12.9 Lagrange Multipliers

One of many challenges in economics and marketing is predicting the behavior of consumers. Basic models of consumer behavior often involve a *utility function* that expresses consumers' combined preference for several different amenities. For example, a simple utility function might have the form $U = f(l, g)$, where l represents the amount of leisure time and g represents the number of consumable goods. The model assumes that consumers try to maximize their utility function, but they do so under certain constraints on the variables of the problem. For example, increasing leisure time may increase utility, but leisure time produces no income for consumable goods. Similarly, consumable goods may also increase utility, but they require income, which reduces leisure time. We first develop a general method for solving such constrained optimization problems and then return to economics problems later in the section.

The Basic Idea

Lagrange Multipliers with Two Independent Variables

Lagrange Multipliers with Three Independent Variables

Quick Quiz

SECTION 12.9 EXERCISES

Review Questions

1. Explain why, at a point that maximizes or minimizes f subject to a constraint $g(x, y) = 0$, the gradient of f is parallel to the gradient of g. Use a diagram.

2. If $f(x, y) = x^2 + y^2$ and $g(x, y) = 2x + 3y - 4 = 0$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to $g(x, y) = 0$.

3. If $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + 3y - 5z + 4 = 0$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to $g(x, y, z) = 0$.

4. Sketch several level curves of $f(x, y) = x^2 + y^2$ and sketch the constraint line $g(x, y) = 2x + 3y - 4 = 0$. Describe the extrema (if any) that f attains on the constraint line.

Basic Skills

5-10. Lagrange multipliers in two variables *Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint.*

5. $f(x, y) = x + 2y$ subject to $x^2 + y^2 = 4$

6. $f(x, y) = xy^2$ subject to $x^2 + y^2 = 1$

7. $f(x, y) = e^{2xy}$ subject to $x^3 + y^3 = 16$

8. $f(x, y) = x^2 + y^2$ subject to $x^6 + y^6 = 1$

9. $f(x, y) = y^2 - 4x^2$ subject to $x^2 + 2y^2 = 4$

10. $f(x, y) = xy + x + y$ subject to $xy = 4$

11-16. Lagrange multipliers in three or more variables *Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint.*

11. $f(x, y, z) = x + 3y - z$ subject to $x^2 + y^2 + z^2 = 4$

12. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

13. $f(x, y, z) = xy^2z^3$ subject to $x^2 + y^2 + 2z^2 = 25$

14. $f(x, y, z) = x^2 + y^2 + z^2$ subject to $z = 1 + 2xy$

15. $f(x, y, z) = x^2 + y^2 + z^2$ subject to $xyz = 4$

16. $f(x, y, z) = (xyz)^{1/2}$ subject to $x + y + z = 1$ with $x \geq 0, y \geq 0, z \geq 0$

17-26. Applications of Lagrange multipliers *Use Lagrange multipliers in the following problems. When the domain of the objective function is unbounded or open, explain why you have found an absolute maximum or minimum value.*

17. Shipping regulations A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (The girth is the perimeter of the smallest base of the box.)

18. Box with minimum surface area Find the rectangular box with a volume of 16 ft^3 that has minimum surface area.

T **19. Extreme distances to an ellipse** Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

20. Maximum area rectangle in an ellipse Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$.

21. Maximum perimeter rectangle in an ellipse Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$.

22. Minimum distance to a plane Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$.

T **23. Minimum distance to a surface** Find the point on the surface $x^2 - 2xy + 2y^2 - x + y = 0$ closest to the point $(1, 2, -3)$.

24. Minimum distance to a cone Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(1, 2, 0)$.

25. Extreme distances to a sphere Find the minimum and maximum distances between the sphere $x^2 + y^2 + z^2 = 9$ and the point $(2, 3, 4)$.

26. Maximum volume cylinder in a sphere Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 16.

27-30. Maximizing utility functions *Find the values of l and g with $l \geq 0$ and $g \geq 0$ that maximize the following utility functions subject to the given constraints. Give the value of the utility function at the optimal point.*

27. $U = f(l, g) = 10\, l^{1/2} g^{1/2}$ subject to $3l + 6g = 18$

28. $U = f(l, g) = 32\, l^{2/3} g^{1/3}$ subject to $4l + 2g = 12$

29. $U = f(l, g) = 8\, l^{4/5} g^{1/5}$ subject to $10l + 8g = 40$

30. $U = f(l, g) = l^{1/6} g^{5/6}$ subject to $4l + 5g = 20$

Further Explorations

31. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Suppose you are standing at the center of a sphere looking at a point P on the surface of the sphere. Your line of sight to P is orthogonal to the plane tangent to the sphere at P.

b. At a point that maximizes f on the curve $g(x, y) = 0$, the dot product $\nabla f \cdot \nabla g$ is zero.

32-37. *Solve the following problems from Section 12.8 using Lagrange multipliers.*

32. Exercise 29

33. Exercise 30

34. Exercise 31

35. Exercise 32

36. Exercise 56

37. Exercise 57

T **38-41. Absolute maximum and minimum values** *Find the absolute maximum and minimum values of the following functions over the given regions R. Use Lagrange multipliers to check for extreme points on the boundary.*

38. $f(x, y) = x^2 + 4y^2 + 1$; $R = \{(x, y) : x^2 + 4y^2 \le 1\}$

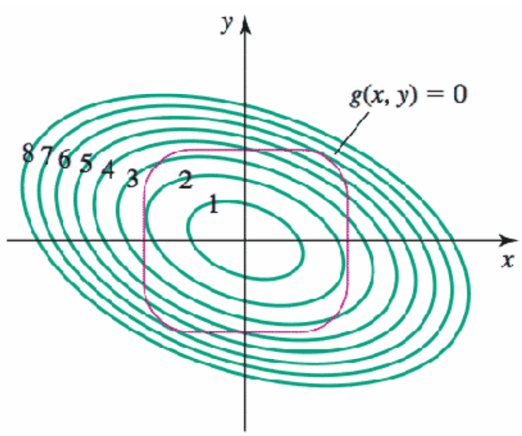
39. $f(x, y) = x^2 - 4y^2 + xy$; $R = \{(x, y) : 4x^2 + 9y^2 \le 36\}$

40. $f(x, y) = 2x^2 + y^2 + 2x - 3y$; $R = \{(x, y) : x^2 + y^2 \le 1\}$

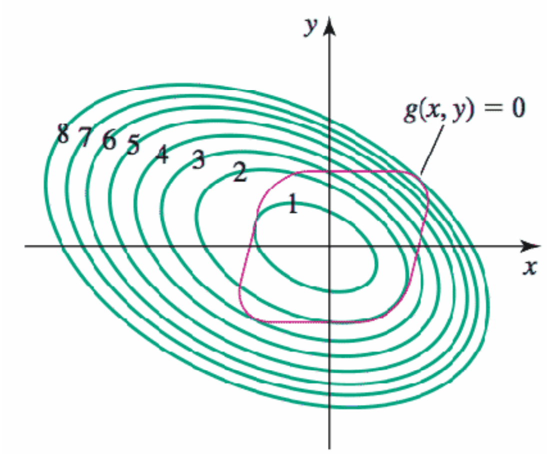
41. $f(x, y) = (x - 1)^2 + (y + 1)^2$; $R = \{(x, y) : x^2 + y^2 \le 4\}$

42-43. Graphical Lagrange multipliers *The following figures show the level curves of f and the constraint curve $g(x, y) = 0$. Estimate the maximum and minimum values of f subject to the constraint. At each point where an extreme value occurs, indicate the direction of ∇f and a possible direction of ∇g.*

42.



43.



44. Extreme points on flattened spheres The equation $x^{2n} + y^{2n} + z^{2n} = 1$, where n is a positive integer, describes a flattened sphere. Define the extreme points to be the points on the flattened sphere with a maximum distance from the origin.

a. Find all the extreme points on the flattened sphere with $n = 2$. What is the distance between the extreme points and the origin?

b. Find all the extreme points on the flattened sphere for integers $n > 2$. What is the distance between the extreme points and the origin?

c. Give the location of the extreme points in the limit as $n \to \infty$. What is the limiting distance between the extreme points and the origin as $n \to \infty$?

Applications

45-47. Production functions *Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form* $P = f(K, L) = C K^a L^{1-a}$, *where K represents capital, L represents labor, and C and a are positive real numbers with* $0 < a < 1$. *If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form* $pK + qL = B$. *Find the values of K and L that maximize the following production functions subject to the given constraint, assuming* $K \geq 0$ *and* $L \geq 0$.

45. $P = f(K, L) = K^{1/2} L^{1/2}$ for $20K + 30L = 300$

46. $P = f(K, L) = 10 K^{1/3} L^{2/3}$ for $30K + 60L = 360$

47. Given the production function $P = f(K, L) = K^a L^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1-a)B/q$.

48. Least squares approximation Find the coefficients in the equation of the plane $z = ax + by + c$ that minimize the sum of the squares of the vertical distances between the plane and the points $(1, 2, 3)$, $(-2, 3, 1)$, $(3, 0, -4)$, and $(0, -2, 6)$.

Additional Exercises

49-51. Maximizing a sum

49. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

50. Generalize Exercise 49 and find the maximum value of $x_1 + x_2 + \cdots + x_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ for a real number c and a positive integer n.

51. Generalize Exercise 49 and find the maximum value of $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for given positive real numbers $a_1, \ldots, a_n$ and a positive integer n.

52. Geometric and arithmetic means Prove that the geometric mean of a set of positive numbers $(x_1 x_2 \cdots x_n)^{1/n}$ is no greater than the arithmetic mean $(x_1 + \cdots + x_n)/n$ in the following cases.

a. Find the maximum value of xyz, subject to $x + y + z = k$, where k is a real number and $x > 0$, $y > 0$, and $z > 0$. Use the result to prove that

$$(xyz)^{1/3} \le \frac{x + y + z}{3}.$$

b. Generalize part (a) and show that

$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + \cdots + x_n}{n}.$$

53. Problems with two constraints Given a differentiable function $w = f(x, y, z)$, the goal is to find its maximum and minimum values subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, where g and h are also differentiable.

a. Imagine a level surface of the function f and the constraint surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Note that g and h intersect (in general) in a curve C on which maximum and minimum values of f must be found. Explain why ∇g and ∇h are orthogonal to their respective surfaces.

b. Explain why ∇f lies in the plane formed by ∇g and ∇h at a point of C where f has a maximum or minimum value.

c. Explain why part (b) implies that $\nabla f = \lambda \nabla g + \mu \nabla h$ at a point of C where f has a maximum or minimum value, where λ and μ (the Lagrange multipliers) are real numbers.

d. Conclude from part (c) that the equations that must be solved for maximum or minimum values of f subject to two constraints are $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$.

54-57. Two-constraint problems *Use the result of Exercise 53 to solve the following problems.*

54. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

55. Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the conditions that $x^2 + y^2 = 4$ and $x + y + z = 1$.

56. The paraboloid $z = x^2 + 2y^2 + 1$ and the plane $x - y + 2z = 4$ intersect in a curve C. Find the points on C that have minimum and maximum distance from the origin.

57. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the intersection between the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$.