12.7 Tangent Planes and Linear Approximation

In Section 4.5, we saw that if we zoom in on a point on a smooth curve (one described by a differentiable function), the curve looks more and more like the tangent line at that point. Once we have the tangent line at a point, it can be used to approximate function values and to estimate changes in the dependent variable. In this section, an analogous story is developed, elevated by one dimension. Now we see that differentiability at a point (as discussed in Section 12.4) implies the existence of a tangent *plane* at that point (Figure 12.76).

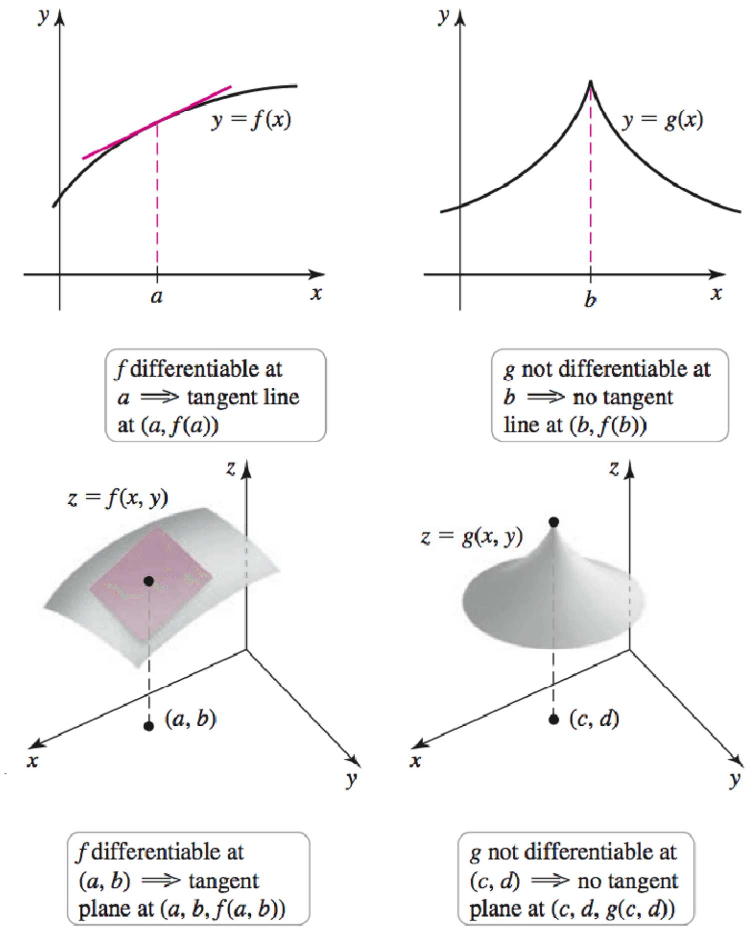


FIGURE 12.76

Consider a smooth surface described by a differentiable function f and focus on a single point on the surface. As we zoom in on that point (Figure 12.77), the surface appears more and more like a plane. The first step is to define this plane carefully; it is called the *tangent plane*. Once we have the tangent plane, we can use it to approximate function values and to estimate changes in the dependent variable.

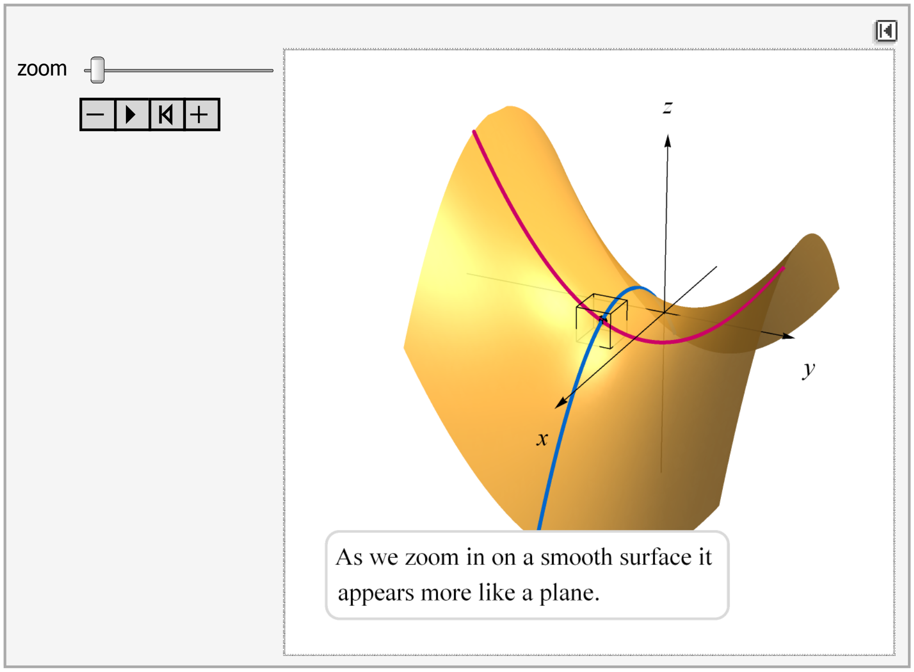


FIGURE 12.77

Tangent Planes

Linear Approximation

Differentials and Change

Quick Quiz

SECTION 12.7 EXERCISES

Review Questions

1. Suppose **n** is a vector normal to the tangent plane of the surface $F(x, y, z) = 0$ at a point. How is **n** related to the gradient of F at that point?

2. Write the explicit function $z = x y^2 + x^2 y - 10$ in the implicit form $F(x, y, z) = 0$.

3. Write an equation for the plane tangent to the surface $F(x, y, z) = 0$ at the point (a, b, c).

4. Write an equation for the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

5. Explain how to approximate a function f at a point near (a, b) where the values of f, f_x, and f_y are known at (a, b).

6. Explain how to approximate the change in a function f when the independent variables change from (a, b) to $(a + \Delta x, b + \Delta y)$.

7. Write the approximate change formula for a function $z = f(x, y)$ at the point (a, b) in terms of differentials.

8. Write the differential dw for the function $w = f(x, y, z)$.

Basic Skills

9-14. Tangent planes for $F(x, y, z) = 0$ *Find an equation of the plane tangent to the following surfaces at the given points.*

9. $xy + xz + yz - 12 = 0$; $(2, 2, 2)$ and $\left(-1, -2, -\frac{10}{3}\right)$

10. $x^2 + y^2 - z^2 = 0$; $(3, 4, 5)$ and $(-4, -3, 5)$

11. $xy \sin z = 1$; $(1, 2, \pi/6)$ and $(-2, -1, 5\pi/6)$

12. $yz\,e^{xz} - 8 = 0$; $(0, 2, 4)$ and $(0, -8, -1)$

13. $z^2 - x^2/16 - y^2/9 - 1 = 0$; $\left(4, 3, -\sqrt{3}\right)$ and $\left(-8, 9, \sqrt{14}\right)$

14. $2x + y^2 - z^2 = 0$; $(0, 1, 1)$ and $(4, 1, -3)$

15-20. Tangent planes for $z = f(x, y)$ *Find an equation of the plane tangent to the following surfaces at the given points.*

15. $z = 4 - 2x^2 - y^2$; $(2, 2, -8)$ and $(-1, -1, 1)$

16. $z = 2 + 2x^2 + y^2/2$; $\left(-\frac{1}{2}, 1, 3\right)$ and $(3, -2, 22)$

17. $z = x^2 e^{x-y}$; $(2, 2, 4)$ and $(-1, -1, 1)$

18. $z = \ln(1 + xy)$; $(1, 2, \ln 3)$ and $(-2, -1, \ln 3)$

19. $z = (x - y)/\left(x^2 + y^2\right)$; $\left(1, 2, -\frac{1}{5}\right)$ and $\left(2, -1, \frac{3}{5}\right)$

20. $z = 2\cos(x - y) + 2$; $(\pi/6, -\pi/6, 3)$ and $(\pi/3, \pi/3, 4)$

21-26. Linear approximation

a. *Find the linear approximation for the following functions at the given point.*

b. *Use part (a) to estimate the given function value.*

21. $f(x, y) = xy + x - y$; $(2, 3)$; estimate $f(2.1, 2.99)$.

22. $f(x, y) = 12 - 4x^2 - 8y^2$; $(-1, 4)$; estimate $f(-1.05, 3.95)$.

23. $f(x, y) = -x^2 + 2y^2$; $(3, -1)$; estimate $f(3.1, -1.04)$.

24. $f(x, y) = \sqrt{x^2 + y^2}$; $(3, -4)$; estimate $f(3.06, -3.92)$.

25. $f(x, y) = \ln(1 + x + y)$; $(0, 0)$; estimate $f(0.1, -0.2)$.

26. $f(x, y) = (x + y)/(x - y)$; $(3, 2)$; estimate $f(2.95, 2.05)$.

27-30. Approximate function change *Use differentials to approximate the change in z for the given changes in the independent variables.*

27. $z = 2x - 3y - 2xy$ when (x, y) changes from $(1, 4)$ to $(1.1, 3.9)$

28. $z = -x^2 + 3\,y^2 + 2$ when $(x,\ y)$ changes from $(-1,\ 2)$ to $(-1.05,\ 1.9)$

29. $z = e^{x+y}$ when $(x,\ y)$ changes from $(0,\ 0)$ to $(0.1,\ -0.05)$

30. $z = \ln(1 + x + y)$ when $(x,\ y)$ changes from $(0,\ 0)$ to $(-0.1,\ 0.03)$

31. Changes in torus surface area The surface area of a torus (an ideal bagel or doughnut) with an inner radius r and an outer radius $R > r$ is $S = 4\,\pi^2\left(R^2 - r^2\right)$.

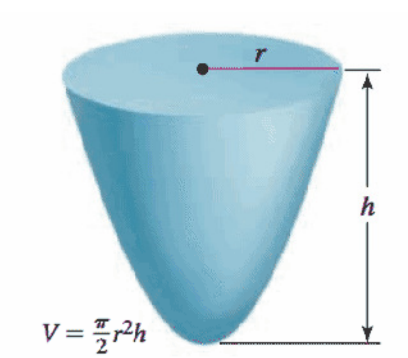
- **a.** If r increases and R decreases, does S increase or decrease, or is it impossible to say?
- **b.** If r increases and R increases, does S increase or decrease, or is it impossible to say?
- **c.** Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 3.05$ and R changes from $R = 5.50$ to $R = 5.65$.
- **d.** Estimate the change in the surface area of the torus when r changes from $r = 3.00$ to $r = 2.95$ and R changes from $R = 7.00$ to $R = 7.04$.
- **e.** Find the relationship between the changes in r and R that leaves the surface area (approximately) unchanged.

32. Changes in cone volume The volume of a right circular cone with radius r and height h is $V = \pi\,r^2\,h/3$.

- **a.** Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$.
- **b.** Approximate the change in the volume of the cone when the radius changes from $r = 5.40$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$.

33. Area of an ellipse The area of an ellipse with axes of length $2\,a$ and $2\,b$ is $A = \pi\,a\,b$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

34. Volume of a paraboloid The volume of a segment of a circular paraboloid (see figure) with radius r and height h is $V = \pi\,r^2\,h/2$. Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%.



35-38. Differentials with more than two variables *Write the differential dw in terms of the differentials of the independent variables.*

35. $w = f(x,\ y,\ z) = x\,y^2 + z\,x^2 + y\,z^2$

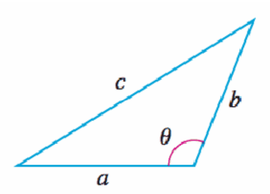
36. $w = f(x,\ y,\ z) = \sin(x + y - z)$

37. $w = f(u,\ x,\ y,\ z) = (u + x)/(y + z)$

38. $w = f(p,\ q,\ r,\ s) = p\,q/(r\,s)$

T **39. Law of Cosines** The side lengths of any triangle are related by the Law of Cosines,

$$c^2 = a^2 + b^2 - 2\,ab\cos\theta\,.$$



a. Estimate the change in the side length c when a changes from $a = 2$ to $a = 2.03$, b changes from $b = 4.00$ to $b = 3.96$ and θ changes from $\theta = \pi/3$ to $\theta = \pi/3 + \pi/90$.

b. If a changes from $a = 2$ to $a = 2.03$ and b changes from $b = 4.00$ to $b = 3.96$, is the resulting change in c greater in magnitude when $\theta = \pi/20$ (small angle) or when $\theta = 9\pi/20$ (close to a right angle)?

40. Travel cost The cost of a trip that is L miles long, driving a car that gets m miles per gallon, with gas costs of $\$p/\text{gal}$ is $C = L\,p/m$ dollars. Suppose you plan a trip of $L = 1500$ mi in a car that gets $m = 32$ mi/gal, with gas costs of $p = \$3.80/\text{gal}$.

a. Explain how the cost function is derived.

b. Compute the partial derivatives C_L, C_m, and C_p. Explain the meaning of the signs of the derivatives in the context of this problem.

c. Estimate the change in the total cost if the trip of L changes from $L = 1500$ to $L = 1520$, m changes from $m = 32$ to 31, and p changes from \$3.80 to $p = \$3.85$.

d. Is the total cost of the trip (with $L = 1500$ mi, $m = 32$ mi/gal, and $p = \$3.80$) more sensitive to a 1% change in L, m, or p (assuming the other two variables are fixed)? Explain.

Further Explorations

41. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The planes tangent to the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ all have the form $a\,x + b\,z + c = 0$.

b. Suppose $w = x\,y/z$ for $x > 0$, $y > 0$, and $z > 0$. A decrease in z with x and y fixed results in an increase in w.

c. The gradient $\nabla F(a, b, c)$ lies in the plane tangent to the surface $F(x, y, z) = 0$ at (a, b, c).

42-45. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given point.*

42. $z = \tan^{-1}(x + y)$; $(0, 0, 0)$

43. $z = \tan^{-1}(x\,y)$; $(1, 1, \pi/4)$

44. $(x + z)/(y - z) = 2$; $(4, 2, 0)$

45. $\sin x\,y\,z = \dfrac{1}{2}$; $\left(\pi, 1, \dfrac{1}{6}\right)$

46-49. Horizontal tangent planes *Find the points at which the following surfaces have horizontal tangent planes.*

46. $z = \sin(x - y)$ in the region $-2\pi \le x \le 2\pi$, $-2\pi \le y \le 2\pi$

47. $x^2 + y^2 - z^2 - 2\,x + 2\,y + 3 = 0$

48. $x^2 + 2\,y^2 + z^2 - 2\,x - 2\,z - 2 = 0$

49. $z = \cos 2\,x \sin y$ in the region $-\pi \le x \le \pi$, $-\pi \le y \le \pi$

50. Heron's formula The area of a triangle with sides of length a, b, and c is given by a formula from antiquity called Heron's formula:

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}\,,$$

where $s = (a+b+c)/2$ is the *semiperimeter* of the triangle.

a. Find the partial derivatives A_a, A_b, and A_c.

b. A triangle has sides of length $a = 2$, $b = 4$, $c = 5$. Estimate the change in the area when a increases by 0.03, b decreases by 0.08, and c increases by 0.6.

c. For an equilateral triangle with $a = b = c$, estimate the percent change in the area when all sides increase in length by $p\%$.

51. Surface area of a cone A cone with height h and radius r has a lateral surface area (the curved surface only, excluding the base) of $S = \pi r \sqrt{r^2 + h^2}$.

a. Estimate the change in the surface area when r increases from $r = 2.50$ to $r = 2.55$ and h decreases from $h = 0.60$ to $h = 0.58$.

b. When $r = 100$ and $h = 200$, is the surface area more sensitive to a small change in r or a small change in h? Explain.

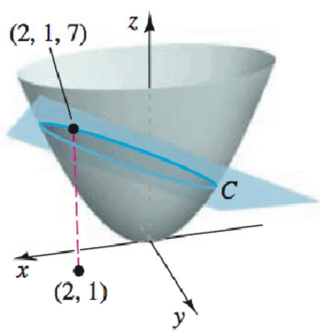
52. Line tangent to an intersection curve Consider the paraboloid $z = x^2 + 3\,y^2$ and the plane $z = x + y + 4$, which intersects the paraboloid in a curve C at $(2, 1, 7)$ (see figure). Find the equation of the line tangent to C at the point $(2, 1, 7)$. Proceed as follows.

a. Find a vector normal to the plane at $(2, 1, 7)$.

b. Find a vector normal to the plane tangent to the paraboloid at $(2, 1, 7)$.

c. Argue that the line tangent to C at $(2, 1, 7)$ is orthogonal to both normal vectors found in parts (a) and (b). Use this fact to find a direction vector for the tangent line.

d. Knowing a point on the tangent line and the direction of the tangent line, write an equation of the tangent line in parametric form.



Applications

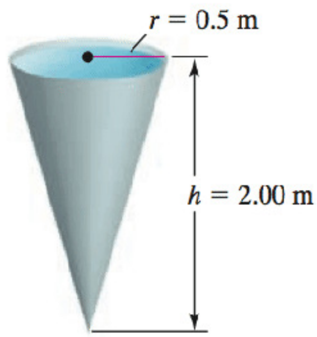
53. Batting averages Batting averages in baseball are defined by $A = x/y$, where $x \geq 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \leq A \leq 1$.

a. Use differentials to estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

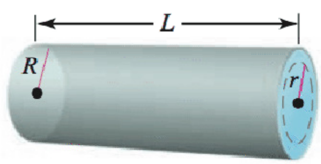
b. If a batter currently has a batting average of $A = 0.350$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

c. Does the answer to part (b) depend on the current batting average? Explain.

54. **Water-level changes** A conical tank with radius 0.50 m and height 2.00 m is filled with water (see figure). Water is released from the tank, and the water level drops by 0.05 m (from 2.00 m to 1.95 m). Approximate the change in the volume of water in the tank. (*Hint:* When the water level drops, both the radius and height of the cone of water change.)



55. **Flow in a cylinder** Poiseuille's Law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \dfrac{P}{4Lv}\left(R^2 - r^2\right)$, where P is the difference in the pressure between the ends of the cylinder and v is the viscosity of the fluid (see figure). Assuming that P and v are constant, the velocity V along the centerline of the cylinder ($r = 0$) is $V = kR^2/L$, where k is a constant that we will take to be $k = 1$.



a. Estimate the change in the centerline velocity ($r = 0$) if the radius of the flow cylinder increases from $R = 3$ cm to $R = 3.05$ cm and the length increases from $L = 50$ cm to $L = 50.5$ cm.

b. Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length L increases by 2%.

c. Complete the following sentence: If the radius of the cylinder increases by p%, then the length of the cylinder must decrease by approximately ___% in order for the velocity to remain constant.

56. **Floating-point operations** In general, real numbers (with infinite decimal expansions) cannot be represented exactly in a computer by floating-point numbers (with finite decimal expansions). Suppose that floating-point numbers on a particular computer carry an error of at most 10^{-16}. Estimate the maximum error that is committed in doing the following arithmetic operations. Express the error in absolute and relative (percent) terms.

a. $f(x, y) = xy$

b. $f(x, y) = x/y$

c. $F(x, y, z) = xyz$

d. $F(x, y, z) = (x/y)/z$

57. **Probability of at least one encounter** Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in n random encounters, you will meet at least one person with flu is $P = f(n, r) = 1 - (1 - r)^n$. Although n is a positive integer, regard it as a positive real number.

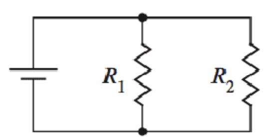
a. Compute f_r and f_n.

b. How sensitive is the probability P to the flu rate r? Suppose you meet $n = 20$ people. Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c. Approximately how much does the probability P increase if the flu rate increases from $r = 0.9$ to $r = 0.91$?

d. Interpret the results of parts (b) and (c).

58. Two electrical resistors When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



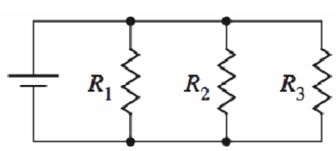
a. Estimate the change in R if R_1 increases from 2 ohms to 2.05 ohms and R_2 decreases from 3 ohms to 2.95 ohms.

b. Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c. Is it true that if R_1 and R_2 increase, then R increases? Explain.

d. Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

59. Three electrical resistors Extending Exercise 58, when three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$, and $R_3 > 0$ are wired in parallel in a circuit (see figure), the combined resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.
Estimate the change in R if R_1 increases from 2 ohms to 2.05 ohms, R_2 decreases from 3 ohms to 2.95 ohms, and R_3 increases from 1.5 ohms to 1.55 ohms.



Additional Exercises

60. Power functions and percent change Suppose that $z = f(x, y) = x^a y^b$, where a and b are real numbers. Let dx/x, dy/y, and dz/z be the approximate relative (percent) changes in x, y, and z, respectively. Show that $(dz)/z = a(dx)/x + b(dy)/y$; that is, the relative changes are additive when weighted by the exponents a and b.

61. Logarithmic differentials Let f be a differentiable function of one or more variables that is positive on its domain.

a. Show that $d(\ln f) = \frac{df}{f}$.

b. Use part (a) to explain the statement that the absolute change in $\ln f$ is approximately equal to the relative change in f.

c. Let $f(x, y) = xy$, note that $\ln f = \ln x + \ln y$, and show that relative changes add; that is $(df)/f = (dx)/x + (dy)/y$.

d. Let $f(x, y) = x/y$, note that $\ln f = \ln x - \ln y$, and show that relative changes subtract; that is $(df)/f = (dx)/x - (dy)/y$.

e. Show that in a product of n numbers, $f = x_1 x_2 \cdots x_n$, the relative change in f is approximately equal to the sum of the relative changes in the variables.

62. Distance from a plane to an ellipsoid (Adapted from 1938 Putnam Exam.) Consider the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and the plane P given by $A\,x + B\,y + C\,z + 1 = 0$. Let $h = \left(A^2 + B^2 + C^2\right)^{-1/2}$ and $m = \left(a^2\,A^2 + b^2\,B^2 + c^2\,C^2\right)^{1/2}$.

a. Find the equation of the plane tangent to the ellipsoid at the point (p, q, r).

b. Find the two points on the ellipsoid at which the tangent plane is parallel to P and find equations of the tangent planes.

c. Show that the distance between the origin and the plane P is h.

d. Show that the distance between the origin and the tangent planes is hm.

e. Find a condition that guarantees the plane P does not intersect the ellipsoid.