12.3 Limits and Continuity

You have now seen examples of functions of several variables, but calculus has not yet entered the picture. In this section we revisit topics encountered in single-variable calculus and see how they apply to functions of several variables. We begin with the fundamental concepts of limits and continuity.

Limit of a Function of Two Variables

Limits at Boundary Points

Continuity of Functions of Two Variables

Functions of Three Variables

Quick Quiz

SECTION 12.3 EXERCISES

Review Questions

1. Describe in words what $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means.

2. Explain why $f(x, y)$ must approach L as (x, y) approaches (a, b) along *all* paths in the domain in order for $\lim_{(x,y)\to(a,b)} f(x, y)$ to exist.

3. Explain what it means to say that limits of polynomials may be evaluated by direct substitution.

4. Suppose (a, b) is on the boundary of the domain of f. Explain how you would determine whether $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.

5. Explain how examining limits along multiple paths may prove the nonexistence of a limit.

6. Explain why evaluating a limit along a finite number of paths does not prove the existence of a limit of a function of several variables.

7. What three conditions must be met for a function f to be continuous at the point (a, b)?

8. Let R be the unit disk $\{(x, y) : x^2 + y^2 \le 1\}$ with $(0, 0)$ removed. Is $(0, 0)$ a boundary point of R? Is R open or closed?

9. At what points of $\mathbb{R}^2$ is a rational function of two variables continuous?

10. Evaluate $\lim_{(x,y,z)\to(1,1,-1)} x y^2 z^3$.

Basic Skills

11-18. Limits of functions *Evaluate the following limits.*

11. $\lim_{(x,y)\to(2,9)} 101$

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12. $\lim\limits_{(x,y)\to(1,-3)} (3x + 4y - 2)$

13. $\lim\limits_{(x,y)\to(-3,3)} \left(4x^2 - y^2\right)$

14. $\lim\limits_{(x,y)\to(2,-1)} \left(x y^8 - 3x^2 y^3\right)$

15. $\lim\limits_{(x,y)\to(0,\pi)} \dfrac{\cos xy + \sin xy}{2y}$

16. $\lim\limits_{(x,y)\to(e^2,4)} \ln\sqrt{xy}$

17. $\lim\limits_{(x,y)\to(2,0)} \dfrac{x^2 - 3xy^2}{x + y}$

18. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{10xy - 2y^2}{x^2 + y^2}$

19-24. Limits at boundary points *Evaluate the following limits.*

19. $\lim\limits_{(x,y)\to(6,2)} \dfrac{x^2 - 3xy}{x - 3y}$

20. $\lim\limits_{(x,y)\to(1,-2)} \dfrac{y^2 + 2xy}{y + 2x}$

21. $\lim\limits_{(x,y)\to(2,2)} \dfrac{y^2 - 4}{xy - 2x}$

22. $\lim\limits_{(x,y)\to(4,5)} \dfrac{\sqrt{x+y} - 3}{x + y - 9}$

23. $\lim\limits_{(x,y)\to(1,2)} \dfrac{\sqrt{y} - \sqrt{x+1}}{y - x - 1}$

24. $\lim\limits_{(x,y)\to(8,8)} \dfrac{x^{1/3} - y^{1/3}}{x^{2/3} - y^{2/3}}$

25-30. Nonexistence of limits *Use the Two-Path Test to prove that the following limits do not exist.*

25. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + 2y}{x - 2y}$

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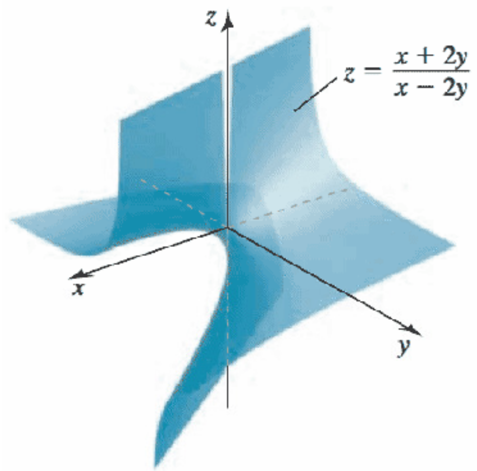
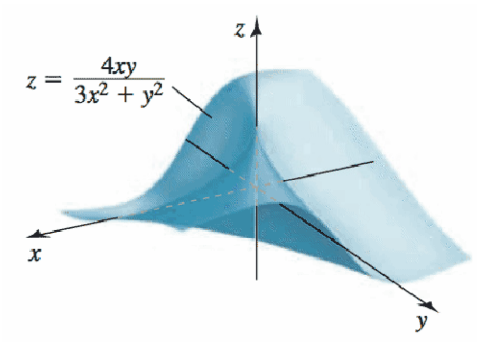


26. $\lim_{(x,y)\to(0,0)} \frac{4\,x\,y}{3\,x^2 + y^2}$



27. $\lim_{(x,y)\to(0,0)} \frac{y^4 - 2\,x^2}{y^4 + x^2}$

28. $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^2}{x^3 + y^2}$

29. $\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{x\,y^2}$

30. $\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2 - y^2}}$

31-34. Continuity *At what points of* $\mathbb{R}^2$ *are the following functions continuous?*

31. $f(x,\ y) = x^2 + 2\,x\,y - y^3$

32. $f(x,\ y) = \frac{x\,y}{x^2\,y^2 + 1}$

33. $p(x,\ y) = \frac{4\,x^2\,y^2}{x^4 + y^2}$

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34. $S(x, y) = \dfrac{4x^2 y^2}{x^2 + y^2}$

35-42. Continuity of composite functions *At what points of* $\mathbb{R}^2$ *are the following functions continuous?*

35. $f(x, y) = \sin xy$

36. $g(x, y) = \ln(x - y)$

37. $h(x, y) = \cos(x + y)$

38. $p(x, y) = e^{x-y}$

39. $f(x, y) = \ln\left(x^2 + y^2\right)$

40. $f(x, y) = \sqrt{4 - x^2 - y^2}$

41. $g(x, y) = \sqrt[3]{x^2 + y^2 - 9}$

42. $h(x, y) = \dfrac{\sqrt{x - y}}{4}$

43-46. Limits of functions of three variables *Evaluate the following limits.*

43. $\displaystyle\lim_{(x,y,z)\to(1,\ln 2,3)} z e^{xy}$

44. $\displaystyle\lim_{(x,y,z)\to(0,1,0)} \ln e^{xz}(1 + y)$

45. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{yz - xy - xz - x^2}{yz + xy + xz - y^2}$

46. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x - \sqrt{xz} - \sqrt{xy} + \sqrt{yz}}{x - \sqrt{xz} + \sqrt{xy} - \sqrt{yz}}$

Further Explorations

47. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the limits $\displaystyle\lim_{(x,0)\to(0,0)} f(x, 0)$ and $\displaystyle\lim_{(0,y)\to(0,0)} f(0, y)$ exist and equal L, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = L$.

b. If $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$, then f is continuous at (a, b).

c. If f is continuous at (a, b), then $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y)$ exists.

d. If P is a boundary point of the domain of f, then P is in the domain of f.

48-55. Miscellaneous limits *Use the method of your choice to evaluate the following limits.*

48. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^8 + y^2}$

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49. $\lim_{(x,y)\to(0,1)} \frac{y \sin x}{x(y+1)}$

50. $\lim_{(x,y)\to(1,1)} \frac{x^2 + xy - 2y^2}{2x^2 - xy - y^2}$

51. $\lim_{(x,y)\to(1,0)} \frac{y \ln y}{x}$

52. $\lim_{(x,y)\to(0,0)} \frac{|xy|}{xy}$

53. $\lim_{(x,y)\to(0,0)} \frac{|x - y|}{|x + y|}$

54. $\lim_{(x,y)\to(-1,0)} \frac{x y e^{-y}}{x^2 + y^2}$

55. $\lim_{(x,y)\to(2,0)} \frac{1 - \cos y}{x y^2}$

56-59. Limits using polar coordinates *Limits at* (0, 0) *may be easier to evaluate by converting to polar coordinates. Remember that the same limit must be obtained as* $r \to 0$ *along all paths to* (0, 0)*. Evaluate the following limits or state that they do not exist.*

56. $\lim_{(x,y)\to(0,0)} \frac{x - y}{\sqrt{x^2 + y^2}}$

57. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$

58. $\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + xy + y^2}$

59. $\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{\left(x^2 + y^2\right)^{3/2}}$

Additional Exercises

60. Sine limits Evaluate the following limits.

a. $\lim_{(x,y)\to(0,0)} \frac{\sin(x + y)}{x + y}$

b. $\lim_{(x,y)\to(0,0)} \frac{\sin x + \sin y}{x + y}$

61. Nonexistence of limits Show that $\lim_{(x,y)\to(0,0)} \frac{a x^m y^n}{b x^{m+n} + c y^{m+n}}$ does not exist when a, b, and c are real numbers and m and n are positive integers.

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62. Nonexistence of limits Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{a\,x^{2(p-n)}\,y^n}{b\,x^{2p} + c\,y^p}$ does not exist when a, b, and c are real numbers and n and p are positive integers with $p \ge n$.

63-66. Limits of composite functions *Evaluate the following limits.*

63. $\lim\limits_{(x,y)\to(1,0)} \dfrac{\sin xy}{xy}$

64. $\lim\limits_{(x,y)\to(4,0)} x^2\,y \ln xy$

65. $\lim\limits_{(x,y)\to(0,2)} (2\,x\,y)^{xy}$

66. $\lim\limits_{(x,y)\to(0,\pi/2)} \dfrac{1-\cos xy}{4\,x^2\,y^3}$

67. Filling in a function value The domain of $f(x,\ y) = e^{-1/\left(x^2+y^2\right)}$ excludes (0, 0). How should f be defined at (0, 0) to make it continuous there?

68. Limit proof Use the formal definition of a limit to prove that $\lim\limits_{(x,y)\to(a,b)} y = b$ (*Hint:* Take $\delta = \epsilon$.)

69. Limit proof Use the formal definition a limit to prove that $\lim\limits_{(x,y)\to(a,b)} (x + y) = a + b$ (*Hint:* Take $\delta = \epsilon/2$.)

70. Proof of Limit Law 1 Use the formal definition of a limit to prove that
$\lim\limits_{(x,y)\to(a,b)} [f(x,\ y) + g(x,\ y)] = \lim\limits_{(x,y)\to(a,.b)} f(x,\ y) + \lim\limits_{(x,y)\to(a,b)} g(x,\ y)$.

71. Proof of Limit Law 3 Use the formal definition of a limit to prove that $\lim\limits_{(x,y)\to(a,b)} [c\,f(x,\ y)] = c \lim\limits_{(x,y)\to(a,.b)} f(x,\ y)$.

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