12.5 The Chain Rule

In this section, we combine ideas based on the Chain Rule (Section 3.6) with what we know about partial derivatives (Section 12.4) to develop new methods for finding derivatives of functions of several variables. To illustrate the importance of these methods, consider the following situation.

An economist modeling the output of a manufacturing system often works with *production functions* that relate the productivity of the system (output) to all the variables on which it depends (input). A simplified production function might take the form $P = F(L, K, R)$, where L, K, and R represent the availability of labor, capital, and natural resources, respectively. However, the variables L, K, and R may be intermediate variables that depend on other variables. For example, it might be that L is a function of the unemployment rate u, K is a function of the prime interest rate i, and R is a function of time t (seasonal availability of resources). Even in this simplified model we see that productivity, which is the dependent variable, is ultimately related to many other variables (Figure 12.53). Of critical interest to an economist is how changes in one variable determine changes in other variables. For instance, if the unemployment rate increases by 0.1% and the interest rate decreases by 0.2%, what is the effect on productivity? In this section we develop the tools needed to answer such questions.

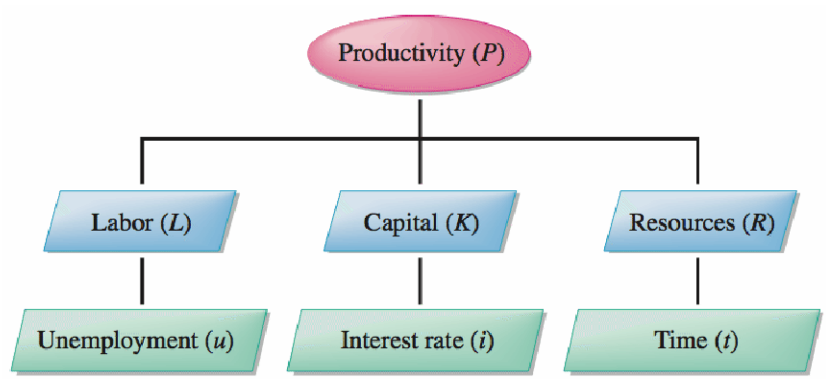


FIGURE 12.53

The Chain Rule with One Independent Variable

The Chain Rule with Several Independent Variables

Implicit Differentiation

Quick Quiz

SECTION 12.5 EXERCISES

Review Questions

1. Suppose $z = f(x, y)$, where x and y are functions of t. How many dependent, intermediate, and independent variables are there?

2. If z is a function of x and y, while x and y are functions of t, explain how to find $\frac{dz}{dt}$.

3. If w is a function of x, y, and z, which are each functions of t, explain how to find $\frac{dw}{dt}$.

4. If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, explain how to find $\partial z/\partial t$.

5. Given that $w = F(x, y, z)$, and x, y, and z are functions of r and s, sketch a tree diagram with branches labeled with the appropriate derivatives.

6. If $F(x, y) = 0$ and y is a differentiable function of x, explain how to find dy/dx.

Basic Skills

7-14. Chain Rule with one independent variable *Use Theorem 12.7 to find the following derivatives. When feasible, express your answer in terms of the independent variable.*

7. dz/dt, where $z = x \sin y$, $x = t^2$, and $y = 4t^3$

8. dz/dt, where $z = x^2 y - x y^3$, $x = t^2$, and $y = t^{-2}$

9. dw/dt, where $w = \cos 2x \sin 3y$, $x = t/2$, and $y = t^4$

10. dz/dt, where $z = \sqrt{r^2 + s^2}$, $r = \cos 2t$, and $s = \sin 2t$

11. dw/dt, where $w = xy \sin z$, $x = t^2$, $y = 4t^3$, and $z = t + 1$

12. dQ/dt, where $Q = \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, and $z = \cos t$

13. dU/dt, where $U = \ln(x + y + z)$, $x = t$, $y = t^2$, and $z = t^3$

14. dV/dt, where $V = \dfrac{x - y}{y + z}$, $x = t$, $y = 2t$, and $z = 3t$

15. Changing cylinder The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$.

a. Assume that r and h are functions of t. Find $V'(t)$.
b. Suppose that $r = e^t$ and $h = e^{-2t}$, for $t \geq 0$. Use part (a) to find $V'(t)$.
c. Does the volume of the cylinder in part (b) increase or decrease as t increases?

16. Changing pyramid The volume of a pyramid with a square base x units on a side and a height of h is $V = \dfrac{1}{3} x^2 h$.

a. Assume that x and h are functions of t. Find $V'(t)$.
b. Suppose that $x = t/(t + 1)$ and $h = 1/(t + 1)$, for $t \geq 0$. Use part (a) to find $V'(t)$.
c. Does the volume of the pyramid in part (b) increase or decrease as t increases?

17-22. Chain Rule with several independent variables *Find the following derivatives.*

17. z_s and z_t, where $z = xy - x^2 y$, $x = s + t$, and $y = s - t$

18. z_s and z_t, where $z = \sin x \cos 2y$, $x = s + t$, and $y = s - t$

19. z_s and z_t, where $z = e^{x+y}$, $x = st$, and $y = s + t$

20. z_s and z_t, where $z = xy - 2x + 3y$, $x = \cos s$, and $y = \sin t$

21. w_s and w_t, where $w = \dfrac{x - z}{y + z}$, $x = s + t$, $y = st$, and $z = s - t$

22. w_r, w_s, and w_t, where $w = \sqrt{x^2 + y^2 + z^2}$, $x = s\,t$, $y = r\,s$, and $z = r\,t$

23-26. Making trees *Use a tree diagram to write the required Chain Rule formula.*

23. w is a function of z, where z is a function of x and y, each of which is a function of t. Find dw/dt.

24. $w = f(x, y, z)$, where $x = g(t)$, $y = h(s, t)$, $z = p(r, s, t)$. Find $\partial w/\partial t$.

25. $u = f(v)$, where $v = g(w, x, y)$, $w = h(z)$, $x = p(t, z)$, $y = q(t, z)$. Find $\partial u/\partial z$.

26. $u = f(v, w, x)$, where $v = g(r, s, t)$, $w = h(r, s, t)$, $x = p(r, s, t)$, $r = F(z)$. Find $\partial u/\partial z$.

27-32. Implicit differentiation *Given the following equations, evaluate dy/dx. Assume that each equation implicitly defines y as a differentiable function of x.*

27. $x^2 - 2y^2 - 1 = 0$

28. $x^3 + 3xy^2 - y^5 = 0$

29. $2\sin xy = 1$

30. $y e^{xy} - 2 = 0$

31. $\sqrt{x^2 + 2xy + y^4} = 3$

32. $y\ln\left(x^2 + y^2 + 4\right) = 3$

33-34. Fluid flow *The x- and y-components of a fluid moving in two dimensions are given by the following functions u and v. The speed of the fluid at (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Use the Chain Rule to find $\partial s/\partial x$ and $\partial s/\partial y$.*

33. $u(x, y) = 2y$ and $v(x, y) = -2x$; $x \geq 0$ and $y \geq 0$

34. $u(x, y) = x(1 - x)(1 - 2y)$ and $v(x, y) = y(y - 1)(1 - 2x)$; $0 \leq x \leq 1$, $0 \leq y \leq 1$

Further Explorations

35. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume all partial derivatives exist.

a. If $z = (x + y)\sin xy$, where x and y are functions of s, then $\dfrac{\partial z}{\partial s} = \dfrac{dz}{dx}\dfrac{dx}{ds}$.

b. Given that $w = f(x(s, t), y(s, t), z(s, t))$, the rate of change of w with respect to t is dw/dt.

36-37. Derivative practice two ways *Find the indicated derivative in two ways:*

a. *Replace x and y to write z as a function of t and differentiate.*

b. *Use the Chain Rule.*

36. $z'(t)$, where $z = \ln(x + y)$, $x = t\,e^t$, and $y = e^t$

37. $z'(t)$, where $z = \dfrac{1}{x} + \dfrac{1}{y}$, $x = t^2 + 2t$, and $y = t^3 - 2$

38-42. Derivative practice *Find the indicated derivative for the following functions.*

38. $\partial z/\partial p$, where $z = x/y$, $x = p + q$, and $y = p - q$

39. dw/dt, where $w = xyz$, $x = 2t^4$, $y = 3t^{-1}$, and $z = 4t^{-3}$

40. $\partial w/\partial x$, where $w = \cos z - \cos x \cos y + \sin x \sin y$, and $z = x + y$

41. $\dfrac{\partial z}{\partial x}$, where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

42. $\partial z/\partial x$, where $xy - z = 1$

43. Change on a line Suppose $w = f(x, y, z)$ and l is the line $\mathbf{r}(t) = \langle at, bt, ct \rangle$, for $-\infty < t < \infty$.

- **a.** Find $w'(t)$ on l (in terms of a, b, c, w_x, w_y, and w_z).
- **b.** Apply part (a) to find $w'(t)$ when $f(x, y, z) = xyz$.
- **c.** Apply part (a) to find $w'(t)$ when $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
- **d.** For a general function $w = f(x, y, z)$, find $w''(t)$.

44. Implicit differentiation rule with three variables Assume that $F(x, y, z(x, y)) = 0$ implicitly defines z as a differentiable function of x and y. Extend Theorem 12.9 to show that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

45-48. Implicit differentiation with three variables *Use the result of Exercise 44 to evaluate* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *for the following relations.*

45. $xy + xz + yz = 3$

46. $x^2 + 2y^2 - 3z^2 = 1$

47. $xyz + x + y - z = 0$

48. More than one way Let $e^{xyz} = 2$. Find z_x and z_y in three ways (and check for agreement).

- **a.** Use the result of Exercise 44.
- **b.** Take logarithms of both sides and differentiate $xyz = \ln 2$.
- **c.** Solve for z and differentiate $z = \ln 2/(xy)$.

49-52. Walking on a surface *Consider the following surfaces specified in the form $z = f(x, y)$ and the curve C in the xy-plane given parametrically in the form $x = g(t)$, $y = h(t)$.*

- ***a.*** *In each case, find $z'(t)$.*
- ***b.*** *Imagine that you are walking on the surface directly above the curve C in the direction of increasing t. Find the values of t for which you are walking uphill (that is, z is increasing).*

49. $z = x^2 + 4y^2 + 1$, C: $x = \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

50. $z = 4x^2 - y^2 + 1$, C: $x = \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

51. $z = \sqrt{1 - x^2 - y^2}$, C: $x = e^{-t}$, $y = e^{-t}$; $t \ge \dfrac{1}{2} \ln 2$

52. $z = 2x^2 + y^2 + 1$, C: $x = 1 + \cos t$, $y = \sin t$; $0 \le t \le 2\pi$

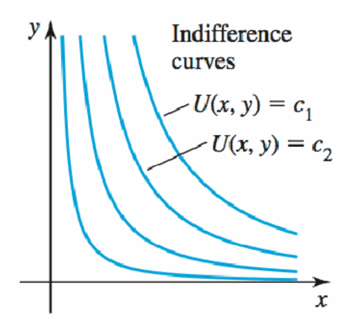
Applications

53. Conservation of energy A projectile is launched into the air on a parabolic trajectory. For $t \ge 0$, its horizontal and vertical coordinates are $x(t) = u_0 t$ and $y(t) = -(1/2) g t^2 + v_0 t$, respectively, where u_0 is the initial horizontal velocity, v_0 is the initial vertical velocity, and g is the acceleration due to gravity. Recalling that $u(t) = x'(t)$ and $v(t) = y'(t)$ are the components of the velocity, the energy of the projectile (kinetic plus potential) is

$$E(t) = \frac{1}{2} m \left(u^2 + v^2\right) + m g y$$

Use the Chain Rule to compute $E'(t)$ and show that $E'(t) = 0$ for all $t \ge 0$. Interpret the result.

54. Utility functions in economics Economists use *utility functions* to describe consumers' relative preference for two or more commodities (for example, vanilla vs. chocolate ice cream or leisure time vs. material goods). The Cobb-Douglas family of utility functions has the form $U(x, y) = x^a y^{1-a}$, where x and y are the amounts of two commodities and $0 < a < 1$ is a parameter. Level curves on which the utility function is constant are called *indifference curves*; the preference is the same for all combinations of x and y along an indifference curve (see figure).

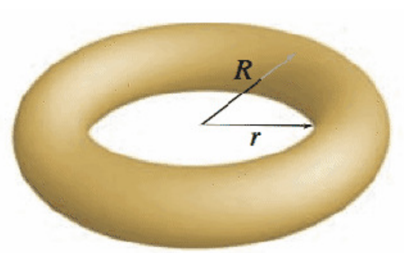


a. The marginal utilities of the commodities x and y are defined to be $\partial U / \partial x$ and $\partial U / \partial y$, respectively. Compute the marginal utilities for the utility function $U(x, y) = x^a y^{1-a}$.

b. The marginal rate of substitution (MRS) is the slope of the indifference curve at the point (x, y). Use the Chain Rule to show that for $U(x, y) = x^a y^{1-a}$, the MRS is $-\frac{a}{1-a} \frac{y}{x}$.

c. Find the MRS for the utility function $U(x, y) = x^{0.4} y^{0.6}$ at $(x, y) = (8, 12)$.

55. Constant volume tori The volume of a solid torus (a bagel or doughnut) is given by $V = \left(\pi^2/4\right)(R + r)(R - r)^2$, where r and R are the inner and outer radii and $R > r$ (see figure).



a. If R and r increase at the same rate, does the volume of the torus increase, decrease, or remain constant?

b. If R and r decrease at the same rate, does the volume of the torus increase, decrease, or remain constant?

56. Body surface area One of several empirical formulas that relates the surface area S of a human body to the height h and weight w of the body is the Mosteller formula $S(h, w) = \frac{1}{60}\sqrt{h\,w}$, where h is measured in cm, w is measured in kg, and S is measured in m^2. Suppose that h and w are functions of t.

a. Find $S'(t)$.

b. Show that the condition that the surface area remains constant as h and w change is $w\,h'(t) + h\,w'(t) = 0$.

c. Show that part (b) implies that for constant surface area, h and w must be inversely related; that is $h = C/w$, where C is a constant.

57. The Ideal Gas Law The pressure, temperature, and volume of an ideal gas are related by $P\,V = k\,T$, where $k > 0$ is a constant. Any two of the variables may be considered independent, which determines the third variable.

a. Use implicit differentiation to compute the partial derivatives $\frac{\partial P}{\partial V}$, $\frac{\partial T}{\partial P}$, and $\frac{\partial V}{\partial T}$.

b. Show that $\frac{\partial P}{\partial V}\frac{\partial T}{\partial P}\frac{\partial V}{\partial T} = -1$. (See Exercise 63 for a generalization.)

58. Variable density The density of a thin circular plate of radius 2 is given by $\rho(x, y) = 4 + x\,y$. The edge of the plate is described by the parametric equations $x = 2\cos t$, $y = 2\sin t$ for $0 \le t \le 2\pi$.

a. Find the rate of change of the density with respect to t on the edge of the plate.

b. At what point(s) on the edge of the plate is the density a maximum?

T **59. Spiral through a domain** Suppose you follow the spiral path $C: x = \cos t,\ y = \sin t,\ z = t$, for $t \ge 0$, through the domain of the function $w = f(x, y, z) = (x y z)/(z^2 + 1)$.

a. Find $w'(t)$ along C.

b. Estimate the point (x, y, z) on C at which w has its maximum value.

Additional Exercises

60. Change of coordinates Recall that Cartesian and polar coordinates are related through the transformation equations

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{or} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

a. Evaluate the partial derivatives x_r, y_r, x_θ, and y_θ.

b. Evaluate the partial derivatives r_x, r_y, θ_x, and θ_y.

c. For a function $z = f(x, y)$, find z_r and z_θ, where x and y are expressed in terms of r and θ.

d. For a function $z = g(r, \theta)$, find z_x and z_y, where r and θ are expressed in terms of x and y.

e. Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

61. Change of coordinates continued An important derivative operation in many applications is called the Laplacian; in Cartesian coordinates, for $z = f(x, y)$, the Laplacian is $z_{xx} + z_{yy}$. Determine the Laplacian in polar coordinates using the following steps.

a. Begin with $z = g(r, \theta)$ and write z_x and z_y in terms of polar coordinates (see Exercise 60).

b. Use the Chain Rule to find $z_{xx} = \frac{\partial}{\partial x}(z_x)$. There should be two major terms, which when expanded and simplified, result in five terms.

c. Use the Chain Rule to find $z_{yy} = \frac{\partial}{\partial y}(z_y)$. There should be two major terms, which when expanded and simplified, result in five terms.

d. Combine part (a) and (b) to show that

$$z_{xx} + z_{yy} = z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}.$$

62. Geometry of implicit differentiation Suppose x and y are related by the equation $F(x, y) = 0$. Interpret the solution of this equation as the set of points (x, y) that lie on the intersection of the surface $z = F(x, y)$ with the xy-plane ($z = 0$).

a. Make a sketch of a surface and its intersection with the xy-plane. Give a geometric interpretation of the result that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

b. Explain geometrically what happens at points where $F_y = 0$.

63. General three-variable relationship In the implicit relationship $F(x, y, z) = 0$, any two of the variables may be considered independent, which then determines the third variable. To avoid confusion, we may use a subscript to indicate which variable is held fixed in a derivative calculation; for example $\left(\frac{\partial z}{\partial x}\right)_y$ means that y is held fixed in taking the partial derivative of z with respect to x. (In this context, the subscript does *not* mean a derivative.)

a. Differentiate $F(x, y, z) = 0$ with respect to x holding y fixed to show that $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.

b. As in part (a), find $\left(\frac{\partial y}{\partial z}\right)_x$ and $\left(\frac{\partial x}{\partial y}\right)_z$.

c. Show that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$.

d. Find the relationship analogous to part (c) for the case $F(w, x, y, z) = 0$.

64. Second derivative Let $f(x, y) = 0$ define y as a twice differentiable function of x.

a. Show that $y''(x) = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}$.

b. Verify part (a) using the function $f(x, y) = x y - 1$.

65. Subtleties of the Chain Rule Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all (x, y, z) in $\mathbb{R}^3$. Suppose that we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

a. We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = f(x, y, z(x, y))$. Differentiate with respect to x holding y fixed to show that $\left(\frac{\partial w}{\partial x}\right)_y = 18$, where the subscript y indicates that y is held fixed.

b. Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = f(x, y(x, z), z)$ and differentiate with respect to x holding z fixed. Show that $\left(\frac{\partial w}{\partial x}\right)_z = 8$, where the subscript z indicates that z is held fixed.

c. Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

d. Repeat the arguments of parts (a) and (b) to find $\left(\frac{\partial w}{\partial y}\right)_x, \left(\frac{\partial w}{\partial y}\right)_z, \left(\frac{\partial w}{\partial z}\right)_x, \left(\frac{\partial w}{\partial z}\right)_y$.