12.2 Graphs and Level Curves

In Chapter 11 we discussed vector-valued functions with one independent variable and several dependent variables. We now reverse the situation and consider functions with several independent variables and one dependent variable. Such functions are aptly called *functions of several variables* or *multivariable functions*.

To set the stage, consider the following practical questions that illustrate a few of the many applications of functions of several variables.

- What is the probability that one man selected randomly from a large group of men weighs more than 200 pounds and is over 6 feet tall?
- Where on the wing of an airliner flying at a speed of 550 mi/hr is the pressure greatest?
- A physician knows the optimal blood concentration of an antibiotic needed by a patient. What dosage of antibiotic is needed and how often should it be given to reach this optimal level?

Although we don't answer these questions immediately, they provide an idea of the scope and importance of the topic. First, we must introduce the idea of a function of several variables.

Functions of Two Variables

Graphs of Functions of Two Variables

Applications of Functions of Two Variables

Functions of More Than Two Variables

Graphs of Functions of More Than Two Variables

Quick Quiz

SECTION 12.2 EXERCISES

Review Questions

1. A function is defined by $z = x^2 y - x y^2$. Identify the independent and dependent variables.

2. What is the domain of $f(x, y) = x^2 y - x y^2$?

3. What is the domain of $g(x, y) = 1/(x y)$?

4. What is the domain of $h(x, y) = \sqrt{x - y}$?

5. How many axes or how many dimensions are needed to graph the function $z = f(x, y)$? Explain.

6. Explain how to graph the level curves of a surface $z = f(x, y)$.

7. Describe in words the level curves of the paraboloid $z = x^2 + y^2$.

8. How many axes (or how many dimensions) are needed to graph the level surfaces of $w = f(x, y, z)$? Explain.

9. The domain of $Q = f(u, v, w, x, y, z)$ lies in $\mathbb{R}^n$ for what value of n? Explain.

10. Give two methods for graphically representing a function with three independent variables.

Basic Skills

11-18. Domains *Find the domain of the following functions.*

11. $f(x, y) = 2xy - 3x + 4y$

12. $f(x, y) = \cos(x^2 - y^2)$

13. $f(x, y) = \sin\left(\frac{x}{y}\right)$

14. $f(x, y) = \frac{12}{y^2 - x^2}$

15. $g(x, y) = \ln(x^2 - y)$

16. $f(x, y) = \tan^{-1}(x + y)$

17. $g(x, y) = \sqrt{\frac{xy}{x^2 + y^2}}$

18. $h(x, y) = \sqrt{x - 2y + 4}$

19-26. Graphs of familiar functions *Use what you learned about surfaces in Section 12.1 to sketch a graph of the following functions. In each case identify the surface, and state the domain and range of the function.*

19. $f(x, y) = 3x - 6y + 18$

20. $h(x, y) = 2x^2 + 3y^2$

21. $p(x, y) = x^2 - y^2$

22. $F(x, y) = \sqrt{1 - x^2 - y^2}$

23. $G(x, y) = -\sqrt{1 + x^2 + y^2}$

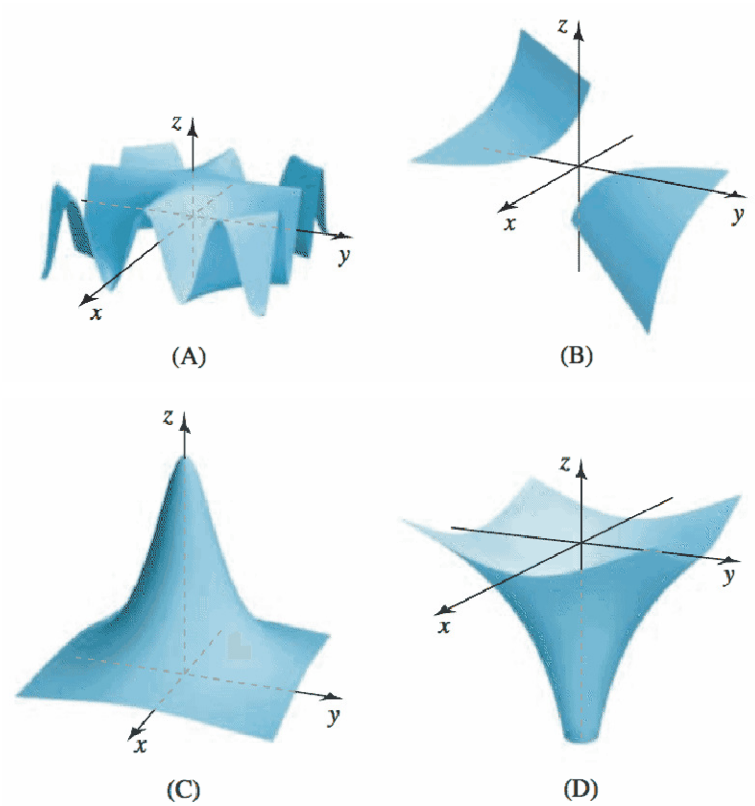
24. $H(x, y) = \sqrt{x^2 + y^2}$

25. $P(x, y) = \sqrt{x^2 + y^2 - 1}$

26. $g(x, y) = y^3 + 1$

27. **Matching surfaces** Match functions a-d with surfaces A-D in the figure.

- **a.** $f(x, y) = \cos xy$
- **b.** $g(x, y) = \ln(x^2 + y^2)$
- **c.** $h(x, y) = 1/(x - y)$
- **d.** $p(x, y) = 1/(1 + x^2 + y^2)$



T **28-33. Level curves** *Graph several level curves of the following functions using the given window. Label at least two level curves with their z-values.*

28. $z = 2x - y$; $[-2, 2] \times [-2, 2]$

29. $z = \sqrt{x^2 + 4y^2}$; $[-8, 8] \times [-8, 8]$

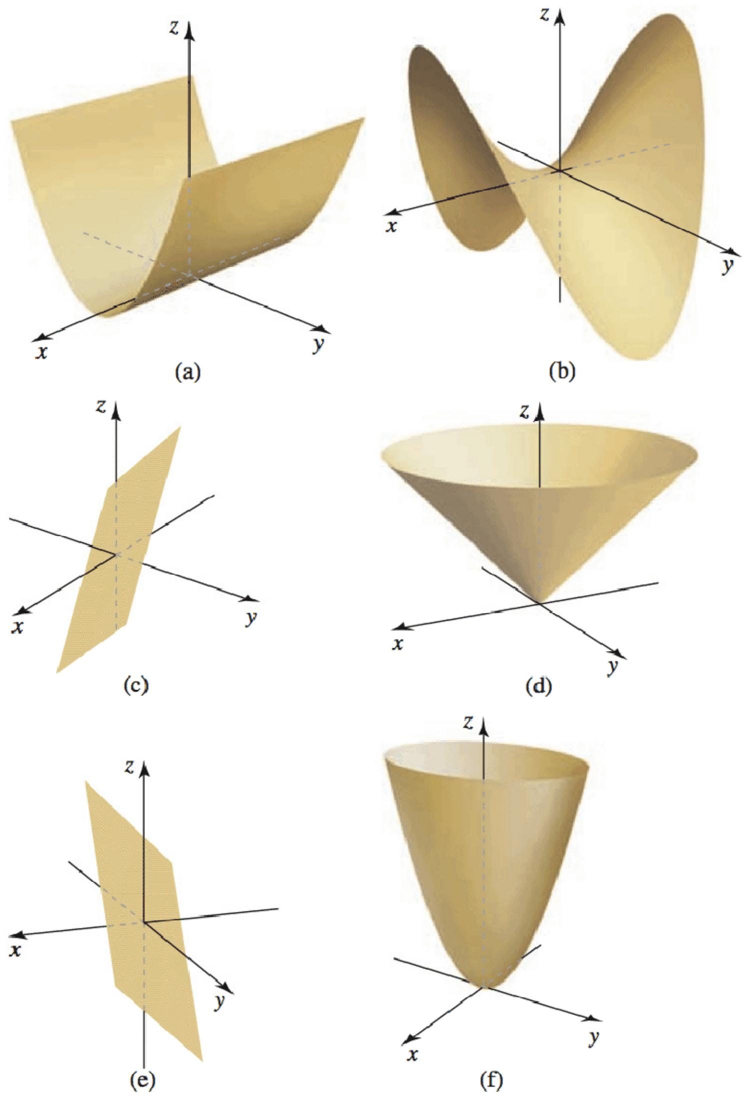
30. $z = e^{-x^2 - 2y^2}$; $[-2, 2] \times [-2, 2]$

31. $z = \sqrt{25 - x^2 - y^2}$; $[-6, 6] \times [-6, 6]$

32. $z = \sqrt{y - x^2 - 1}$; $[-5, 5] \times [-5, 5]$

33. $z = 3\cos(2x + y)$; $[-2, 2] \times [-2, 2]$

34. Matching level curves with surfaces Match surfaces a-f in the figure with level curves A-F.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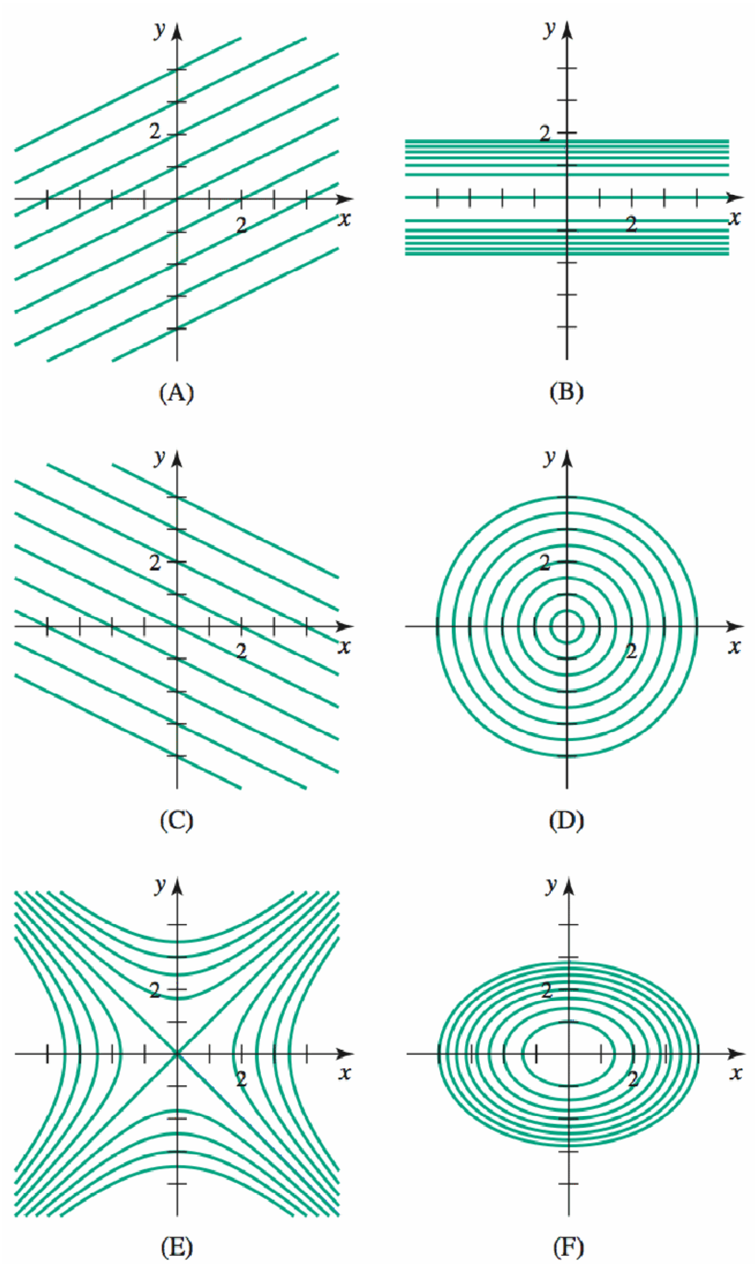


T **35. A volume function** The volume of a right circular cone of radius r and height h is $V(r, h) = \pi r^2 h/3$.

a. Graph the function in the window $[0, 5]\times[0, 5]\times[0, 150]$.

b. What is the domain of the volume function?

c. What is the relationship between the values of r and h when $V = 100$?

T **36. Earned run average** A baseball pitcher's earned run average (ERA) is $A(e, i) = 9\,e/i$, where e is the number of earned runs given up by the pitcher and i is the number of innings pitched. Good pitchers have low ERAs. Assume that $e \geq 0$ and $i > 0$ are real numbers.

a. The single-season major league record for the lowest ERA was set by Dutch Leonard of the Detroit Tigers in 1914. During that season, Dutch pitched a total of 224 innings and gave up just 24 earned runs. What was his ERA?

b. Determine the ERA of a relief pitcher who gives up 4 earned runs in one-third of an inning.

c. Graph the level curve $A(e, i) = 3$, and describe the relationship between e and i in this case.

T **37. Electric potential function** The electric potential function for two positive charges, one at (0, 1) with twice the strength as the charge at (0, −1), is given by

$$\phi(x, y) = \frac{2}{\sqrt{x^2 + (y-1)^2}} + \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

a. Graph the electric potential using the window $[-5, 5] \times [-5, 5] \times [0, 10]$.

b. For what values of x and y is the potential ϕ defined?

c. Is the electric potential greater at (3, 2) or (2, 3)?

d. Describe how the electric potential varies along the line $y = x$.

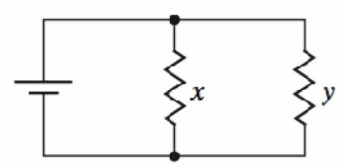
T **38. Cobb-Douglas production function** The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = c\, L^a K^b$, where a, b, and c are positive real numbers. When $a + b = 1$, the case is called *constant returns to scale*. Suppose $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = 40$.

a. Graph the output function using the window $[0, 20] \times [0, 20] \times [0, 500]$.

b. If L is held constant at $L = 10$, write the function that gives the dependence of Q on K.

c. If K is held constant at $K = 15$, write the function that gives the dependence of Q on L.

T **39. Resistors in parallel** Two resistors wired in parallel in an electrical circuit give an effective resistance of $R(x, y) = \frac{x\,y}{x + y}$, where x and y are the positive resistances of the individual resistors (typically measured in ohms).



a. Graph the resistance function using the window $[0, 10] \times [0, 10] \times [0, 5]$.

b. Estimate the maximum value of R for $0 < x \le 10$ and $0 < y \le 10$.

c. Explain what it means to say that the resistance function is symmetric in x and y.

T **40. Water waves** A snapshot of a water wave moving toward shore is described by the function $z = 10 \sin(2x - 3y)$, where z is the height of the water surface above (or below) the xy-plane, which is the level of undisturbed water.

a. Graph the height function using the window $[-5, 5] \times [-5, 5] \times [-15, 15]$.

b. For what values of x and y is z defined?

c. What are the maximum and minimum values of the water height?

d. Give a vector in the xy-plane that is orthogonal to the level curves of the crests and troughs of the wave (which also gives the direction of wave propagation).

T **41. Approximate mountains** Suppose the elevation of Earth's surface over a 16-mi by 16-mi region is approximated by the function

$$z = 10\, e^{-(x^2+y^2)} + 5\, e^{-((x+5)^2+(y-3)^2)/10} + 4\, e^{-2((x-4)^2+(y+1)^2)}.$$

a. Graph the height function using the window $[-8, 8] \times [-8, 8] \times [0, 15]$.

b. Approximate the points (x, y) where the peaks in the landscape appear.

c. What are the approximate elevations of the peaks?

42-48. Domains of functions of three or more variables *Find the domain of the following functions. If possible, give a description of the domain in words (for example, all points outside a sphere of radius 1 centered at the origin).*

42. $f(x, y, z) = 2xyz - 3xz + 4yz$

43. $g(x, y, z) = \dfrac{1}{x - z}$

44. $p(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$

45. $f(x, y, z) = \sqrt{y - z}$

46. $Q(x, y, z) = \dfrac{10}{1 + x^2 + y^2 + 4z^2}$

47. $F(x, y, z) = \sqrt{y - x^2}$

48. $f(w, x, y, z) = \sqrt{1 - w^2 - x^2 - y^2 - z^2}$

Further Explorations

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The domain of the function $f(x, y) = 1 - |x - y|$ is $\{(x, y) : x \geq y\}$.

b. The domain of the function $Q = g(w, x, y, z)$ is a region in $\mathbb{R}^3$.

c. All level curves of the plane $z = 2x - 3y$ are lines.

T **50-56. Graphing functions**

a. *Determine the domain and range of the following functions.*

b. *Graph each function using a graphing utility. Be sure to experiment with the window and orientation to give the best perspective on the surface.*

50. $g(x, y) = e^{-xy}$

51. $f(x, y) = |xy|$

52. $p(x, y) = 1 - |x - 1| + |y + 1|$

53. $h(x, y) = (x + y)/(x - y)$

54. $G(x, y) = \ln[2 + \sin(x + y)]$

55. $F(x, y) = \tan^2(x - y)$

56. $P(x, y) = \cos x \sin 2y$

T **57-60. Peaks and valleys** *The following functions have exactly one isolated peak or one isolated depression (one local maximum or minimum). Use a graphing utility to approximate the coordinates of the peak or depression.*

57. $f(x, y) = x^2 y^2 - 8x^2 - y^2 + 6$

58. $g(x, y) = \left(x^2 - x - 2\right)\left(y^2 + 2y\right)$

59. $h(x, y) = 1 - e^{-(x^2 + y^2 - 2x)}$

60. $p(x, y) = 2 + |x - 1| + |y - 1|$

61. Level curves of planes Prove that the level curves of the plane $a x + b y + c z = d$ are parallel lines in the xy-plane, provided $a^2 + b^2 \neq 0$ and $c \neq 0$.

Applications

T **62. Level curves of a savings account** Suppose you make a one-time deposit of P dollars into a savings account that earns interest at an annual rate of $p\%$ compounded continuously. The balance in the account after t years is $B(P, r, t) = P e^{r t}$, where $r = p/100$ (for example, if the annual interest rate is 4%, then $r = 0.04$). Let the interest rate be fixed at $r = 0.04$.

- **a.** With a target balance of \$2000, find the set of all points (P, t) that satisfy $B = 2000$. This curve gives all deposits P and times t that result in a balance of \$2000.
- **b.** Repeat part (a) with B = \$500, \$1000, \$1500, and \$2500, and draw the resulting level curves of the balance function.
- **c.** In general, on one level curve, if t increases, does P increase or decrease?

T **63. Level curves of a savings plan** Suppose you make monthly deposits of P dollars into an account that earns interest at a *monthly* rate of $p\%$. The balance in the account after t years is $B(P, r, t) = P\left[\frac{(1 + r)^{12t} - 1}{r}\right]$, where $r = p/100$ (for example, if the annual interest rate is 9%, then $p = \frac{9}{12} = 0.75$ and $r = 0.0075$). Let the time of investment be fixed at $t = 20$ years.

- **a.** With a target balance of \$20,000, find the set of all points (P, r) that satisfy $B = 20{,}000$. This curve gives all deposits P and monthly interest rates r that result in a balance of \$20,000 after 20 years.
- **b.** Repeat part (a) with B = \$5000, \$10,000, \$15,000, and \$25,000, and draw the resulting level curves of the balance function.

64. Quarterback ratings One measurement of the quality of a quarterback in the National Football League is known as the *quarterback rating*. The rating formula is $R(c, t, i, y) = \frac{50 + 20c + 80t - 100i + 100y}{24}$, where c is the percentage of passes completed, t is the percentage of passes thrown for touchdowns, i is the percentage of intercepted passes, and y is the yards gained per attempted pass.

- **a.** In his career, Hall of Fame quarterback Johnny Unitas completed 54.57% of his passes, 5.59% of his passes were thrown for touchdowns, 4.88% of his passes were intercepted, and he gained an average of 7.76 yards per attempted pass. What was his quarterback rating?
- **b.** If c, t, and y remained fixed, what happens to the quarterback rating as i increases? Explain your answer with and without mathematics.

[*Source: The College Mathematics Journal* (November 1993).]

T **65. Ideal Gas Law** Many gases can be modeled by the Ideal Gas Law, $P V = n R T$, which relates the temperature (T, measured in Kelvin (K)), pressure (P, measured in Pascals (Pa)), and volume (V, measured in m^3) of a gas. Assume that the quantity of gas in question is $n = 1$ mole (mol). The gas constant has a value of $R = 8.3\ \text{m}^3\ \text{Pa/mol–}K$.

- **a.** Consider T to be the dependent variable and plot several level curves (called *isotherms*) of the temperature surface in the region $0 \le P \le 100{,}000$ and $0 \le V \le 0.5$.
- **b.** Consider P to be the dependent variable and plot several level curves (called *isobars*) of the pressure surface in the region $0 \le T \le 900$ and $0 < V \le 0.5$.
- **c.** Consider V to be the dependent variable and plot several level curves of the volume surface in the region $0 \le T \le 900$ and $0 < P \le 100{,}000$.

Additional Exercises

66-69. Challenge domains *Find the domains of the following functions. Specify the domain mathematically and then describe it in words or with a sketch.*

66. $g(x, y, z) = \dfrac{10}{x^2 - (y+z)x + yz}$

67. $f(x, y) = \sin^{-1}(x - y)^2$

68. $f(x, y, z) = \ln\left(z - x^2 - y^2 + 2x + 3\right)$

69. $h(x, y, z) = \sqrt[4]{z^2 - xz + yz - xy}$

70. Other balls The closed unit ball in $\mathbb{R}^3$ centered at the origin is the set $\left\{(x, y, z) : x^2 + y^2 + z^2 \le 1\right\}$. Describe in words the following alternative unit balls.

a. $\{(x, y, z) : |x| + |y| + |z| \le 1\}$

b. $\{(x, y, z) : \max\{|x|, |y|, |z|\} \le 1\}$, where $\max\{a, b, c\}$ is the maximum value of a, b, and c.