13.2 Double Integrals over General Regions

Evaluating double integrals over rectangular regions is a useful place to begin our study of multiple integrals. Problems of practical interest, however, usually involve nonrectangular regions of integration. The goal of this section is to extend the methods presented in Section 13.1 so that they apply to more general regions of integration.

General Regions of Integration

Iterated Integrals

Choosing and Changing the Order of Integration

Regions Between Two Surfaces

Decomposition of Regions

Finding Area by Double Integrals

Quick Quiz

SECTION 13.2 EXERCISES

Review Questions

1. Describe and sketch a region that is bounded above and below by two curves.

2. Describe and a sketch a region that is bounded on the left and on the right by two curves.

3. Which order of integration is preferable to integrate $f(x, y) = x\,y$ over $R = \{(x, y) : y - 1 \le x \le 1 - y,\ 0 \le y \le 1\}$?

4. Which order of integration would you use to find the area of the region bounded by the x-axis and the lines $y = 2\,x + 3$ and $y = 3\,x - 4$ using a double integral?

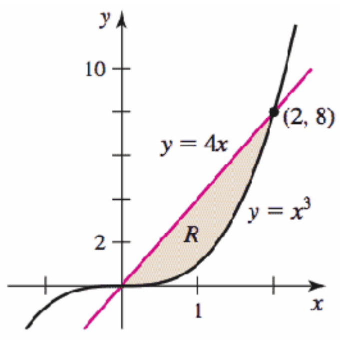
5. Change the order of integration in the integral $\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\,dx\,dy$.

6. Sketch the region of integration for $\int_{-2}^{2} \int_{x^2}^{4} e^{x\,y}\,dy\,dx$.

Basic Skills

7-8. Regions of integration *Consider the regions R shown in the figures and write an iterated integral of a continuous function f over R.*

7.



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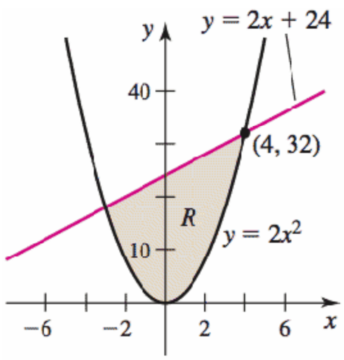


9-12. Regions of integration *Sketch the following regions and write an iterated integral of a continuous function f over the region.*

9. $R = \{(x,\ y): 0 \le x \le \pi/4,\ \sin x \le y \le \cos x\}$

10. $R = \left\{(x,\ y): 0 \le x \le 2,\ 3\,x^2 \le y \le -6\,x + 24\right\}$

11. $R = \{(x,\ y): 1 \le x \le 2,\ x + 1 \le y \le 2\,x + 4\}$

12. $R = \left\{(x,\ y): 0 \le x \le 4,\ x^2 \le y \le 8\,\sqrt{x}\,\right\}$

13-18. Evaluating integrals *Evaluate the following integrals as they are written.*

13. $\displaystyle\int_0^2\int_{x^2}^{2\,x} x\,y\,d\,y\,d\,x$

14. $\displaystyle\int_0^3\int_{2\,x^2}^{2\,x+12} (x + y)\,d\,y\,d\,x$

15. $\displaystyle\int_{-\pi/4}^{\pi/4}\int_{\sin x}^{\cos x} d\,y\,d\,x$

16. $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\,x^2\,y\,d\,y\,d\,x$

17. $\displaystyle\int_{-2}^2\int_{x^2}^{8-x^2} x\,d\,y\,d\,x$

18. $\int_0^{\ln 2} \int_{e^x}^{2} d\,y\,d\,x$

19-22. Evaluating integrals *Evaluate the following integrals.*

19. $\iint_R x\,y\,d\,A$; R is bounded by $x = 0$, $y = 2\,x + 1$, and $y = -2\,x + 5$

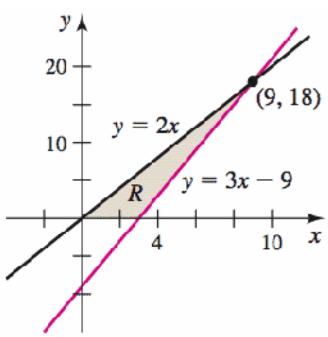
20. $\iint_R (x + y)\,d\,A$; R is the region in the first quadrant bounded by $x = 0$, $y = x^2$, and $y = 8 - x^2$.

21. $\iint_R y^2\,d\,A$; R is bounded by $x = 1$, $y = 2\,x + 2$, and $y = -x - 1$.

22. $\iint_R x^2\,y\,d\,A$; R is the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at (0, 0).

23-24. Regions of integration *Write an iterated integral of a continuous function f over the region R shown in the figure.*

23.



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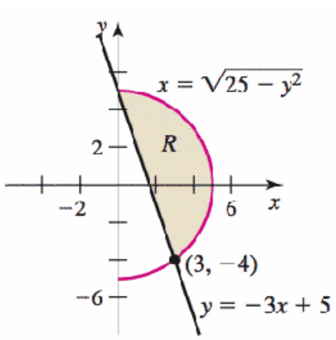


25-28. Regions of integration *Write an iterated integral of a continuous function f over the following regions.*

25. The region bounded by $y = 2\,x + 3$, $y = 3\,x - 7$, and $y = 0$

26. $R = \{(x,\ y) : 0 \le x \le y\,(1 - y)\}$

27. The region bounded by $y = 4 - x$, $y = 1$, and $x = 0$

28. The region in quadrants 2 and 3 bounded by the semicircle with radius 3 centered at (0, 0)

29-34. Evaluating integrals *Sketch the region of integration and evaluate the following integrals as they are written.*

29. $\int_{-1}^{2}\int_{y}^{4-y} dx\,dy$

30. $\int_{0}^{2}\int_{0}^{4-y^2} (x+y)\,dx\,dy$

31. $\int_{0}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2xy\,dx\,dy$

32. $\int_{0}^{1}\int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} 2x\,dx\,dy$

33. $\int_{0}^{\ln 2}\int_{e^y}^{2} \frac{y}{x}\,dx\,dy$

34. $\int_{0}^{4}\int_{y}^{2y} xy\,dx\,dy$

35-38. Evaluating integrals *Sketch the regions of integration and evaluate the following integrals.*

35. $\iint_R xy\,dA$; R is bounded by $x=0$, $y=0$, and $y=9-x^2$.

36. $\iint_R (x+y)\,dA$; R is bounded by $y=|x|$ and $y=4$.

37. $\iint_R y^2\,dA$; R is bounded by $y=0$, $y=2x+4$, and $y=x^3$.

38. $\iint_R x^2 y\,dA$; R is bounded by $y=0$, $y=\sqrt{x}$, and $y=x-2$.

39-42. Volumes *Use double integrals to calculate the volume of the following regions.*

39. The tetrahedron bounded by the coordinate planes ($x=0$, $y=0$, $z=0$) and the plane $z=8-2x-4y$

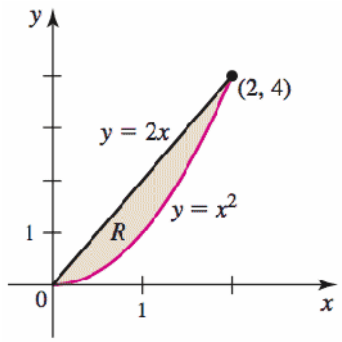
40. The solid in the first octant bounded by the coordinate planes and the surface $z=8-x^2-2y^2$

41. The segment of the cylinder $x^2+y^2=1$ bounded above by the plane $z=12+x+y$ and below by $z=0$

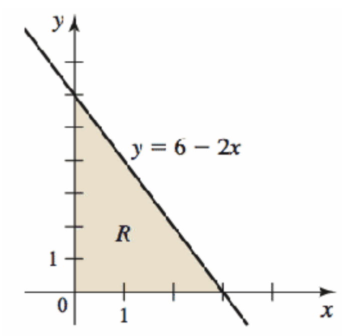
42. The solid beneath the cylinder $z=y^2$ and above the region $R=\{(x,y): 0\le y\le 1,\ y\le x\le 1\}$

43-48. Changing order of integration *Reverse the order of integration in the following integrals.*

43. $\int_{0}^{2}\int_{x^2}^{2x} f(x,y)\,dy\,dx$



44. $\int_0^3 \int_0^{6-2x} f(x,\ y)\, dy\, dx$



45. $\int_{1/2}^1 \int_0^{-\ln y} f(x,\ y)\, dx\, dy$

46. $\int_0^1 \int_1^{e^y} f(x,\ y)\, dx\, dy$

47. $\int_0^1 \int_0^{\cos^{-1} y} f(x,\ y)\, dx\, dy$

48. $\int_1^e \int_0^{\ln x} f(x,\ y)\, dy\, dx$

49-54. Changing order of integration *The following integrals can be evaluated only by reversing the order of integration. Sketch the region of integration, reverse the order of integration, and evaluate the integral.*

49. $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

50. $\int_0^\pi \int_x^\pi \sin y^2\, dy\, dx$

51. $\int_0^{1/2} \int_{y^2}^{1/4} y \cos\left(16\pi x^2\right) dx\, dy$

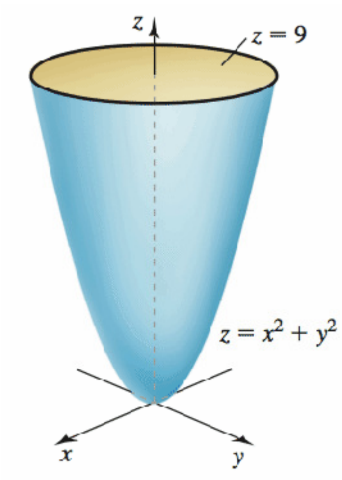
52. $\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 1}\, dy\, dx$

53. $\displaystyle\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos\left(x^2 y\right) dx\, dy$

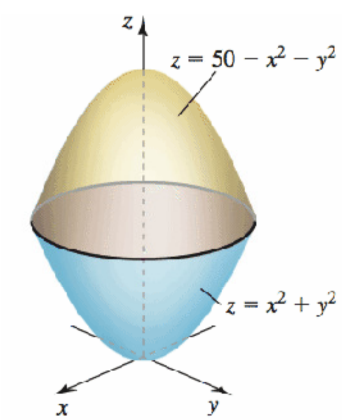
54. $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y}\, dy\, dx$

55-58. Regions between two surfaces *Find the volume of the following solid regions.*

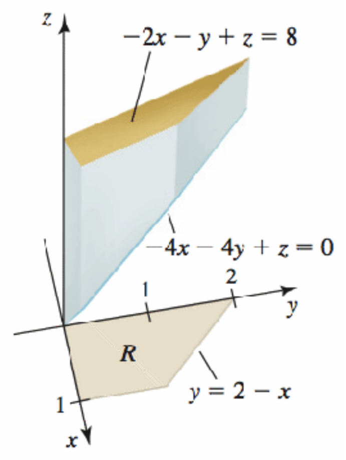
55. The solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$



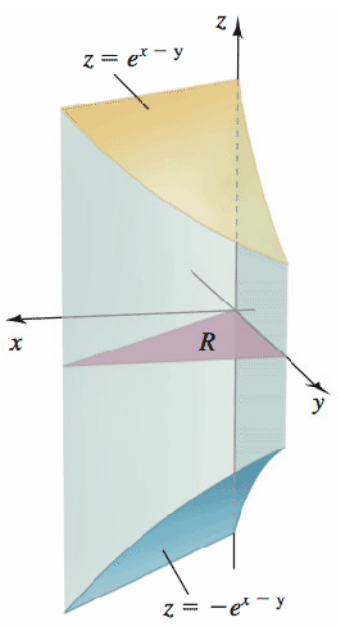
56. The solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 50 - x^2 - y^2$



57. The solid above the region $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2 - x\}$ and between the planes $-4x - 4y + z = 0$ and $-2x - y + z = 8$



58. The solid S between the surfaces $z = e^{x-y}$ and $z = -e^{x-y}$, where S intersects the xy-plane in the region $R = \{(x, y) : 0 \le x \le y, 0 \le y \le 1\}$



59-64. Area of plane regions *Use a double integral to compute the area of the following regions. Make a sketch of the region.*

59. The region bounded by the parabola $y = x^2$ and the line $y = 4$

60. The region bounded by the parabola $y = x^2$ and the line $y = x + 2$

61. The region in the first quadrant bounded by $y = e^x$ and $x = \ln 2$

62. The region bounded by $y = 1 + \sin x$ and $y = 1 - \sin x$ on the interval $[0, \pi]$

63. The region in the first quadrant bounded by $y = x^2$, $y = 5x + 6$, and $y = 6 - x$

64. The region bounded by the lines $x = 0$, $x = 4$, $y = x$, and $y = 2x + 1$

Further Explorations

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. In the iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$, the limits a and b must be constants or functions of x.

b. In the iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$, the limits c and d must be constants or functions of y.

c. Changing the order of integration gives $\int_0^2 \int_1^y f(x, y)\, dx\, dy = \int_1^y \int_0^2 f(x, y)\, dy\, dx$.

66-69. Miscellaneous integrals *Evaluate the following integrals.*

66. $\iint_R y\, dA;\ R = \{(x, y) : 0 \le y \le \sec x,\ 0 \le x \le \pi/3\}$

67. $\iint_R (x + y)\, dA;$ R is the region bounded by $y = 1/x$ and $y = 5/2 - x$.

68. $\iint_R \frac{xy}{1 + x^2 + y^2}\, dA;\ R = \{(x, y) : 0 \le y \le x,\ 0 \le x \le 2\}$

69. $\iint_R x \sec^2 y\, dA;\ R = \left\{(x, y) : 0 \le y \le x^2,\ 0 \le x \le \sqrt{\pi}/2\right\}$

70. Paraboloid sliced by plane Find the volume of the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 1 - 2y$.

71. Two integrals to one Draw the regions of integration and write the following integrals as a single iterated integral:

$$\int_0^1 \int_{e^y}^e f(x, y)\, dx\, dy + \int_{-1}^0 \int_{e^{-y}}^e f(x, y)\, dx\, dy$$

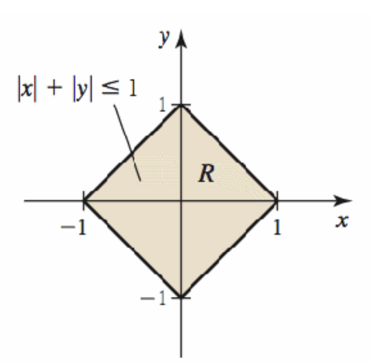
72. Diamond region Consider the region $R = \{(x, y) : |x| + |y| \le 1\}$ shown in the figure.

a. Use a double integral to show that the area of R is 2.

b. Find the volume of the square column whose base is R and whose upper surface is $z = 12 - 3x - 4y$.

c. Find the volume of the solid above R and beneath the cylinder $x^2 + z^2 = 1$.

d. Find the volume of the pyramid whose base is R and whose vertex is on the z-axis at (0, 0, 6).



73-74. Average value *Use the definition for the average value of a function over a region R (Section 13.1),*

$$\overline{f} = \frac{1}{\text{area of } R} \iint_R f(x,\ y)\, dA.$$

73. Find the average value of $a - x - y$ over the region $R = \{(x,\ y) : x + y \le a,\ x \ge 0,\ y \ge 0\}$, where $a > 0$.

74. Find the average value of $z = a^2 - x^2 - y^2$ over the region $R = \left\{(x,\ y) : x^2 + y^2 \le a^2\right\}$, where $a > 0$.

75-76. Area integrals *Consider the following regions R.*

- ***a.*** *Sketch the region R.*
- ***b.*** *Evaluate* $\iint_R dA$ *to determine the area of the region.*
- ***c.*** *Evaluate* $\iint_R x\, y\, dA$.

75. R is the region between both branches of $y = 1/x$ and the lines $y = x + 3/2$ and $y = x - 3/2$.

76. R is the region bounded by the ellipse $x^2 \big/ 18 + y^2 \big/ 36 = 1$ with $y \le 4\,x/3$.

77-80. Improper integrals *Many improper double integrals may be handled using the techniques for improper integrals in one variable (Section 7.7). For example, under suitable conditions on f,*

$$\int_a^\infty \int_{g(x)}^{h(x)} f(x,\ y)\, dy\, dx = \lim_{b \to \infty} \int_a^b \int_{g(x)}^{h(x)} f(x,\ y)\, dy\, dx.$$

Use or extend the one-variable methods for improper integrals to evaluate the following integrals.

77. $\displaystyle\int_1^\infty \int_0^{e^{-x}} x\, y\, dy\, dx$

78. $\displaystyle\int_1^\infty \int_0^{1/x^2} \frac{2\,y}{x}\, dy\, dx$

79. $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}\, dy\, dx$

80. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\left(x^2 + 1\right)\left(y^2 + 1\right)}\, dy\, dx$

81-85. Volumes *Compute the volume of the following solids.*

81. Sliced block The solid bounded by the planes $x = 0$, $x = 5$, $z = y - 1$, $z = -2\,y - 1$, $z = 0$, $z = 2$.

82. Tetrahedron A tetrahedron with vertices $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(b,\ c,\ 0)$, and $(0,\ 0,\ d)$, where a, b, c, and d are positive real numbers.

83. Square column The column with a square base $R = \{(x,\ y) : |x| \le 1,\ |y| \le 1\}$ cut by the plane $z = 4 - x - y$.

84. Wedge The wedge sliced from the cylinder $x^2 + y^2 = 1$ by the planes $z = 1 - x$ and $z = x - 1$.

85. Wedge The wedge sliced from the cylinder $x^2 + y^2 = 1$ by the planes $z = a\,(2 - x)$ and $z = a\,(x - 2)$, where $a > 0$.

Additional Exercises

86. Existence of improper double integral For what values of m and n does the integral $\int_1^{\infty}\int_0^{1/x} \frac{y^m}{x^n}\,dy\,dx$ have a finite value?

87. Existence of improper double integral Let $R_1 = \{(x, y) : x \geq 1, 1 \leq y \leq 2\}$ and $R_2 = \{(x, y) : 1 \leq x \leq 2, y \geq 1\}$. For $n > 1$, which integral(s) have finite values: $\iint_{R_1} x^{-n}\,dA$ or $\iint_{R_2} x^{-n}\,dA$?