12.8 Maximum/Minimum Problems

In Chapter 4 we showed how to use derivatives to find maximum and minimum values of functions of a single variable. When those techniques are extended to functions of two variables, we discover both similarities and differences. The landscape of a surface is far more complicated than the profile of a curve in the plane, so we see more interesting features when working with several variables. In addition to peaks (maximum values) and hollows (minimum values), we encounter winding ridges, long valleys, and mountain passes. Yet despite these complications, many of the ideas used for single-variable functions reappear in higher dimensions. For example, the Second Derivative Test, suitably adapted for two variables, plays a central role. As with single-variable functions, the techniques developed here are useful for solving practical optimization problems.

Local Maximum/Minimum Values

Second Derivative Test

Absolute Maximum and Minimum Values

Quick Quiz

SECTION 12.8 EXERCISES

Review Questions

1. Describe the appearance of a smooth surface with a local maximum at a point.

2. Describe the usual appearance of a smooth surface at a saddle point.

3. What are the conditions for a critical point of a function f?

4. If $f_x(a, b) = f_y(a, b) = 0$, does it follow that f has a local maximum or local minimum at (a, b)? Explain.

5. What is the discriminant and how do you compute it?

6. Explain how the Second Derivative Test is used.

7. What is an absolute minimum value of a function f on a set R in $\mathbb{R}^2$?

8. What is the procedure for locating absolute maximum and minimum values on a closed bounded domain?

Basic Skills

9-14. Critical points *Find all critical points of the following functions.*

9. $f(x, y) = 1 + x^2 + y^2$

10. $f(x, y) = x^2 - 6x + y^2 + 8y$

11. $f(x, y) = (3x - 2)^2 + (y - 4)^2$

12. $f(x, y) = 3x^2 - 4y^2$

13. $f(x, y) = x^4 + y^4 - 16xy$

14. $f(x, y) = x^3/3 - y^3/3 + 3xy$

15-28. Analyzing critical points *Find the critical points of the following functions. Use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point. Confirm your results using a graphing utility.*

15. $f(x, y) = 4 + 2x^2 + 3y^2$

16. $f(x, y) = (4x - 1)^2 + (2y + 4)^2 + 1$

17. $f(x, y) = -4x^2 + 8y^2 - 3$

18. $f(x, y) = x^4 + y^4 - 4x - 32y + 10$

19. $f(x, y) = x^4 + 2y^2 - 4xy$

20. $f(x, y) = xye^{-x-y}$

21. $f(x, y) = \sqrt{x^2 + y^2 - 4x + 5}$

22. $f(x, y) = \tan^{-1}(xy)$

23. $f(x, y) = 2xye^{-x^2-y^2}$

24. $f(x, y) = x^2 - x^4/2 - y^2 - xy$

25. $f(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$

26. $f(x, y) = \dfrac{xy(x - y)}{x^2 + y^2}$

27. $f(x, y) = ye^x - e^y$

28. $f(x, y) = \sin(2\pi x)\cos(\pi y)$, for $|x| \le \dfrac{1}{2}$ and $|y| \le \dfrac{1}{2}$.

29. Shipping regulations A shipping company handles rectangular boxes provided the sum of the height and the girth of the box does not exceed 96 in. (The girth is the perimeter of the smallest base of the box.) Find the dimensions of the box that meets this condition and has the largest volume.

30. Cardboard boxes A lidless box is to be made using 2 m^2 of cardboard. Find the dimensions of the box with the largest possible volume.

31. Cardboard boxes A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that requires the least amount of cardboard.

32. Optimal box Find the dimensions of the largest rectangular box in the first octant of the *xyz*-coordinate system that has one vertex at the origin and the opposite vertex on the plane $x + 2y + 3z = 6$.

33-36. Inconclusive tests *Show that the Second Derivative test is inconclusive when applied to the following functions at* (0, 0). *Describe the behavior of the function at the critical point.*

33. $f(x, y) = 4 + x^4 + 3y^4$

34. $f(x, y) = x^2 y - 3$

35. $f(x, y) = x^4 y^2$

36. $f(x, y) = \sin(x^2 y^2)$

37-44. Absolute maxima and minima *Find the absolute maximum and minimum values of the following functions on the given set R.*

37. $f(x, y) = x^2 + y^2 - 2y + 1$; $R = \{(x, y) : x^2 + y^2 \leq 4\}$

38. $f(x, y) = -x^2 - y^2 + \sqrt{3}\, x - y - 1$; $R = \{(x, y) : x^2 + y^2 \leq 6\}$

39. $f(x, y) = 4 + 2x^2 + y^2$; $R = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$

40. $f(x, y) = 6 - x^2 - 4y^2$; $R = \{(x, y) : -2 \leq x \leq 2, -1 \leq y \leq 1\}$

41. $f(x, y) = x^2 + y^2 + 4x - 2y$; $R = \{(x, y) : x^2 + y^2 \leq 16\}$

42. $f(x, y) = x^2 + y^2 - 2x - 2y$; R is the closed set bounded by the triangle with vertices (0, 0), (2, 0), and (0, 2).

43. $f(x, y) = x^2 + 4y^2 + 2x + 4y$; R is the closed set bounded by the ellipse $\{(x, y) : x = 4\cos\theta,\ y = \sin\theta,\ \text{for } 0 \leq \theta \leq 2\pi\}$.

44. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}$; R is the closed half disk $\{(x, y) : x^2 + y^2 \leq 4 \text{ with } y \geq 0\}$.

45-48. Absolute extrema on open and/or unbounded sets *If possible, find the absolute maximum and minimum values of the following functions on the set R.*

45. $f(x, y) = x^2 + y^2 - 4$; $R = \{(x, y) : x^2 + y^2 < 4\}$

46. $f(x, y) = x + 3y$; $R + \{(x, y) : |x| < 1, |y| < 2\}$

47. $f(x, y) = 2e^{-x-y}$; $R = \{(x, y) : x \geq 0, y \geq 0\}$

48. $f(x, y) = x^2 - y^2$; $R = \{(x, y); |x| < 1, |y| < 1\}$

49-52. Absolute extrema on open and/or unbounded sets

49. Find the point on the plane $x + y + z = 4$ nearest the point $P(0, 3, 6)$.

50. Find the point(s) on the cone $z^2 = x^2 + y^2$ nearest the point $P(1, 4, 0)$.

51. Find the point on the surface $f(x, y) = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$. Identify the point on the plane.

52. Rectangular boxes with a volume of $10\ \text{m}^3$ are to be made of two materials. The material for the top and bottom of the box costs $\$8/\text{m}^2$ and the material for the sides of the box costs $\$1/\text{m}^2$. What are the dimensions of the box that minimizes the cost of the box?

Further Explorations

53. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample. Assume that f is differentiable at the points in question.

a. The fact that $f_x(2, 2) = f_y(2, 2) = 0$ implies that f has a local maximum, local minimum, or saddle point at (2, 2).

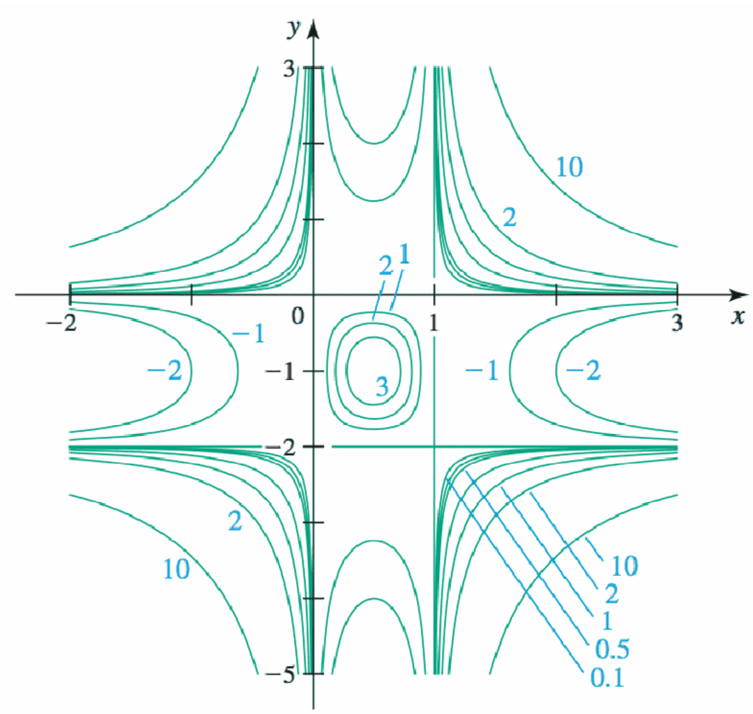
b. The function f could have a local maximum at (a, b) where $f_y(a, b) \neq 0$.

c. The function f could have both an absolute maximum and an absolute minimum at two different points that are not critical points.

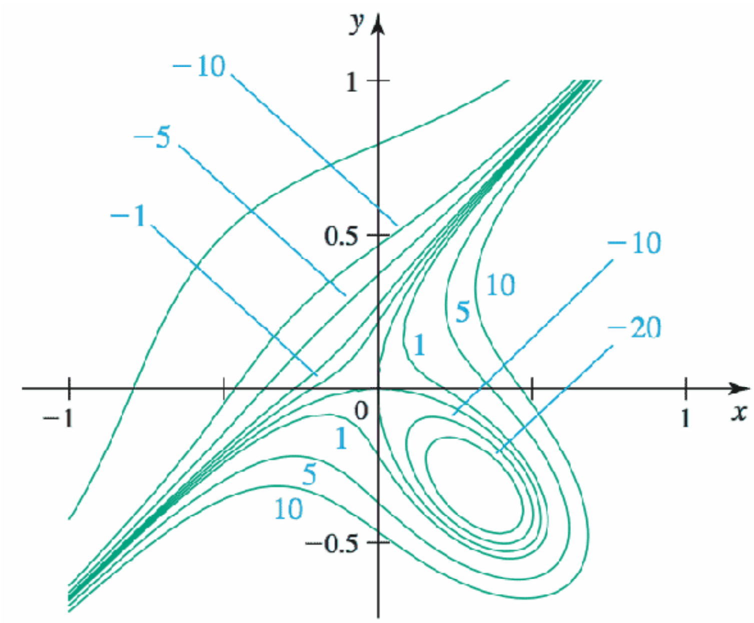
d. The tangent plane is horizontal at a point on a surface corresponding to a critical point.

54-55. Extreme points from contour plots *Based on the level curves that are visible in the following graphs, identify the approximate locations of the local maxima, local minima, and saddle points.*

54.



55.



56. Optimal box Find the dimensions of the rectangular box with maximum volume in the first octant with one vertex at the origin and the opposite vertex on the ellipsoid $36 x^2 + 4 y^2 + 9 z^2 = 36$.

57. Least distance What point on the plane $x - y + z = 2$ is closest to the point $(1, 1, 1)$?

58. Maximum/Minimum of linear functions Let R be a closed bounded set in $\mathbb{R}^2$ and let $f(x, y) = a x + b y + c$, where a, b, and c are real numbers, with a and b not both zero. Give a geometrical argument explaining why the absolute maximum and minimum values of f over R occur on the boundaries of R.

59. Magic triples Let x, y, and z be nonnegative numbers with $x + y + z = 200$.

a. Find the values of x, y, and z that minimize $x^2 + y^2 + z^2$.

b. Find the values of x, y, and z that minimize $\sqrt{x^2 + y^2 + z^2}$.

c. Find the values of x, y, and z that maximize $x y z$.

d. Find the values of x, y, and z that maximize $x^2 y^2 z^2$.

60. Powers and roots Assume that $x + y + z = 1$ with $x \ge 0$, $y \ge 0$, and $z \ge 0$.

a. Find the maximum and minimum values of $\left(1 + x^2\right)\left(1 + y^2\right)\left(1 + z^2\right)$.

b. Find the maximum and minimum values of $\left(1 + \sqrt{x}\right)\left(1 + \sqrt{y}\right)\left(1 + \sqrt{z}\right)$.

[*Source: Math Horizons* (April 2004).]

Applications

T **61. Optimal locations** Suppose n houses are located at the distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. A power substation must be located at a point such that the *sum of the squares* of the distances between the houses and the substation is minimized.

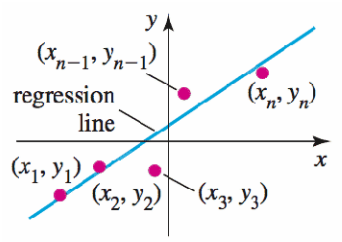
a. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at $(0, 0)$, $(2, 0)$, and $(1, 1)$.

b. Find the optimal location of the substation in the case that $n = 3$ and the houses are located at distinct points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

c. Find the optimal location of the substation in the general case of n houses located at distinct points (x_1, y_1), (x_2, y_2), …, (x_n, y_n).

d. You might argue that the locations found in parts (a), (b) and (c) are not optimal because they result from minimizing the sum of the *squares* of the distances, not the sum of the distances themselves. Use the locations in part (a) and write the function that gives the sum of the distances. Note that minimizing this function is much more difficult than in part (a). Then use a graphing utility to determine whether the optimal location is the same in the two cases. (Also see Exercise 69 about Steiner's problem.)

62-65. Least squares approximation *In its many guises, least squares approximation arises in numerous areas of mathematics and statistics. Suppose you collect data for two variables (for example, height and shoe size) in the form of pairs* (x_1, y_1), (x_2, y_2), …, (x_n, y_n). *The data may be plotted as a scatterplot in the xy-plane, as shown in the figure. The technique known as* linear regression *asks the question: What is the equation of the line that "best fits" the data? The least squares criterion for best fit requires that the sum of the squares of the vertical distances between the line and the data points is a minimum.*



62. Let the equation of the best-fit line be $y = m x + b$, where the slope m and the y-intercept b must be determined using the least squares condition. First assume that there are three data points (1, 2), (3, 5), and (4, 6). Show that the function of m and b that gives the sum of the squares of the vertical distances between the line and the three data points is

$$E(m, b) = [(m + b) - 2]^2 + [(3 m + b) - 5]^2 + [(4 m + b) - 6]^2.$$

Find the critical points of E and find the values of m and b that minimize E. Graph the three data points and the best-fit line.

T **63.** Generalize the procedure in Exercise 62 by assuming that n data points (x_1, y_1), (x_2, y_2), …, (x_n, y_n) are given. Write the function $E(m, b)$ (summation notation allows for a more compact calculation). Show that the coefficients of the best-fit line are

$$m = \frac{(\sum x_k)(\sum y_k) - n \sum x_k\, y_k}{(\sum x_k)^2 - n \sum {x_k}^2},$$
$$b = \frac{1}{n}\left(\sum y_k - m \sum x_k\right),$$

where all sums run from $k = 1$ to $k = n$.

T **64-65. Least squares practice** *Use the results of Exercise 63 to find the best-fit line for the following data sets. Plot the points and the best-fit line.*

64. (0, 0), (2, 3), (4, 5)

65. (−1, 0), (0, 6), (3, 8)

Additional Exercises

66. Second Derivative Test Prove that if (a, b) is a critical point of f at which $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}(a, b) < 0 < f_{yy}(a, b)$ or $f_{yy}(a, b) < 0 < f_{xx}(a, b)$, then f has a saddle point at (a, b).

67. Maximum area triangle Among all triangles with a perimeter of 9 units, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

68. Ellipsoid inside a tetrahedron (1946 Putnam Exam) Let P be a plane tangent to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at a point in the first octant. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x = 0$, $y = 0$, and $z = 0$. Find the minimum volume of T. (The volume of a tetrahedron is one-third the area of the base times the height.)

T **69. Steiner's problem for three points** Given three distinct noncollinear points A, B, and C in the plane, find the point P in the plane such that the sum of the distances $|AP| + |BP| + |CP|$ is a minimum. Here is how to proceed with three points, assuming that the triangle formed by the three points has no angle greater than $2\pi/3$ ($120°$).

a. Assume the coordinates of the three given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b. Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of the line between the two points.

c. The goal is to minimize $f(x, y) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d. Explain why part (c) implies that the optimal point P has the property that the three line segments AP, BP, and CP all intersect symmetrically in angles of $2\pi/3$.

e. What is the optimal solution if one of the angles in the triangle is greater than $2\pi/3$ (just draw a picture)?

f. Estimate the Steiner point for the three points $(0, 0)$, $(0, 1)$, $(2, 0)$.

70. Slicing plane Find an equation of the plane passing through the point $(3, 2, 1)$ that slices off the region in the first octant with the least volume.

T **71. Two mountains without a saddle** Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

a. $f(x, y) = -(x^2 - 1)^2 - (x^2 - e^y)^2$

b. $f(x, y) = 4x^2 e^y - 2x^4 - e^{4y}$

Source: Proposed by Ira Rosenholtz, *Mathematics Magazine* (February, 1987).

T **72. Solitary critical points** A function of *one* variable has the property that a local maximum (or minimum) occurring at the only critical point is also the absolute maximum (or minimum) (for example, $f(x) = x^2$). Does the same result hold for a function of *two* variables? Show that the following functions have the property that they have a single local maximum (or minimum), occurring at the only critical point, but that the local maximum (or minimum) is not an absolute maximum (or minimum) on $\mathbb{R}^2$.

a. $f(x, y) = 3x e^y - x^3 - e^{3y}$

b. $f(x, y) = (2y^2 - y^4)\left(e^x + \dfrac{1}{1+x^2}\right) - \dfrac{1}{1+x^2}$

This property has the following interpretation. Suppose that a surface has a single local minimum that is not the absolute minimum. Then water can be pourerd into the basin around the local minimum and the surface never overflows, even though there are points on the surface below the local minimum.

Source: See three articles in *Mathematics Magazine* (May 1985) and *Calculus and Analytical Geometry*, 2nd ed., Philip Gillett.