12.6 Directional Derivatives and the Gradient

Partial derivatives tell us a lot about the rate of change of a function on its domain. However, they do not *directly* answer some important questions. For example, suppose you are standing at a point $(a, b, f(a, b))$ on the surface $z = f(x, y)$. The partial derivatives f_x and f_y tell you the rate of change (or slope) of the surface at that point in the directions parallel to the x-axis and y-axis, respectively. But you could walk in an infinite number of directions from that point and find a different rate of change in every direction. With this observation in mind, we pose several questions.

- Suppose you are standing on a surface and you walk in a direction *other* than a coordinate direction—say, northwest or south-southeast. What is the rate of change of the function in such a direction?
- Suppose you are standing on a surface and you release a ball at your feet and let it roll. In which direction will it roll?
- If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the steepest path?

These questions will be answered in this section as we introduce the *directional derivative*, followed by one of the central concepts of calculus—the *gradient*.

Directional Derivatives

The Gradient Vector

Interpretations of the Gradient

The Gradient and Level Curves

The Gradient in Three Dimensions

Quick Quiz

SECTION 12.6 EXERCISES

Review Questions

1. Explain how a directional derivative is formed from the two partial derivatives f_x and f_y.

2. How do you compute the gradient of the functions $f(x, y)$ and $f(x, y, z)$?

3. Interpret the direction of the gradient vector at a point.

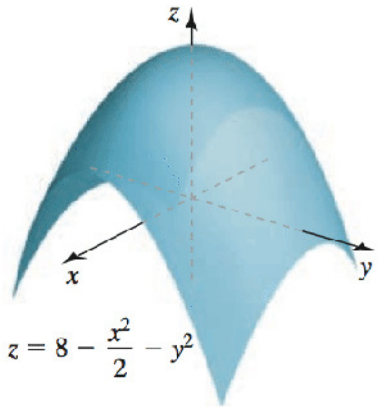
4. Interpret the magnitude of the gradient vector at a point.

5. Given a function f, explain the relationship between the gradient and the level curves of f.

6. The level curves of the surface $z = x^2 + y^2$ are circles in the xy-plane centered at the origin. Without computing the gradient, what is the direction of the gradient at $(1, 1)$ and $(-1, -1)$ (determined up to a scalar multiple)?

Basic Skills

7. **Directional derivatives** Consider the function $f(x, y) = 8 - x^2/2 - y^2$, whose graph is a paraboloid (see figure).

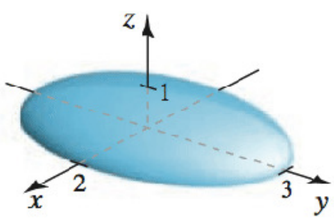


a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions $\langle \cos\theta, \sin\theta \rangle$.

	$(a, b) = (2, 0)$	$(a, b) = (0, 2)$	$(a, b) = (1, 1)$
$\theta = \pi/4$			
$\theta = 3\pi/4$			
$\theta = 5\pi/4$			

b. Sketch the xy-plane and indicate the direction of the directional derivative for each of the table entries in part (a).

8. **Directional derivatives** Consider the function $f(x, y) = \sqrt{1 - \left(\frac{x}{4}\right)^2 - \left(\frac{y}{9}\right)^2}$, whose graph is the upper half of an ellipsoid (see figure).



a. Fill in the table with the values of the directional derivative at the points (a, b) in the directions $\langle \cos\theta, \sin\theta \rangle$.

	$(a, b) = (1, 0)$	$(a, b) = (0, 2)$	$(a, b) = (1, 1)$
$\theta = \pi/4$			
$\theta = 3\pi/4$			
$\theta = 5\pi/4$			

b. Sketch the xy-plane and indicate the direction of the directional derivative for each of the table entries in part (a).

9-14. Computing gradients *Compute the gradient of the following functions and evaluate it at the given point P.*

9. $f(x, y) = 2 + 3x^2 - 5y^2$; $P(2, -1)$

10. $f(x, y) = 4x^2 - 2xy + y^2$; $P(-1, -5)$

11. $g(x, y) = x^2 - 4x^2 y - 8xy^2$; $P(-1, 2)$

12. $p(x, y) = \sqrt{12 - 4x^2 - y^2}$; $P(-1, -1)$

13. $F(x, y) = e^{-x^2 - 2y^2}$; $P(-1, 2)$

14. $h(x, y) = \ln\left(1 + x^2 + 2y^2\right)$; $P(2, -3)$

15-20. Computing directional derivatives with the gradient *Compute the directional derivative of the following functions at the given point P in the direction of the given vector. Be sure to use a unit vector for the direction vector.*

15. $f(x, y) = 10 - 3x^2 + y^4/4$; $P(2, -3)$; $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

16. $g(x, y) = \sin \pi(2x - y)$; $P(-1, -1)$; $\left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$

17. $f(x, y) = \sqrt{4 - x^2 - 2y}$; $P(2, -2)$; $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

18. $h(x, y) = e^{-x-y}$; $P(\ln 2, \ln 3)$; $\langle 1, 1 \rangle$

19. $P(x, y) = \ln\left(4 + x^2 + y^2\right)$; $P(-1, 2)$; $\langle 2, 1 \rangle$

20. $f(x, y) = x/(x - y)$; $P(4, 1)$; $\langle -1, 2 \rangle$

21-26. Direction of steepest ascent and descent *Consider the following functions and points P.*

- ***a.*** *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*
- ***b.*** *Find a vector that points in a direction of no change in the function at P.*

21. $f(x, y) = x^2 - 4y^2 - 9$; $P(1, -2)$

22. $f(x, y) = 6x^2 + 4xy - 3y^2$; $P(6, -1)$

23. $f(x, y) = x^4 - x^2 y + y^2 + 6$; $P(-1, 1)$

24. $p(x, y) = \sqrt{20 + x^2 + 2xy - y^2}$; $P(1, 2)$

25. $F(x, y) = e^{-x^2/2 - y^2/2}$; $P(-1, 1)$

26. $f(x, y) = 2\sin(2x - 3y)$; $P(0, \pi)$

T **27-32. Interpreting directional derivatives** *A function f and a point P are given. Let θ correspond to the direction of the directional derivative.*

- ***a.*** *Find the gradient and evaluate it at P.*
- ***b.*** *Find the angles θ (with respect to the positive x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*
- ***c.*** *Write the directional derivative at P as a function of θ; call this function $g(\theta)$.*
- ***d.*** *Find the value of θ that maximizes $g(\theta)$ and find the maximum value.*
- ***e.*** *Verify that the value of θ that maximizes g corresponds to the direction of the gradient. Verify that the maximum value of g equals the magnitude of the gradient.*

27. $f(x, y) = 10 - 2x^2 - 3y^2$; $P(3, 2)$

28. $f(x, y) = 8 + x^2 + 3y^2$; $P(-3, -1)$

29. $f(x, y) = \sqrt{2 + x^2 + y^2}$; $P\left(\sqrt{3}, 1\right)$

30. $f(x, y) = \sqrt{12 - x^2 - y^2}$; $P\left(-1, -1\big/\sqrt{3}\right)$

31. $f(x, y) = e^{-x^2-2y^2}$; $P(-1, 0)$

32. $f(x, y) = \ln\left(1 + 2x^2 + 3y^2\right)$; $P\left(\frac{3}{4}, -\sqrt{3}\right)$

33-36. Directions of change *Consider the following functions f and points P. Sketch the xy-plane showing P and the level curve through P. Indicate (as in Figure 12.70) the directions of maximum increase, maximum decrease, and no change for f.*

33. $f(x, y) = 8 + 4x^2 + 2y^2$; $P(2, -4)$

34. $f(x, y) = -4 + 6x^2 + 3y^2$; $P(-1, -2)$

35. $f(x, y) = x^2 + xy + y^2 + 7$; $P(-3, 3)$

36. $f(x, y) = \tan(2x + 2y)$; $P(\pi/16, \pi/16)$

37-40. Level curves *Consider the paraboloid $f(x, y) = 16 - x^2\big/4 - y^2\big/16$ and the point P on the given level curve of f. Compute the slope of the line tangent to the level curve at P and verify that the tangent line is orthogonal to the gradient at that point.*

37. $f(x, y) = 0$; $P(0, 16)$

38. $f(x, y) = 0$; $P(8, 0)$

39. $f(x, y) = 12$; $P(4, 0)$

40. $f(x, y) = 12$; $P\left(2\sqrt{3}, 4\right)$

41-44. Level curves *Consider the ellipsoid $f(x, y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}$ and the point P on the given level curve of f.*

Compute the slope of the line tangent to the level curve at P and verify that the tangent line is orthogonal to the gradient at that point.

41. $f(x, y) = \sqrt{3}\big/2$; $P\left(1/2, \sqrt{3}\right)$

42. $f(x, y) = 1\big/\sqrt{2}$; $P\left(0, \sqrt{8}\right)$

43. $f(x, y) = 1\big/\sqrt{2}$; $P\left(\sqrt{2}, 0\right)$

44. $f(x, y) = 1\big/\sqrt{2}$; $P(1, 2)$

45-48. Path of steepest descent *Consider each of the following surfaces and the point P on the surface.*

- **a.** *Find the gradient of f.*
- **b.** *Let C' be the path of steepest descent on the surface beginning at P and let C be the projection of C' on the xy-plane. Find an equation of C in the xy-plane.*

45. $f(x, y) = 4 + x$ (a plane); $P(4, 4, 8)$

46. $f(x, y) = y + x$ (a plane); $P(2, 2, 4)$

47. $f(x, y) = 4 - x^2 - 2y^2$; $P(1, 1, 1)$

48. $f(x, y) = y + x^{-1}$; $P(1, 2, 3)$

49-56. Gradients in three dimensions *Consider the following functions f, points P, and unit vectors* **u**.

- **a.** *Compute the gradient of f and evaluate it at P.*
- **b.** *Find the unit vector in the direction of maximum increase of f at P.*
- **c.** *Find the rate of change of the function in the direction of maximum increase at P.*
- **d.** *Find the directional derivative at P in the direction of the given vector.*

49. $f(x, y, z) = x^2 + 2y^2 + 4z^2 + 10$; $P(1, 0, 4)$; $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$

50. $f(x, y, z) = 4 - x^2 + 3y^2 + z^2/2$; $P(0, 2, -1)$; $\left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

51. $f(x, y, z) = 1 + 4xyz$; $P(1, -1, -1)$; $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

52. $f(x, y, z) = xy + yz + xz + 4$; $P(2, -2, 1)$; $\left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

53. $f(x, y, z) = 1 + \sin(x + 2y - z)$; $P\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)$; $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

54. $f(x, y, z) = e^{xyz-1}$; $P(0, 1, -1)$; $\left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{1}{3} \right\rangle$

55. $f(x, y, z) = \ln\left(1 + x^2 + y^2 + z^2\right)$; $P(1, 1, -1)$; $\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

56. $f(x, y, z) = \frac{x - z}{y - z}$; $P(3, 2, -1)$; $\left\langle \frac{1}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

Further Explorations

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** If $f(x, y) = x^2 + y^2 - 10$, then $\nabla f(x, y) = 2x + 2y$.
- **b.** Because the gradient gives the direction of maximum increase of a function, the gradient is always positive.
- **c.** The gradient of $f(x, y, z) = 1 + xyz$ has four components.

d. If $f(x, y, z) = 4$, then $\nabla f = \mathbf{0}$.

58. Gradient of a composite function Consider the function $F(x, y, z) = e^{xyz}$.

a. Write F as a composite function $f \circ g$, where f is a function of one variable and g is a function of three variables.

b. Relate ∇F to ∇g.

59-63. Directions of zero change *Find the directions in the xy-plane in which the following functions have zero change at the given point. Express the directions in terms of unit vectors.*

59. $f(x, y) = 12 - 4x^2 - y^2$; $P(1, 2, 4)$

60. $f(x, y) = x^2 - 4y^2 - 8$; $P(4, 1, 4)$

61. $f(x, y) = \sqrt{3 + 2x^2 + y^2}$; $P(1, -2, 3)$

62. $f(x, y) = e^{1-xy}$; $P(1, 0, e)$

63. Steepest ascent on a plane Suppose a long sloping hillside is described by the plane $z = ax + by + c$, where a, b, and c are constants. Find the path in the xy-plane, beginning at (x_0, y_0), that corresponds to the path of steepest ascent on the hillside.

64. Gradient of a distance function Let (a, b) be a fixed point in $\mathbb{R}^2$ and let $d = f(x, y)$ be the distance between (a, b) and an arbitrary point (x, y).

a. Show that the graph of f is a cone.

b. Show that the gradient of f at any point other than (a, b) is a unit vector.

c. Interpret the direction and magnitude of ∇f.

65-68. Looking ahead—tangent planes *Consider the following surfaces $f(x, y, z) = 0$, which may be regarded as a level surface of the function $w = f(x, y, z)$. A point $P(a, b, c)$ on the surface is also given.*

a. *Find the (three-dimensional) gradient of f and evaluate it at P.*

b. *The heads of all vectors orthogonal to the gradient with their tails at P form a plane. Find an equation of that plane (soon to be called the* tangent plane *to the surface at f).*

65. $f(x, y, z) = x^2 + y^2 + z^2 - 3 = 0$; $P(1, 1, 1)$

66. $f(x, y, z) = 8 - xyz = 0$; $P(2, 2, 2)$

67. $f(x, y, z) = e^{x+y+z} - 1 = 0$; $P(1, 1, 2)$

68. $f(x, y, z) = xy + xz - yz - 1$; $P(1, 1, 1)$

Applications

T **69. A traveling wave** A snapshot (frozen in time) of a water wave is described by the function $z = 1 + \sin(x - y)$, where z gives the height of the wave relative to a reference point and (x, y) are coordinates in a horizontal plane.

a. Use a graphing utility to graph $z = 1 + \sin(x - y)$.

b. The crests and the troughs of the waves are aligned in the direction in which the height function has zero change. Find the direction in which the crests and troughs are aligned.

c. If you were surfing on this wave and wanted the steepest descent from a crest to a trough, in which direction would you point your surfboard (given in terms of a unit vector in the xy-plane)?

d. Check that your answers to parts (b) and (c) are consistent with the graph of part (a).

70. Traveling waves in general Generalize Exercise 69 by considering a wave described by the function $z = A + \sin(a\,x - b\,y)$, where a, b, and A are real numbers.

a. Find the direction in which the crests and troughs of the wave are aligned. Express your answer as a unit vector in terms of a and b.

b. Find the surfer's direction—that is, the direction of steepest descent from a crest to a trough. Express your answer as a unit vector in terms of a and b.

71-73. Potential functions *Potential functions arise frequently in physics and engineering. A potential function has the property that a field of interest (for example, an electric field, a gravitational field, or a velocity field) is the gradient of the potential (or sometimes the negative of the gradient of the potential). (Potential functions are considered in depth in Chapter 14.)*

71. Electric potential due to a point charge The electric field due to a point charge of strength Q at the origin has a potential function $V = k\,Q/r$, where $r^2 = x^2 + y^2 + z^2$ is the square of the distance between a variable point $P(x, y, z)$ and the charge and k is a physical constant. The electric field is given by $\mathbf{E} = -\nabla V$, where ∇V is the gradient in three dimensions.

a. Show that the three-dimensional electric field due to a point charge is given by

$$\mathbf{E}(x, y, z) = k\,Q\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle.$$

b. Show that the electric field at a point has a magnitude $|\mathbf{E}| = k\,Q/r^2$. Explain why this relationship is called an inverse square law.

72. Gravitational potential The gravitational potential associated with two objects of mass M and m is $V = -G\,M\,m/r$, where G is the gravitational constant. If one of the objects is at the origin and the other object is at $P(x, y, z)$, then $r^2 = x^2 + y^2 + z^2$ is the square of the distance between the objects. The gravitational field at a point is given by $\mathbf{F} = -\nabla V$, where ∇V is the gradient in three dimensions. Show that the force has a magnitude $|\mathbf{F}| = G\,M\,m/r^2$. Explain why this relationship is called an inverse square law.

73. Velocity potential In two dimensions, the motion of an ideal fluid (an incompressible and irrotational fluid) is governed by a velocity potential ϕ. The velocity components of the fluid, u in the x-direction and v in the y-direction, are given by $\langle u, v\rangle = \nabla\phi$. Find the velocity components associated with the velocity potential $\phi(x, y) = \sin \pi x \sin 2\pi y$.

Additional Exercises

74. Gradients for planes Prove that for the plane described by $f(x, y) = A\,x + B\,y$, where A and B are nonzero constants, the gradient is constant (independent of (x, y)). Interpret this result.

75. Rules for gradients Use the definition of the gradient (in two or three dimensions), assume that f and g are differentiable functions on $\mathbb{R}^2$ or $\mathbb{R}^3$, and let c be a constant. Prove the following gradient rules.

a. Constants Rule: $\nabla(c\,f) = c\,\nabla f$

b. Sum Rule: $\nabla(f + g) = \nabla f + \nabla g$

c. Product Rule: $\nabla(f\,g) = (\nabla f)\,g + f\,\nabla g$

d. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$

e. Chain Rule: $\nabla(f \circ g) = f'(g)\,\nabla g$, where f is a function of one variable

76-81. Using gradient rules *Use the gradient rules of Exercise 75 to find the gradient of the following functions.*

76. $f(x, y) = x\,y\cos(x\,y)$

77. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$

78. $f(x, y) = \ln\left(1 + x^2 + y^2\right)$

79. $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$

80. $f(x, y, z) = (x + y + z)\, e^{xyz}$

81. $f(x, y, z) = \dfrac{x + yz}{y + xz}$