le 2\pi$

12. $\mathbf{r}(t) = \langle 5\cos t,\ 3\sin t,\ 4\sin t \rangle$, for $0 \le t \le \pi$

13. $\mathbf{r}(t) = \left\langle \cos t^2,\ \sin t^2 \right\rangle$, for $0 \le t \le \sqrt{\pi}$

14. $\mathbf{r}(t) = \left\langle t^2,\ 2t^2,\ 4t^2 \right\rangle$, for $1 \le t \le 4$

15-22. Curvature *Find the unit tangent vector* **T** *and the curvature* κ *for the following parametrized curves.*

15. $\mathbf{r}(t) = \langle 2t + 1, 4t - 5, 6t + 12 \rangle$

16. $\mathbf{r}(t) = \langle 2\cos t, -2\sin t \rangle$

17. $\mathbf{r}(t) = \langle 2t, 4\sin t, 4\cos t \rangle$

18. $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$

19. $\mathbf{r}(t) = \langle \sqrt{3}\sin t, \sin t, 2\cos t \rangle$

20. $\mathbf{r}(t) = \langle t, \ln(\cos t) \rangle$

21. $\mathbf{r}(t) = \langle t, 2t^2 \rangle$

22. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$

23-28. Alternative curvature formula *Use the alternative curvature formula* $\kappa = |\mathbf{a} \times \mathbf{v}| / |\mathbf{v}|^3$ *to find the curvature of the following parametrized curves.*

23. $\mathbf{r}(t) = \langle -3\cos t, 3\sin t, 0 \rangle$

24. $\mathbf{r}(t) = \langle t, 8\sin t, 8\cos t \rangle$

25. $\mathbf{r}(t) = \langle 4 + t^2, t, 0 \rangle$

26. $\mathbf{r}(t) = \langle \sqrt{3}\sin t, \sin t, 2\cos t \rangle$

27. $\mathbf{r}(t) = \langle 7\cos t, \sqrt{3}\sin t, 2\cos t \rangle$

28. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, t \rangle$

29-36. Principal unit normal vector *Find the unit tangent vector* **T** *and the principal unit normal vector* **N** *for the following parametrized curves. In each case, verify that* $|\mathbf{T}| = |\mathbf{N}| = 1$ *and* $\mathbf{T} \cdot \mathbf{N} = 0$.

29. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t \rangle$

30. $\mathbf{r}(t) = \langle 4\sin t, 4\cos t, 10t \rangle$

31. $\mathbf{r}(t) = \langle t^2/2, 4 - 3t, 1 \rangle$

32. $\mathbf{r}(t) = \langle t^2/2, t^3/3 \rangle$

33. $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$

34. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$

35. $\mathbf{r}(t) = \langle t^2, t \rangle$

36. $\mathbf{r}(t) = \langle t, \ln(\cos t) \rangle$

37-42. Components of the acceleration *Consider the following trajectories of moving objects. Find the tangential and normal components of the acceleration.*

37. $\mathbf{r}(t) = \langle t,\ 1 + 4t,\ 2 - 6t \rangle$

38. $\mathbf{r}(t) = \langle 10 \cos t,\ -10 \sin t \rangle$

39. $\mathbf{r}(t) = \left\langle \cos t,\ 6 \sin t,\ \sqrt{5}\ \cos t \right\rangle$

40. $\mathbf{r}(t) = \left\langle t,\ t^2 + 1 \right\rangle$

41. $\mathbf{r}(t) = \left\langle t^3,\ t^2 \right\rangle$

42. $\mathbf{r}(t) = \langle 20 \cos t,\ 20 \sin t,\ 30 t \rangle$

Further Explorations

43. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** The position, unit tangent, and principal unit normal vectors (**r**, **T**, and **N**) at a point lie in the same plane.
- **b.** The vectors **T** and **N** at a point depend on the orientation of a curve.
- **c.** The curvature at a point depends on the orientation of a curve.
- **d.** An object with unit speed ($|\mathbf{v}| = 1$) on a circle of radius R has an acceleration of $\mathbf{a} = \mathbf{N}/R$.
- **e.** If the speedometer of a car reads a constant 60 mi/hr, the car is not accelerating.

44. Special formula: Curvature for $y = f(x)$ Assume that f is twice differentiable and prove that the curve $y = f(x)$ has curvature

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}.$$

(*Hint:* Use the parametric description $x = t,\ y = f(t)$.)

45-48. Curvature for $y = f(x)$ *Use the result of Exercise 44 to find the curvature function of the following curves.*

45. $f(x) = x^2$

46. $f(x) = \sqrt{a^2 - x^2}$

47. $f(x) = \ln x$

48. $f(x) = \ln(\cos x)$

49. Special formula: Curvature for plane curves Show that the curve $\mathbf{r}(t) = \langle f(t),\ g(t) \rangle$, where f and g are twice differentiable, has curvature

$$\kappa(t) = \frac{|f'g'' - f''g'|}{\left((f')^2 + (g')^2\right)^{3/2}},$$

where all derivatives are taken with respect to t.

50-53. Curvature for plane curves *Use the result of Exercise 49 to find the curvature function of the following curves.*

50. $\mathbf{r}(t) = \langle a \sin t,\ a \cos t \rangle$ (circle)

51. $\mathbf{r}(t) = \langle a \sin t,\ b \cos t \rangle$ (ellipse)

52. $\mathbf{r}(t) = \langle a \cos^3 t,\ a \sin^3 t \rangle$ (astroid)

53. $\mathbf{r}(t) = \langle t,\ a\, t^2 \rangle$ (parabola)

When appropriate, consider using the special formulas derived in Exercises 44 and 49 in the remaining exercises.

54-57. Same paths, different velocity *The position functions of objects A and B describe different motion along the same path for $t \geq 0$.*

- ***a.*** *Sketch the path followed by both A and B.*
- ***b.*** *Find the velocity and acceleration of A and B and discuss the differences.*
- ***c.*** *Express the acceleration of A and B in terms of the tangential and normal components and discuss the differences.*

54. $A : \mathbf{r}(t) = \langle 1 + 2\,t,\ 2 - 3\,t,\ 4\,t \rangle,\ B : \mathbf{r}(t) = \langle 1 + 6\,t,\ 2 - 9\,t,\ 12\,t \rangle$

55. $A : \mathbf{r}(t) = \langle t,\ 2\,t,\ 3\,t \rangle,\ B : \mathbf{r}(t) = \langle t^2,\ 2\,t^2,\ 3\,t^2 \rangle$

56. $A : \mathbf{r}(t) = \langle \cos t,\ \sin t \rangle,\ B : \mathbf{r}(t) = \langle \cos 3\,t,\ \sin 3\,t \rangle$

57. $A : \mathbf{r}(t) = \langle \cos t,\ \sin t \rangle,\ B : \mathbf{r}(t) = \langle \cos t^2,\ \sin t^2 \rangle$

T **58-61. Graphs of the curvature** *Consider the following curves.*

- ***a.*** *Graph the curve.*
- ***b.*** *Compute the curvature (using either of the two formulas).*
- ***c.*** *Graph the curvature as a function of the parameter.*
- ***d.*** *Identify the points (if any) at which the curve has a minimum and maximum curvature.*
- ***e.*** *Verify that the graph of the curvature is consistent with the graph of the curve.*

58. $\mathbf{r}(t) = \langle t,\ t^2 \rangle$, for $-2 \leq t \leq 2$ (parabola)

59. $\mathbf{r}(t) = \langle t - \sin t,\ 1 - \cos t \rangle$, for $0 \leq t \leq 2\pi$ (cycloid)

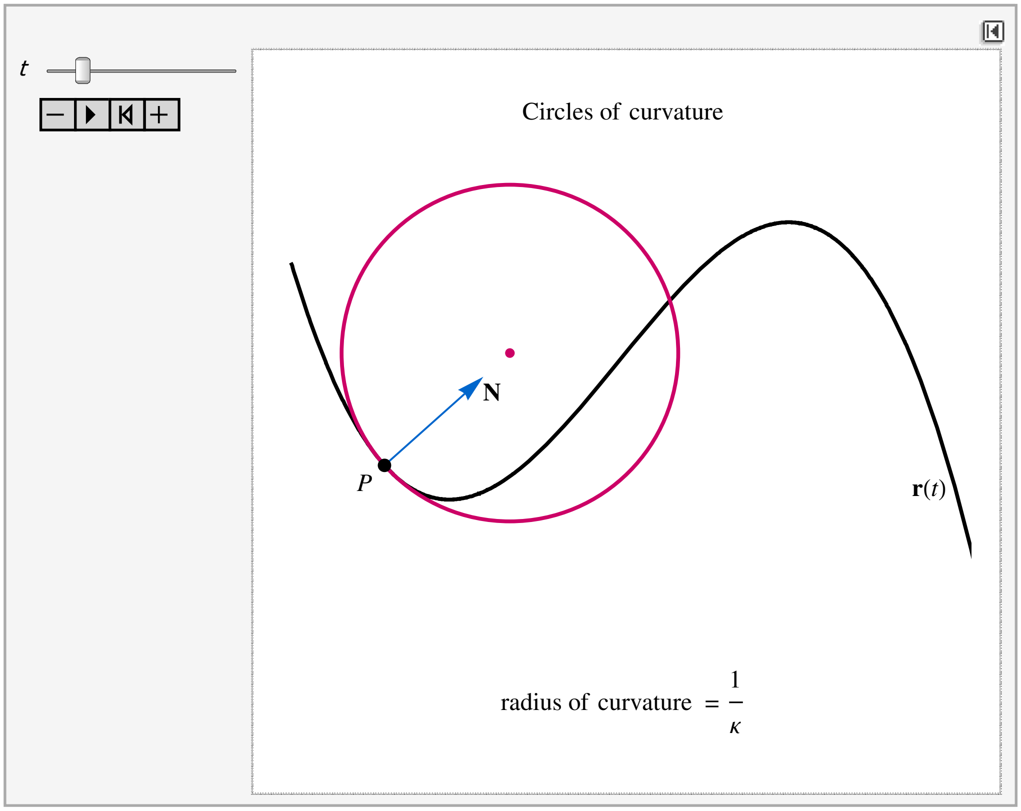
60. $\mathbf{r}(t) = \langle t,\ \sin t \rangle$, for $0 \leq t \leq \pi$ (sine curve)

61. $\mathbf{r}(t) = \langle t^2/2,\ t^3/3 \rangle$, for $t > 0$

62. **Curvature of ln *x*** Find the curvature of $f(x) = \ln x$, for $x > 0$, and find the point at which it is a maximum. What is the value of the maximum curvature?

63. **Curvature of e^x** Find the curvature of $f(x) = e^x$ and find the point at which it is a maximum. What is the value of the maximum curvature?

64. **Circle and radius of curvature** Choose a point P on a smooth curve C in the plane. The **circle of curvature** (or **osculating circle**) at the point P is the circle that (a) is tangent to C at P, (b) has the same curvature as C at P, and (c) lies on the same side of C as the principal unit normal $\mathbf{N}$ (see figure). The **radius of curvature** is the radius of the circle of curvature. Show that the radius of curvature is $1/\kappa$, where κ is the curvature of C at P.



65-68. Finding radii of curvature *Find the radius of curvature (see Exercise 64) of the following curves at the given point. Then write the equation of the circle of curvature at the point.*

65. $\mathbf{r}(t) = \langle t, t^2 \rangle$ (parabola) at $t = 0$

66. $y = \ln x$ at $x = 1$

67. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ (cycloid) at $t = \pi$

68. $y = \sin x$ at $x = \pi/2$

69. Curvature of the sine curve The function $f(x) = \sin nx$, where n is a positive real number, has a local maximum at $x = \pi/(2n)$. Compute the curvature κ of f at this point. How does κ vary (if at all) as n varies?

Applications

70. Parabolic trajectory In Example 8 it was shown that for the parabolic trajectory $\mathbf{r}(t) = \langle t, t^2 \rangle$, $\mathbf{a} = \langle 0, 2 \rangle$ and $\mathbf{a} = \frac{2}{\sqrt{1 + 4t^2}}(\mathbf{N} + 2t\,\mathbf{T})$. Show that the second equation for $\mathbf{a}$ reduces to the first equation.

T **71. Parabolic trajectory** Consider the parabolic trajectory

$$x = (V_0 \cos \alpha)\, t, \quad y = (V_0 \sin \alpha)\, t - \frac{1}{2} g t^2,$$

where V_0 is the initial speed, α is the angle of launch, and g is the acceleration due to gravity. Consider all times $[0, T]$ for which $y \geq 0$.

a. Find and graph the speed for $0 \le t \le T$.

b. Find and graph the curvature for $0 \le t \le T$.

c. At what times (if any) do the speed and curvature have minimum and maximum values?

72. Relationship between T, N, and a Show that if an object accelerates in the sense that $d^2 s/dt^2 > 0$ and $\kappa \ne 0$, then the acceleration vector lies between **T** and **N** in the plane of **T** and **N**. If an object decelerates in the sense that $d^2 s/dt^2 < 0$, then the acceleration vector lies in the plane of **T** and **N**, but not between **T** and **N**.

Additional Exercises

73. Arc length parametrization Prove that the line $\mathbf{r}(t) = \langle x_0 + a\,t,\ y_0 + b\,t,\ z_0 + c\,t \rangle$ is parametrized by arc length provided $a^2 + b^2 + c^2 = 1$.

74. Arc length parametrization Prove that the curve $\mathbf{r}(t) = \langle a\cos t,\ b\sin t,\ c\sin t \rangle$ is parameterized by arc length provided $a^2 = b^2 + c^2 = 1$.

75. Zero curvature Prove that the curve

$$\mathbf{r}(t) = \langle a + b\,t^p,\ c + d\,t^p,\ e + f\,t^p \rangle,$$

where a, b, c, d, e, f are real numbers and p is a positive integer, has zero curvature. Give an explanation.

76. Practical formula for N Show that the definition of the principal unit normal vector $\mathbf{N} = \dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$ implies the practical formula $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$. Use the Chain Rule and recall that $|\mathbf{v}| = ds/dt > 0$.

T **77. Maximum curvature** Consider the "superparabolas" $f_n(x) = x^{2n}$, where n is a positive integer.

a. Find the curvature function of f_n, for $n = 1, 2, 3$.

b. Plot f_n and their curvature functions, for $n = 1, 2, 3$, and check for consistency.

c. At what points does the maximum curvature occur, for $n = 1, 2, 3$?

d. Let the maximum curvature for f_n occur at $x = \pm z_n$. Using either analytical methods or a calculator determine $\lim_{n\to\infty} z_n$. Interpret your result.

78. Descartes' four-circle solution Consider the four mutually tangent circles shown in the figure that have radii a, b, c, and d, and curvatures $A = 1/a$, $B = 1/b$, $C = 1/c$, and $D = 1/d$. Prove Descartes' result (1643) that

$$(A + B + C + D)^2 = 2\left(A^2 + B^2 + C^2 + D^2\right).$$