Chapter Preview

We have now generalized limits and derivatives to functions of several variables. The next step is to carry out a similar process with respect to integration. As you know, single (one-variable) integrals are developed from Riemann sums and are used to compute areas of regions in $\mathbb{R}^2$. In an analogous way, we use Riemann sums to develop double (two-variable) and triple (three-variable) integrals, which are used to compute volumes of solid regions in $\mathbb{R}^3$. These multiple integrals have many applications in statistics, science, and engineering, including calculating the mass, the center of mass, and moments of inertia of solids with a variable density. Another significant development in this chapter is the appearance of cylindrical and spherical coordinates. These alternative coordinate systems often simplify the evaluation of integrals in three-dimensional space. The chapter closes with the two- and three-dimensional versions of the substitution (change of variables) rule. The overall lesson of the chapter is that we can integrate functions over most geometrical objects, from intervals on the x-axis to regions in the plane bounded by curves to complicated three-dimensional solids.

13.1 Double Integrals over Rectangular Regions

In Chapter 12 the concept of differentiation was extended to functions of several variables. In this chapter we extend integration to multivariable functions. By the close of the chapter, we will have completed Table 13.1, which is a basic road map for calculus.

Table 13.1

	Derivatives	**Integrals**
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ and $f(x, y, z)$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA, \iiint_D f(x, y, z)\,dV$

Volumes of Solids

Iterated Integrals

Average Value

Quick Quiz

SECTION 13.1 EXERCISES

Review Questions

1. Write an iterated integral that gives the volume of the solid bounded by the surface $f(x, y) = xy$ over the square $R = \{(x, y) : 0 \le x \le 2, 1 \le y \le 3\}$.

2. Write an iterated integral that gives the volume of a box with height 10 and base $\{(x, y) : 0 \le x \le 5, -2 \le y \le 4\}$.

3. Write two iterated integrals that equal $\iint_R f(x, y)\,dA$, where $R = \{(x, y) : -2 \le x \le 4, 1 \le y \le 5\}$.

4. Consider the integral $\int_1^3 \int_{-1}^1 \left(2\,y^2 + x\,y\right) d\,y\,d\,x$. Give the variable of integration in the first (inner) integral and the limits of integration. Give the variable of integration in the second (outer) integral and the limits of integration.

Basic Skills

5-12. Iterated integrals *Evaluate the following iterated integrals.*

5. $\int_1^3 \int_0^2 x^2\,y\,d\,x\,d\,y$

6. $\int_0^3 \int_{-2}^1 (2\,x + 3\,y)\,d\,x\,d\,y$

7. $\int_1^3 \int_0^{\pi/2} x \sin y\,d\,y\,d\,x$

8. $\int_1^3 \int_1^2 \left(y^2 + y\right) d\,x\,d\,y$

9. $\int_1^4 \int_0^4 \sqrt{u\,v}\;d\,u\,d\,v$

10. $\int_0^{\pi/2} \int_0^1 x \cos x y\,d\,y\,d\,x$

11. $\int_1^{\ln 5} \int_0^{\ln 3} e^{x+y}\,d\,x\,d\,y$

12. $\int_0^{\pi/4} \int_0^3 r \sec\theta\,d\,r\,d\theta$

13-19. Iterated integrals *Evaluate the following double integrals over the region R.*

13. $\iint_R (x + 2\,y)\,d\,A;\;\; R = \{(x,\ y) : 0 \le x \le 3,\ 1 \le y \le 4\}$

14. $\iint_R \left(x^2 + x\,y\right) d\,A;\;\; R = \{(x,\ y) : 1 \le x \le 2,\ -1 \le y \le 1\}$

15. $\iint_R \sqrt{\frac{x}{y}}\;d\,A;\;\; R = \{(x,\ y) : 0 \le x \le 1,\ 1 \le y \le 4\}$

16. $\iint_R x\,y \sin x^2\,d\,A;\;\; R = \left\{(x,\ y) : 0 \le x \le \sqrt{\pi/2}\,,\ 0 \le y \le 1\right\}$

17. $\iint_R e^{x+2\,y}\,d\,A;\;\; R = \{(x,\ y) : 0 \le x \le \ln 2,\ 1 \le y \le \ln 3\}$

18. $\iint_R \left(x^4 + y^4\right)^2 d\,A;\;\; R = \{(x,\ y) : -1 \le x \le 1,\ 0 \le y \le 1\}$

19. $\displaystyle\iint_R \left(x^5 - y^5\right)^2 dA$; $R = \{(x, y) : 0 \le x \le 1, -1 \le y \le 1\}$

20-23. Choose a convenient order *When converted to an iterated integral, the following double integrals are easier to evaluate in one order than the other. Find the best order and evaluate the integral.*

20. $\displaystyle\iint_R x \sec^2 xy \, dA$; $R = \{(x, y) : 0 \le x \le \pi/3, 0 \le y \le 1\}$

21. $\displaystyle\iint_R x^5 e^{x^3 y} \, dA$; $R = \{(x, y) : 0 \le x \le \ln 2, 0 \le y \le 1\}$

22. $\displaystyle\iint_R y^3 \sin xy^2 \, dA$; $R = \left\{(x, y) : 0 \le x \le 1, 0 \le y \le \sqrt{\pi/2}\right\}$

23. $\displaystyle\iint_R \frac{x}{(1 + xy)^2} \, dA$; $R = \{(x, y) : 0 \le x \le 4, 1 \le y \le 2\}$

24-26. Average value *Compute the average value of the following functions over the region R.*

24. $f(x, y) = 4 - x - y$; $R = \{(x, y) : 0 \le x \le 2, 0 \le y \le 2\}$

25. $f(x, y) = e^{-y}$; $R = \{(x, y) : 0 \le x \le 6, 0 \le y \le \ln 2\}$

26. $f(x, y) = \sin x \sin y$; $R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le \pi\}$

27-28. Average value

27. Find the average squared distance between the points of $R = \{(x, y) : -2 \le x \le 2, 0 \le y \le 2\}$ and the origin.

28. Find the average squared distance between the points of $R = \{(x, y) : 0 \le x \le 3, 0 \le y \le 3\}$ and the point $(3, 3)$.

Further Explorations

29. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The region of integration for $\displaystyle\int_4^6 \int_1^3 4 \, dx \, dy$ is a square.

b. If f is continuous on $\mathbb{R}^2$, then $\displaystyle\int_4^6 \int_1^3 f(x, y) \, dx \, dy = \int_4^6 \int_1^3 f(x, y) \, dy \, dx$.

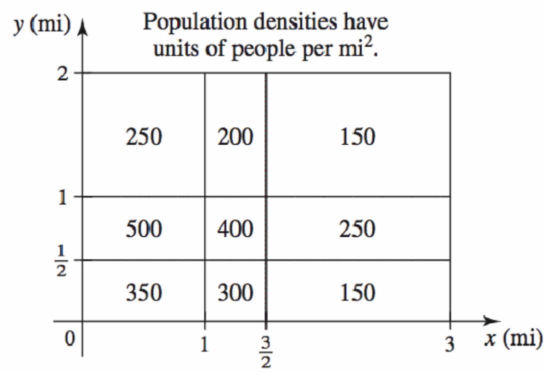
c. If f is continuous on $\mathbb{R}^2$, then $\displaystyle\int_4^6 \int_1^3 f(x, y) \, dx \, dy = \int_1^3 \int_4^6 f(x, y) \, dy \, dx$.

30. **Symmetry** Evaluate the following integrals using symmetry arguments. Let $R = \{(x, y) : -a \le x \le a, -b \le y \le b\}$, where a and b are positive real numbers.

a. $\displaystyle\iint_R x y e^{-\left(x^2+y^2\right)} \, dA$

b. $\displaystyle\iint_R \frac{\sin(x - y)}{x^2 + y^2 + 1} \, dA$

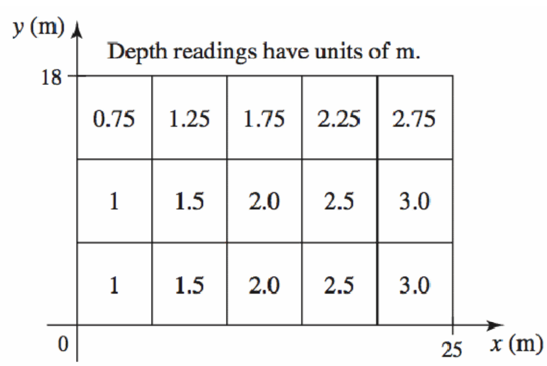
31. Computing populations The population densities in nine districts of a rectangular county are shown in the figure.



a. Use the fact that population $=$ (population density) $\times$ (area) to estimate the population of the county.

b. Explain how the calculation of part (a) is related to Riemann sums and double integrals.

T **32. Approximating water volume** The varying depth of an $18\text{ m} \times 25\text{ m}$ swimming pool is measured in 15 different rectangles of equal area (see figure). Approximate the volume of water in the pool.



33-34. Pictures of solids *Draw the solid region whose volume is given by the following double integrals. Then find the volume of the solid.*

33. $\displaystyle\int_0^6 \int_1^2 10\, dy\, dx$

34. $\displaystyle\int_0^1 \int_{-1}^1 \left(4 - x^2 - y^2\right) dx\, dy$

35-38. More integration practice *Evaluate the following iterated integrals.*

35. $\displaystyle\int_1^e \int_0^1 \frac{x}{x+y}\, dy\, dx$

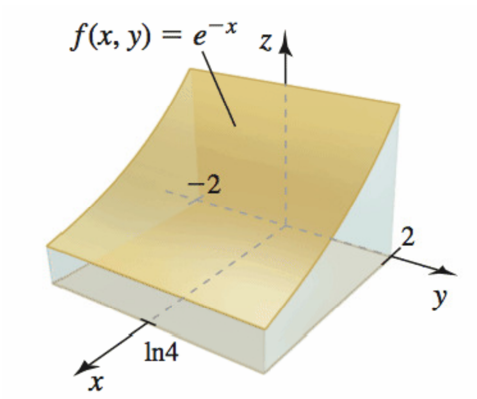
36. $\displaystyle\int_0^2 \int_0^1 x^5 y^2 e^{x^3 y^3}\, dy\, dx$

37. $\displaystyle\int_0^1 \int_1^4 \frac{3y}{\sqrt{x+y^2}}\, dx\, dy$

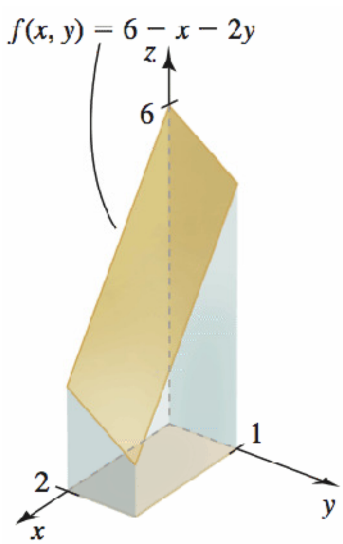
38. $\int_1^4 \int_0^2 e^{y\sqrt{x}}\, dy\, dx$

39-42. Volumes of solids *Find the volume of the following solids.*

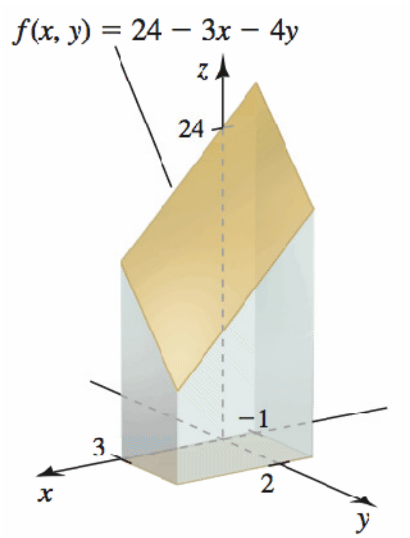
39. The solid between the cylinder $f(x, y) = e^{-x}$ and the region $R = \{(x, y) : 0 \le x \le \ln 4, -2 \le y \le 2\}$



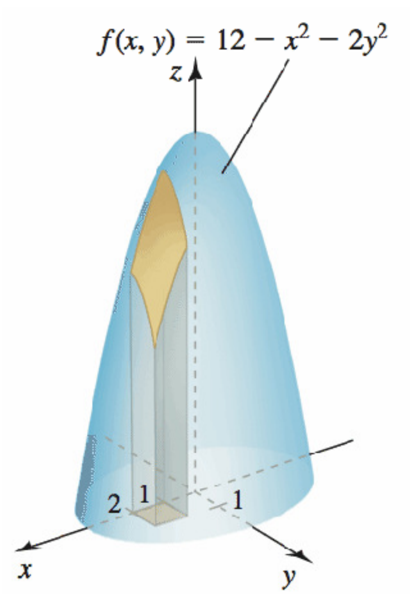
40. The solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y) : 0 \le x \le 2, 0 \le y \le 1\}$



41. The solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y) : -1 \le x \le 3, 0 \le y \le 2\}$



42. The solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y) : 1 \le x \le 2, 0 \le y \le 1\}$



43. **Net volume** Let $R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le a\}$. For what values of a, with $0 \le a \le \pi$, is $\iint_R \sin(x + y)\, dA$ equal to 1?

44-45. Zero average value *Let* $R = \{(x, y) : 0 \le x \le a, 0 \le y \le a\}$. *Find the value of* $a > 0$ *such that the average value of the following functions over* R *is zero.*

44. $f(x, y) = x + y - 8$

45. $f(x, y) = 4 - x^2 - y^2$

46. **Maximum integral** Consider the plane $x + 3y + z = 6$ over the rectangle R with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), where the vertex (a, b) lies on the line where the plane intersects the xy-plane (so $a + 3b = 6$). Find the point (a, b) for which the volume of the solid between the plane and R is a maximum.

Applications

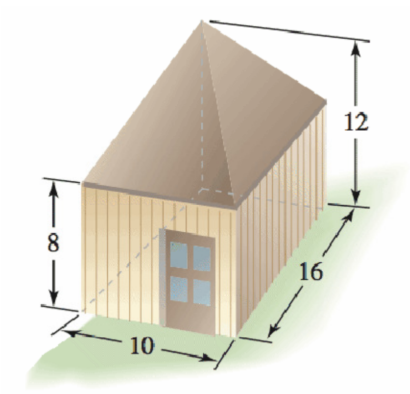
47. **Density and mass** Suppose a thin rectangular plate, represented by a region R in the xy-plane, has a density given by the function $\rho(x, y)$; this function gives the *area density* in units such as g/cm^2. The mass of the plate is $\iint_R \rho(x, y)\, dA$. Assume that $R = \{(x, y) : 0 \le x \le \pi/2,\ 0 \le y \le \pi\}$ and find the mass of the plates with the following density functions.

a. $\rho(x, y) = 1 + \sin x$

b. $\rho(x, y) = 1 + \sin y$

c. $\rho(x, y) = 1 + \sin x \sin y$

48. **Approximating volume** Propose a method based on Riemann sums to approximate the volume of the shed shown in the figure (the peak of the roof is directly above the rear corner of the shed). Carry out the method and provide an estimate of the volume.



Additional Exercises

49. **Cylinders** Let S be the solid in $\mathbb{R}^3$ between the cylinder $z = f(x)$ and the region $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$, where $f(x) \ge 0$ on R. Explain why $\int_c^d \int_a^b f(x)\, dx\, dy$ equals the area of the constant cross section of S multiplied by $(d - c)$, which is the volume of S.

50. **Product of integrals** Suppose $f(x, y) = g(x)\, h(y)$, where g and h are continuous functions for all real values.

a. Show that $\int_c^d \int_a^b f(x, y)\, dx\, dy = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$. Interpret this result geometrically.

b. Write $\left(\int_a^b g(x)\, dx\right)^2$ as an iterated integral.

c. Use the result of part (a) to evaluate $\int_0^{2\pi} \int_{10}^{30} (\cos x)\, e^{-4y^2}\, dy\, dx$.

51. **An identity** Suppose the second partial derivatives of f are continuous on $R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le b\}$. Simplify $\iint_R \frac{\partial^2 f}{\partial x\, \partial y}\, dA$.

52. **Two integrals** Let $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$.

a. Evaluate $\int\int_R \cos\left(x\sqrt{y}\right) dA$

b. Evaluate $\int\int_R x^3\, y \cos\left(x^2\, y^2\right) dA$

53. A generalization Let R be as in Exercise 52, let F be an antiderivative of f with $F(0) = 0$ and let G be an antiderivative of F. Show that if f and F are integrable, and $r \geq 1$ and $s \geq 1$ are real numbers, then

$$\int\int_R x^{2r-1}\, y^{s-1}\, f\left(x^r\, y^s\right) dA = \frac{G(1) - G(0)}{r\, s}.$$