Chapter Preview

Chapter 11 was devoted to vector-valued functions, which generally have one independent variable and two or more dependent variables. In this chapter, we step into three-dimensional space along a different path by considering functions with several independent variables and one dependent variable. All the familiar properties of single-variable functions—domains, graphs, limits, continuity, and derivatives—have generalizations for multivariable functions, although there are often subtle differences when compared to single-variable functions. With functions of several independent variables, we work with *partial derivatives*, which, in turn, give rise to directional derivatives and the *gradient*, a fundamental concept in calculus. Partial derivatives allow us to find maximum and minimum values of multivariable functions. We define tangent planes, rather than tangent lines, that allow us to make linear approximations. The chapter ends with a survey of optimization problems in several variables.

12.1 Planes and Surfaces

Functions with one independent variable, such as $f(x) = x e^{-x}$, or *equations* in two variables, such as $x^2 + y^2 = 4$, describe curves in $\mathbb{R}^2$. We now add a third variable to the picture and consider functions of two independent variables (for example, $f(x, y) = x^2 + 2y^2$) and equations in three variables (for example, $x^2 + y^2 + 2z^2 = 4$). We see in this chapter that such functions and equations describe *surfaces* that may be displayed in $\mathbb{R}^3$. Just as a line is the simplest curve in $\mathbb{R}^2$, a plane is the fundamental surface in $\mathbb{R}^3$.

Equations of Planes

Parallel and Orthogonal Planes

Cylinders and Traces

Quadric Surfaces

Quick Quiz

SECTION 12.1 EXERCISES

Review Questions

1. Give two pieces of information which, taken together, uniquely determine a plane.

2. Find a vector normal to the plane $-2x - 3y + 4z = 12$.

3. Where does the plane $-2x - 3y + 4z = 12$ intersect the coordinate axes?

4. What is the equation of the plane with a normal vector $\mathbf{n} = \langle 1, 1, 1 \rangle$ that passes through the point $(1, 0, 0)$?

5. To which coordinate axes are the following cylinders in $\mathbb{R}^3$ parallel: $x^2 + 2y^2 = 8$, $z^2 + 2y^2 = 8$, and $x^2 + 2z^2 = 8$?

6. Describe the graph of $x = z^2$ in $\mathbb{R}^3$.

7. What are the traces of a surface?

8. What is the name of the type of surface defined by the equation $y = \frac{x^2}{4} + \frac{z^2}{8}$?

9. What is the name of the type of surface defined by the equation $x^2 + \frac{y^2}{3} + 2z^2 = 1$?

10. What is the name of the type of surface defined by the equation $-y^2 - \frac{z^2}{2} + x^2 = 1$?

Basic Skills

11-14. Equations of planes *Find an equation of the plane that passes through the point P_0 with a normal vector **n**.*

11. $P_0(0, 2, -2)$; $\mathbf{n} = \langle 1, 1, -1 \rangle$

12. $P_0(1, 0, -3)$; $\mathbf{n} = \langle 1, -1, 2 \rangle$

13. $P_0(2, 3, 0)$; $\mathbf{n} = \langle -1, 2, -3 \rangle$

14. $P_0(1, 2, -3)$; $\mathbf{n} = \langle -1, 4, -3 \rangle$

15-18. Equations of planes *Find an equation of the following planes.*

15. The plane passing through the points $(1, 0, 3)$, $(0, 4, 2)$, and $(1, 1, 1)$

16. The plane passing through the points $(-1, 1, 1)$, $(0, 0, 2)$, and $(3, -1, -2)$

17. The plane passing through the points $(2, -1, 4)$, $(1, 1, -1)$, and $(-4, 1, 1)$

18. The plane passing through the points $(5, 3, 1)$, $(1, 3, -5)$, and $(-1, 3, 1)$

19-22. Properties of planes *Find the points at which the following planes intersect the coordinate axes and find equations of the lines where the planes intersect the coordinate planes. Sketch a graph of the plane.*

19. $3x - 2y + z = 6$

20. $-4x + 8z = 16$

21. $x + 3y - 5z - 30 = 0$

22. $12x - 9y + 4z + 72 = 0$

23-24. Equations of planes *For the following sets of planes, determine which pairs of planes in the set are parallel, orthogonal, or identical.*

23. $Q: 3x - 2y + z = 12$; $R: -x + 2y/3 - z/3 = 0$; $S: -x + 2y + 7z = 1$; $T: 3x/2 - y + z/2 = 6$

24. $Q: x + y - z = 0$; $R: y + z = 0$; $S: x - y = 0$; $T: x + y + z = 0$

25-28. Parallel planes *Find an equation of the plane parallel to the plane Q passing through the point P_0.*

25. $Q: -x + 2y - 4z = 1$; $P_0(1, 0, 4)$

26. $Q: 2x + y - z = 1$; $P_0(0, 2, -2)$

27. $Q: 4x + 3y - 2z = 12$; $P_0(1, -1, 3)$

28. $Q: x - 5y - 2z = 1$; $P_0(1, 2, 0)$

29-32. Intersecting planes *Find an equation of the line where the planes Q and R intersect.*

29. $Q: -x + 2y + z = 1;\ R: x + y + z = 0$

30. $Q: x + 2y - z = 1;\ R: x + y + z = 1$

31. $Q: 2x - y + 3z - 1 = 0;\ R: -x + 3y + z - 4 = 0$

32. $Q: x - y - 2z = 1;\ R: x + y + z = -1$

33-36. Cylinders in $\mathbb{R}^3$ *Consider the following cylinders in $\mathbb{R}^3$.*

a. *Identify the coordinate axis to which the cylinder is parallel.*

b. *Sketch the cylinder.*

33. $y - x^3 = 0$

34. $x - 2z^2 = 0$

35. $z - \ln y = 0$

36. $x - 1/y = 0$

37-60. Quadric surfaces *Consider the following equations of quadric surfaces.*

a. *Find the intercepts with the three coordinate axes, when they exist.*

b. *Find the equations of the xy-, xz-, and yz-traces, when they exist.*

c. *Sketch a graph of the surface.*

Ellipsoids

37. $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$

38. $4x^2 + y^2 + \frac{z^2}{2} = 1$

39. $\frac{x^2}{3} + 3y^2 + \frac{z^2}{12} = 3$

40. $\frac{x^2}{6} + 24y^2 + \frac{z^2}{24} - 6 = 0$

Elliptic paraboloids

41. $x = y^2 + z^2$

42. $z = \frac{x^2}{4} + \frac{y^2}{9}$

43. $9x - 81y^2 - \frac{z^2}{4} = 0$

44. $2y - \frac{x^2}{8} - \frac{z^2}{18} = 0$

Hyperboloids of one sheet

45. $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 1$

46. $\frac{y^2}{2} + \frac{z^2}{36} - 4x^2 = 1$

47. $\frac{y^2}{16} + 36z^2 - \frac{x^2}{4} - 9 = 0$

48. $9z^2 + x^2 - \frac{y^2}{3} - 1 = 0$

Hyperbolic paraboloids

49. $z = \frac{x^2}{9} - y^2$

50. $y = \frac{x^2}{16} - 4z^2$

51. $5x - \frac{y^2}{5} + \frac{z^2}{20} = 0$

52. $6y + \frac{x^2}{6} - \frac{z^2}{24} = 0$

Elliptic cones

53. $x^2 + \frac{y^2}{4} = z^2$

54. $4y^2 + \frac{z^2}{4} = 9x^2$

55. $\frac{z^2}{32} + \frac{y^2}{18} = 2x^2$

56. $\frac{x^2}{3} + \frac{z^2}{12} = 3y^2$

Hyperboloids of two sheets

57. $-x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$

58. $1 - 4x^2 + y^2 + \frac{z^2}{2} = 0$

59. $-\frac{x^2}{3} + 3y^2 - \frac{z^2}{12} = 1$

60. $-\frac{x^2}{6} - 24y^2 + \frac{z^2}{24} - 6 = 0$

Further Explorations

61. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

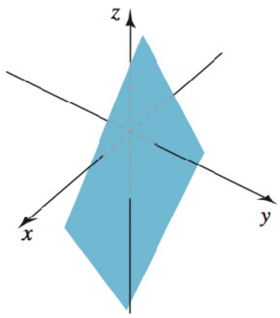
- **a.** The plane passing through the point (1, 1, 1) with a normal vector $\mathbf{n} = \langle 1, 2, -3 \rangle$ is the same as the plane passing through the point (3, 0, 1) with a normal vector $\mathbf{n} = \langle -2, -4, 6 \rangle$.
- **b.** The equations $x + y - z = 1$ and $-x - y + z = 1$ describe the same plane.
- **c.** Given a plane Q, there is exactly one plane orthogonal to Q.
- **d.** Given a line l and a point P_0 not on l, there is exactly one plane that contains l and passes through P_0.
- **e.** Given a plane R and a point P_0, there is exactly one plane that is orthogonal to R and passes through P_0.
- **f.** Any two distinct lines in $\mathbb{R}^3$ determine a unique plane.
- **g.** If plane Q is orthogonal to plane R and plane R is orthogonal to plane S, then plane Q is orthogonal to plane S.

62. **Plane containing a line and a point** Find an equation of the plane that passes through the point P_0 and contains the line l.

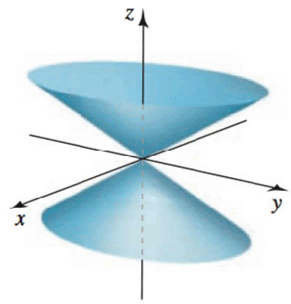
- **a.** $P_0(1, -2, 3)$; $l : \mathbf{r} = \langle t, -t, 2t \rangle$, $-\infty < t < \infty$
- **b.** $P_0(-4, 1, 2)$; $l : \mathbf{r} = \langle 2t, -2t, -4t \rangle$, $-\infty < t < \infty$

63. **Matching graphs with equations** Match equations a—f with surfaces A—F.

- **a.** $y - z^2 = 0$
- **b.** $2x + 3y - z = 5$
- **c.** $4x^2 + \frac{y^2}{9} + z^2 = 1$
- **d.** $x^2 + \frac{y^2}{9} - z^2 = 1$
- **e.** $x^2 + \frac{y^2}{9} = z^2$
- **f.** $y = |x|$



(A)



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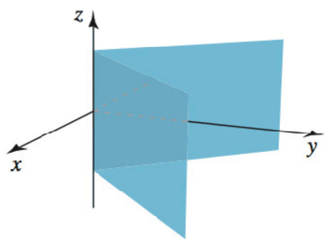

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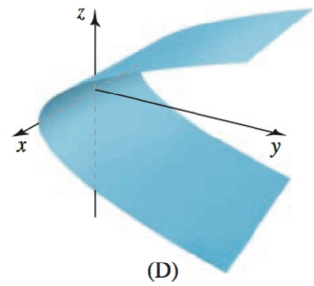

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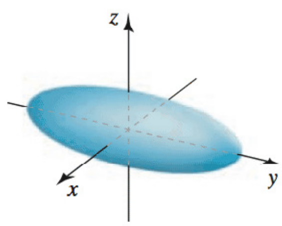

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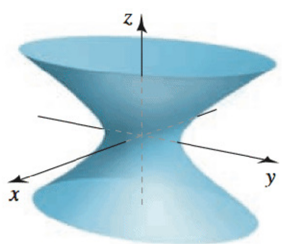


(F)

64-73. Identifying surfaces *Identify and briefly describe the surfaces defined by the following equations.*

64. $z^2 + 4y^2 - x^2 = 1$

65. $y = 4z^2 - x^2$

66. $-y^2 - 9z^2 + x^2/4 = 1$

67. $y = x^2/6 + z^2/16$

68. $x^2 + y^2 + 4z^2 + 2x = 0$

69. $9x^2 + y^2 - 4z^2 + 2y = 0$

70. $x^2 + 4y^2 = 1$

71. $y^2 - z^2 = 2$

72. $-x^2 - y^2 + z^2/9 + 6x - 8y = 26$

73. $x^2/4 + y^2 - 2x - 10y - z^2 + 41 = 0$

74-77. Curve-plane intersections *Find the points (if they exist) at which the following planes and curves intersect.*

74. $y = 2x + 1$; $\mathbf{r}(t) = \langle 10\cos t, 2\sin t, 1\rangle$, for $0 \le t \le 2\pi$

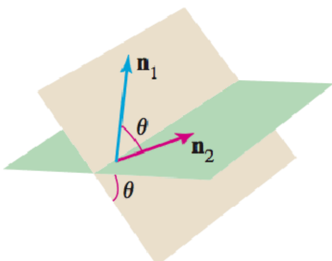
75. $8x + y + z = 60$; $\mathbf{r}(t) = \langle t, t^2, 3t^2\rangle$, for $-\infty < t < \infty$

76. $8x + 15y + 3z = 20$; $\mathbf{r}(t) = \langle 1, \sqrt{t}, -t\rangle$, for $t > 0$

77. $2x + 3y - 12z = 0$; $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, \cos t\rangle$, for $0 \le t \le 2\pi$

78. Intercepts Let a, b, c, and d be constants. Find the points at which the plane $ax + by + cz = d$ intersects the x-, y-, and z-axes.

79. Angle between planes The angle between two planes is the angle θ between the normal vectors of the planes, where the directions of the normal vectors are chosen so that $0 \le \theta \le \pi$. Find the angle between the planes $5x + 2y - z = 0$ and $-3x + y + 2z = 0$.



80. Solids of revolution Consider the ellipse $x^2 + 4y^2 = 1$ in the xy-plane.

a. If this ellipse is revolved about the x-axis, what is the equation of the resulting ellipsoid?

b. If this ellipse is revolved about the y-axis, what is the equation of the resulting ellipsoid?

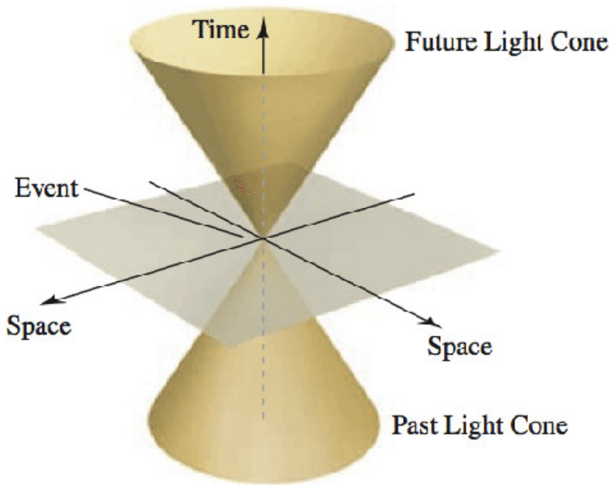
81. Solids of revolution Which of the quadric surfaces in Table 12.1 can be generated by revolving a curve in one of the coordinate planes about a coordinate axis?

Applications

82. Light cones The idea of a *light cone* appears in the Special Theory of Relativity. The xy-plane (see figure) represents all of three-dimensional space, and the z-axis is the time axis (t-axis). If an event E occurs at the origin, the interior of the future light cone ($t > 0$) represents all events in the future that are affected by E, assuming that no signal travels faster than the speed of light. The interior of the past light cone ($t < 0$) represents all events in the past that could have affected E, again assuming that no signal travels faster than the speed of light.

a. If time is measured in seconds and distance (x and y) is measured in light-seconds (the distance light travels in 1 s), the light cone makes a 45° angle with the xy-plane. Write the equation of the light cone in this case.

b. Suppose distance is measured in meters and time is measured in seconds. Write the equation of the light cone in this case given that the speed of light is 3×10^8 m/s.



83. T-shirt profits A clothing company makes a profit of \$10 on its long-sleeved T-shirts and \$5 on its short-sleeved T-shirts. Assuming there is a \$200 setup cost, the profit on T-shirt sales is $z = 10x + 5y - 200$, where x is the number of long-sleeved T-shirts sold and y is the number of short-sleeved T-shirts sold. Assume x and y are nonnegative.

a. Graph the plane that gives the profit using the window $[0, 40] \times [0, 40] \times [-400, 400]$.

b. If $x = 20$ and $y = 10$, is the profit positive or negative?

c. Describe the values of x and y for which the company breaks even (for which the profit is zero). Mark this set on your graph.

Additional Exercises

84. Parallel line and plane Show that the plane $ax + by + cz = d$ and the line $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t$, not in the plane, have no points of intersection if and only if $\mathbf{v} \cdot \langle a, b, c \rangle = 0$. Give a geometric explanation of the result.

85. Tilted ellipse Consider the curve $\mathbf{r}(t) = \langle \cos t, \sin t, c \sin t \rangle$ for $0 \le t \le 2\pi$, where c is a real number.

a. What is the equation of the plane P in which the curve lies?

b. What is the angle between P and the xy-plane?

c. Prove that the curve is an ellipse in P.

86. Distance from a point to a plane

a. Show that the point in the plane $ax + by + cz = d$ nearest the origin is $P(ad/D^2, bd/D^2, cd/D^2)$, where $D^2 = a^2 + b^2 + c^2$. Conclude that the least distance from the plane to the origin is $|d|/D$. (*Hint:* The least distance is along a normal to the plane.)

b. Show that the least distance from the point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is $|ax_0 + by_0 + cz_0 - d|/D$. (*Hint:* Find the point P on the plane closest to P_0.)

87. Projections Find the projection of the position vector $\langle 2, 3, -4 \rangle$ on the plane $3x + 2y - z = 0$.

88. Ellipsoid-plane intersection Let E be the ellipsoid $x^2/9 + y^2/4 + z^2 = 1$, P be the plane $z = Ax + By$, and C be the intersection of E and P.

a. Is C an ellipse for all values of A and B? Explain.

b. Sketch and interpret the situation in which $A = 0$ and $B \ne 0$.

c. Find an equation of the projection of C on the xy-plane.

d. Assume $A = \frac{1}{6}$ and $B = \frac{1}{2}$. Find a parametric description of C as a curve in $\mathbb{R}^3$. (*Hint:* Assume C is described by

$\langle a\cos t + b\sin t,\ c\cos t + d\sin t,\ e\cos t + f\sin t\rangle$ and find a, b, c, d, e, and f.)