12.4 Partial Derivatives

The derivative of a function of one variable, $y = f(x)$, measures the rate of change of y with respect to x, and it gives slopes of tangent lines. The analogous idea for functions of several variables presents a new twist: Derivatives may be defined with respect to any of the independent variables. For example, we can compute the derivative of $f(x, y)$ with respect to x or y. The resulting derivatives are called *partial derivatives*; they still represent rates of change and they are associated with slopes of tangents. So, much of what you have learned about derivatives applies to functions of several variables. However, much is also different.

Derivatives with Two Variables

Higher-Order Partial Derivatives

Functions of Three Variables

Differentiability

Quick Quiz

SECTION 12.4 EXERCISES

Review Questions

1. Suppose you are standing on the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$. Interpret the meaning of $f_x(a, b)$ and $f_y(a, b)$ in terms of slopes or rates of change.

2. Find f_x and f_y when $f(x, y) = 3x^2 y + x y^3$.

3. Find f_x and f_y when $f(x, y) = x \cos(x y)$.

4. Find the four second partial derivatives of $f(x, y) = 3x^2 y + x y^3$.

5. Explain how you would evaluate f_z for the differentiable function $w = f(x, y, z)$.

6. The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$. Is the volume an increasing or decreasing function of the radius at a fixed height (assume $r > 0$ and $h > 0$)?

Basic Skills

7-16. Partial derivatives *Find the first partial derivatives of the following functions.*

7. $f(x, y) = 3x^2 y + 2$

8. $f(x, y) = y^8 + 2x^6 + 2xy$

9. $g(x, y) = \cos 2xy$

10. $h(x, y) = \left(y^2 + 1\right) e^x$

11. $f(w, z) = \dfrac{w}{w^2 + z^2}$

12. $g(x,\ z) = x \ln\left(z^2 + x^2\right)$

13. $s(y,\ z) = z^2 \tan yz$

14. $F(p,\ q) = \sqrt{p^2 + pq + q^2}$

15. $G(s,\ t) = \dfrac{\sqrt{st}}{s+t}$

16. $h(u,\ v) = \sqrt{\dfrac{uv}{u-v}}$

17-24. Second partial derivatives *Find the four second partial derivatives of the following functions.*

17. $h(x,\ y) = x^3 + x\,y^2 + 1$

18. $f(x,\ y) = 2\,x^5\,y^2 + x^2\,y$

19. $f(x,\ y) = y^3 \sin 4\,x$

20. $f(x,\ y) = \cos xy$

21. $p(u,\ v) = \ln\left(u^2 + v^2 + 4\right)$

22. $Q(r,\ s) = r/s$

23. $F(r,\ s) = r\,e^s$

24. $H(x,\ y) = \sqrt{4 + x^2 + y^2}$

25-30. Equality of mixed partial derivatives *Verify that $f_{xy} = f_{yx}$ for the following functions.*

25. $f(x,\ y) = 2\,x^3 + 3\,y^2 + 1$

26. $f(x,\ y) = x\,e^y$

27. $f(x,\ y) = \cos xy$

28. $f(x,\ y) = 3\,x^2\,y^{-1} - 2\,x^{-1}\,y^2$

29. $f(x,\ y) = e^{x+y}$

30. $f(x,\ y) = \sqrt{x\,y}$

31-40. Partial derivatives with more than two variables *Find the first partial derivatives of the following functions.*

31. $f(x,\ y,\ z) = x\,y + x\,z + y\,z$

32. $g(x,\ y,\ z) = 2\,x^2\,y - 3\,x\,z^4 + 10\,y^2\,z^2$

33. $h(x,\ y,\ z) = \cos(x + y + z)$

34. $Q(x,\ y,\ z) = \tan xyz$

35. $F(u, v, w) = \dfrac{u}{v + w}$

36. $G(r, s, t) = \sqrt{r s + r t + s t}$

37. $f(w, x, y, z) = w^2 x y^2 + x y^3 z^2$

38. $g(w, x, y, z) = \cos(w + x) \sin(y - z)$

39. $h(w, x, y, z) = \dfrac{w z}{x y}$

40. $F(w, x, y, z) = w \sqrt{x + 2 y + 3 z}$

41. **Gas law calculations** Consider the Ideal Gas Law $P V = k T$, where $k > 0$ is a constant. Solve this equation for V in terms of P and T.

- **a.** Determine the rate of change of the volume with respect to the pressure at constant temperature. Interpret the result.
- **b.** Determine the rate of change of the volume with respect to the temperature at constant pressure. Interpret the result.
- **c.** Assuming $k = 1$, draw several level curves of the volume function and interpret the results as in Example 5.

42. **Volume of a box** A box with a square base of length x and height h has a volume $V = x^2 h$.

- **a.** Compute the partial derivatives V_x and V_h.
- **b.** For a box with $h = 1.5$ m, use linear approximation to estimate the change in volume if x increases from $x = 0.5$ m to $x = 0.51$ m.
- **c.** For a box with $x = 0.5$ m, use linear approximation to estimate the change in volume if h decreases from $h = 1.5$ m to $h = 1.49$ m.
- **d.** For a fixed height, does a 10% change in x always produce (approximately) a 10% change in V? Explain.
- **e.** For a fixed base length, does a 10% change in h always produce (approximately) a 10% change in V? Explain.

43-46. Nondifferentiability? *Consider the following functions f.*

- ***a.*** *Is f continuous at* $(0, 0)$?
- ***b.*** *Is f differentiable at* $(0, 0)$?
- ***c.*** *If possible, evaluate* $f_x(0, 0)$ *and* $f_y(0, 0)$.
- ***d.*** *Determine whether f_x and f_y are continuous at* $(0, 0)$.
- ***e.*** *Explain why Theorems 12.5 and 12.6 are consistent with the results in part (a)—(d).*

43. $f(x, y) = \begin{cases} -\dfrac{x y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

44. $f(x, y) = \begin{cases} \dfrac{2 x y^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

45. $f(x, y) = 1 - |x y|$

46. $f(x, y) = \sqrt{|x y|}$

Further Explorations

47. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\frac{\partial}{\partial x}\left(y^{10}\right) = 10\, y^9$

b. $\frac{\partial^2}{\partial x\,\partial y}\left(\sqrt{x\,y}\right) = \frac{1}{\sqrt{x\,y}}$

c. If f has continuous partial derivatives of all orders, then $f_{xxy} = f_{yxx}$.

48-52. Miscellaneous partial derivatives *Compute the first partial derivatives of the following functions.*

48. $f(x,\ y) = \ln\left(1 + e^{-x\,y}\right)$

49. $f(x,\ y) = 1 - \tan^{-1}\left(x^2 + y^2\right)$

50. $f(x,\ y) = 1 - \cos(2\,(x + y)) + \cos^2(x + y)$

51. $h(x,\ y,\ z) = (1 + x + 2\,y)^z$

52. $g(x,\ y,\ z) = \frac{4\,x - 2\,y - 2\,z}{3\,y - 6\,x - 3\,z}$

53. Partial derivatives and level curves Consider the function $z = x \big/ y^2$.

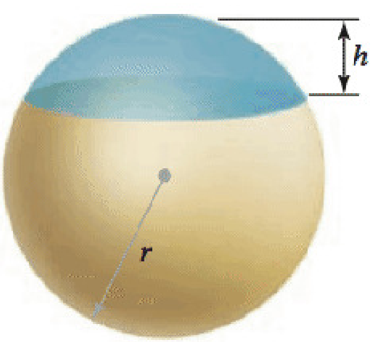
a. Compute z_x and z_y.

b. Sketch the level curves for $z = 1, 2, 3,$ and 4.

c. Move along the horizontal line $y = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_x as computed in part (a).

d. Move along the vertical linie $x = 1$ in the xy-plane and describe how the corresponding z-values change. Explain how this observation is consistent with z_y as computed in part (a).

54. Spherical caps The volume of the cap of a sphere of radius r and thickness h is $V = \frac{\pi}{3} h^2(3\,r - h)$, for $0 \le h \le r$.



$V = \frac{\pi}{3}h^2(3r - h)$

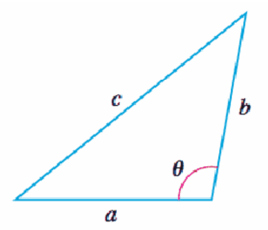
a. Compute the partial derivatives V_h and V_r.

b. For a sphere of any radius, is the rate of change of volume with respect to r greater when $h = 0.2\,r$ or when $h = 0.8\,r$?

c. For a sphere of any radius, for what value of h is the rate of change of volume with respect to r equal to 1?

d. For a fixed radius r, for what value of h $(0 \le h \le r)$ is the rate of change of volume with respect to h the greatest?

55. Law of Cosines All triangles satisfy the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$ (see figure). Notice that when $\theta = \pi/2$, the Law of Cosines becomes the Pythagorean Theorem. Consider all triangles with a fixed angle $\theta = \pi/3$, in which case, c is a function of a and b, where $a > 0$ and $b > 0$.



a. Compute $\dfrac{\partial c}{\partial a}$ and $\dfrac{\partial c}{\partial b}$ by solving for c and differentiating.

b. Compute $\dfrac{\partial c}{\partial a}$ and $\dfrac{\partial c}{\partial b}$ by implicit differentiation. Check for agreement with part (a).

c. What relationship between a and b makes c an increasing function of a (for constant b)?

Applications

56. Body mass index The body mass index (BMI) for an adult human is given by the function $B = w/h^2$, where w is the weight measured in kilograms and h is the height measured in meters. (The BMI for units of pounds and inches is $B + 703\,w/h^2$.)

a. Find the rate of change of the BMI with respect to weight at a constant height.

b. For fixed h, is the BMI an increasing or decreasing function of w? Explain.

c. Find the rate of change of the BMI with respect to height at a constant weight.

d. For fixed w, is the BMI an increasing or decreasing function of h? Explain.

57. Electric potential function The electric potential in the xy-plane associated with two positive charges, one at $(0, 1)$ with twice the magnitude as the charge at $(0, -1)$, is

$$\phi(x, y) = \frac{2}{\sqrt{x^2 + (y-1)^2}} + \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

a. Compute ϕ_x and ϕ_y.

b. Describe how ϕ_x and ϕ_y behave as $x, y \to \pm\infty$.

c. Evaluate $\phi_x(0, y)$ for all $y \ne \pm 1$. Interpret this result.

d. Evaluate $\phi_y(x, 0)$ for all x. Interpret this result.

T **58. Cobb-Douglas production function** The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = c\,L^a K^b$. Suppose $a = \dfrac{1}{3}$, $b = \dfrac{2}{3}$, and $c = 1$.

a. Evaluate the partial derivatives Q_L and Q_K.

b. If $L = 10$ is fixed and K increases from $K = 20$ to $K = 20.5$, use linear approximation to estimate the change in Q.

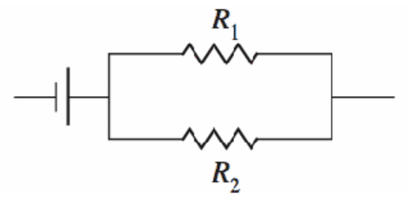
c. If $K = 20$ is fixed and L decreases from $L = 10$ to $L = 9.5$, use linear approximation to estimate the change in Q.

d. Graph the level curves of the production function in the first quadrant of the LK-plane for $Q = 1, 2, 3$.

e. If you move along the vertical line $L = 2$ in the positive K-direction, how does Q change? Is this consistent with Q_K computed in part (a)?

f. If you move along the horizontal line $K = 2$ in the positive L-direction, how does Q change? Is this consistent with Q_L computed in part (a)?

59. Resistors in parallel Two resistors in an electrical circuit with resistance R_1 and R_2 wired in parallel with a constant voltage give an effective resistance of R, where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



a. Find $\frac{\partial R}{\partial R_1}$ and $\frac{\partial R}{\partial R_2}$ by solving for R and differentiating.

b. Find $\frac{\partial R}{\partial R_1}$ and $\frac{\partial R}{\partial R_2}$ by differentiating implicitly.

c. Describe how an increase in R_1 with R_2 constant affects R.

d. Describe how a decrease in R_2 with R_1 constant affects R.

60. Wave on a string Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x, t) = 2\sin(\pi x)\sin(\pi t/2)$ for $0 \le x \le 1$ and $t \ge 0$ (see figure). At a fixed point in time, the string forms a wave on $[0, 1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

a. What is the period of the motion in time?

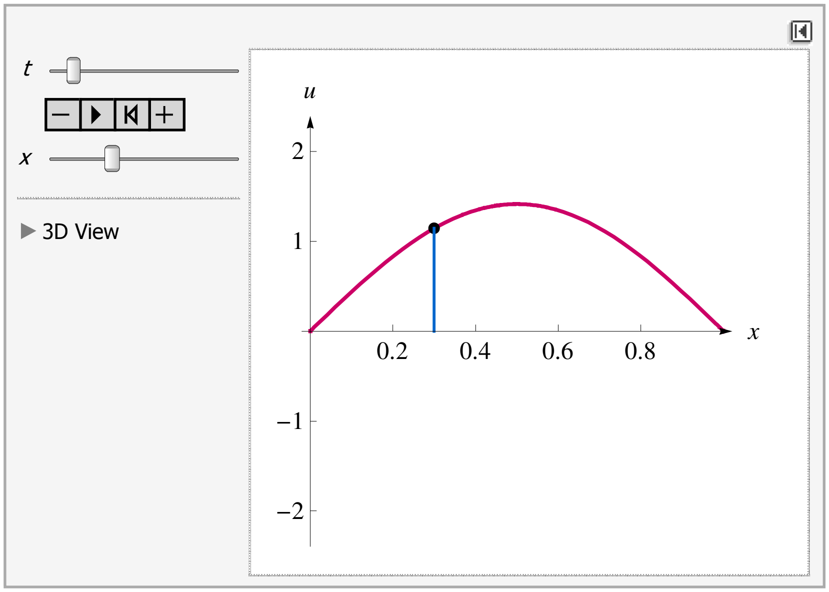
b. Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).

c. At a fixed time, what point on the string is moving fastest?

d. At a fixed position on the string, when is the string moving fastest?

e. Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).

f. At a fixed time, where is the slope of the string greatest?



61–63. Wave equation *Traveling waves (for example, water waves or electromagnetic waves) exhibit periodic motion in both time and position. In one dimension (for example, a wave on a string) wave motion is governed by the one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the height or displacement of the wave surface at position x and time t, and c is the constant speed of the wave. Show that the following functions are solutions of the wave equation.

61. $u(x, t) = \cos(2(x + c t))$

62. $u(x, t) = 5\cos(2(x + c t)) + 3\sin(x - c t)$

63. $u(x, t) = A f(x + c t) + B g(x - c t)$, where A and B are constants, and f and g are twice differentiable functions of one variable.

64–67. Laplace's equation *A classical equation of mathematics is Laplace's equation, which arises in both theory and applications. It governs ideal fluid flow, electrostatic potentials, and the steady-state distribution of heat in a conducting medium. In two dimensions, Laplace's equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that the following functions are **harmonic***; that is, they satisfy Laplace's equation.*

64. $u(x, y) = e^{-x} \sin y$

65. $u(x, y) = x(x^2 - 3y^2)$

66. $u(x, y) = e^{a x} \cos a y$ for any real number a

67. $u(x, y) = \tan^{-1}\left(\frac{y}{x - 1}\right) - \tan^{-1}\left(\frac{y}{x + 1}\right)$

68–71. Heat equation *The flow of heat along a thin conducting bar is governed by the one-dimensional heat equation (with analogs for thin plates in two dimensions and for solids in three dimensions)*

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},$$

where u is a measure of the temperature at a location x on the bar at time t and the positive constant k is related to the conductivity of the material. Show that the following functions satisfy the heat equation with k = 1.

68. $u(x, t) = 10\,e^{-t}\sin x$

69. $u(x, t) = 4\,e^{-4t}\cos 2x$

70. $u(x, t) = e^{-t}(2\sin x + 3\cos x)$

71. $u(x, t) = A\,e^{-a^2 t}\cos a x$ for any real numbers a and A

Additional Exercises

72–73. Differentiability *Use the definition of differentiability to prove that the following functions are differentiable at* (0, 0). *You must produce functions* ϵ_1 *and* ϵ_2 *with the required properties.*

72. $f(x, y) = x + y$

73. $f(x, y) = x\,y$

74. Mixed partial derivatives

a. Consider the function $w = f(x, y, z)$. List all possible second partial derivatives that could be computed.
b. Let $f(x, y, z) = x^2 y + 2xz^2 - 3y^2 z$ and determine which second partial derivatives are equal.
c. How many second partial derivatives does $p = g(w, x, y, z)$ have?

75. Derivatives of an integral Let h be continuous for all real numbers.

a. Find f_x and f_y when $f(x, y) = \int_x^y h(s)\,ds$.
b. Find f_x and f_y when $f(x, y) = \int_1^{xy} h(s)\,ds$.

76. An identity Show that if $f(x, y) = \dfrac{ax + by}{cx + dy}$, where a, b, c, and d are real numbers with $ad - bc = 0$, then $f_x = f_y = 0$, for all x and y in the domain of f. Give an explanation.

77. Cauchy-Riemann equations In the advanced subject of complex variables, a function typically has the form $f(x, y) = u(x, y) + i\,v(x, y)$, where u and v are real-valued functions and $i = \sqrt{-1}$ is the imaginary unit. A function $f = u + i\,v$ is said to be *analytic* (analogous to differentiable) if it satisfies the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

a. Show that $f(x, y) = \left(x^2 - y^2\right) + i\,(2xy)$ is analytic.
b. Show that $f(x, y) = x\left(x^2 - 3y^2\right) + i\,y\left(3x^2 - y^2\right)$ is analytic.
c. Show that if $f = u + i\,v$ is analytic, then $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.