13.3 Double Integrals in Polar Coordinates

In Chapter 10, we explored polar coordinates and saw that in certain situations they simplify problems considerably. The same is true when it comes to integration over plane regions. In this section, we learn how to formulate double integrals in polar coordinates and how to change double integrals from Cartesian coordinates to polar coordinates.

Note

Polar Rectangular Regions

More General Polar Regions

Areas of Regions

Average Value over a Planar Polar Region

Quick Quiz

SECTION 13.3 EXERCISES

Review Questions

1. Draw the region $\{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi/2\}$. Why is it called a polar rectangle?

2. Write the double integral $\iint_R f(x, y)\, dA$ as an iterated integral in polar coordinates when $R = \{(r, \theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$.

3. Sketch the region of integration for the integral $\int_{-\pi/6}^{\pi/6} \int_{1/2}^{\cos 2\theta} f(r, \theta)\, r\, dr\, d\theta$.

4. Explain why the element of area in Cartesian coordinates $dx\, dy$ becomes $r\, dr\, d\theta$ in polar coordinates.

5. How do you find the area of a region $R = \{(r, \theta) : g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta\}$?

6. How do you find the average value of a function over a region that is expressed in polar coordinates?

Basic Skills

7-10. Polar rectangles *Sketch the following polar rectangles.*

7. $R = \{(r, \theta) : 0 \le r \le 5,\ 0 \le \theta \le \pi/2\}$

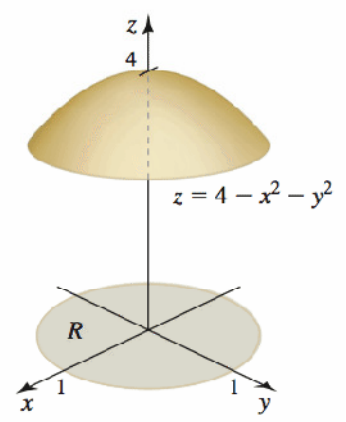
8. $R = \{(r, \theta) : 2 \le r \le 3,\ \pi/4 \le \theta \le 5\pi/4\}$

9. $R = \{(r, \theta) : 1 \le r \le 4,\ -\pi/4 \le \theta \le 2\pi/3\}$

10. $R = \{(r, \theta) : 4 \le r \le 5,\ -\pi/3 \le \theta \le \pi/2\}$

11-14. Solids bounded by paraboloids *Find the volume of the solid below the paraboloid* $z = 4 - x^2 - y^2$ *and above the following regions.*

11. $R = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$



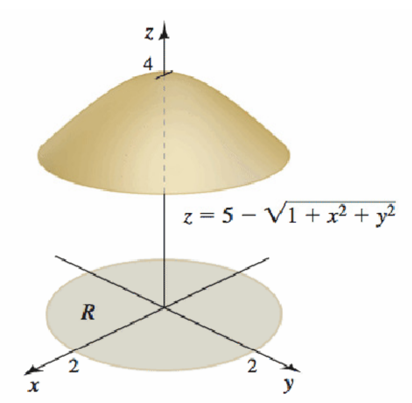
12. $R = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

13. $R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

14. $R = \{(r, \theta) : 1 \le r \le 2,\ -\pi/2 \le \theta \le \pi/2\}$

15-18. Solids bounded by hyperboloids *Find the volume of the solid below the hyperboloid* $z = 5 - \sqrt{1 + x^2 + y^2}$ *and above the following regions.*

15. $R = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$



16. $R = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi\}$

17. $R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

18. $R = \{(r, \theta) : 1 \le r \le 3,\ -\pi/2 \le \theta \le \pi/2\}$

19-24. Cartesian to polar coordinates *Sketch the given region of integration R and evaluate the integral over R using polar coordinates.*

19. $\displaystyle\iint_R \left(x^2 + y^2\right) dA;\ R = \{(r, \theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

20. $\displaystyle\iint_R 2\,x\,y\,dA;\ R = \{(r,\,\theta) : 1 \le r \le 3,\ 0 \le \theta \le \pi/2\}$

21. $\displaystyle\iint_R 2\,x\,y\,dA;\ R = \left\{(x,\ y) : x^2 + y^2 \le 9,\ y \le 0\right\}$

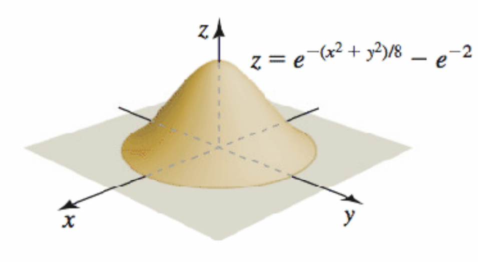
22. $\displaystyle\iint_R \frac{1}{1 + x^2 + y^2}\,dA;\ R = \{(r,\,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

23. $\displaystyle\iint_R \frac{1}{\sqrt{16 - x^2 - y^2}}\,dA;\ R = \left\{(x,\ y) : x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0\right\}$

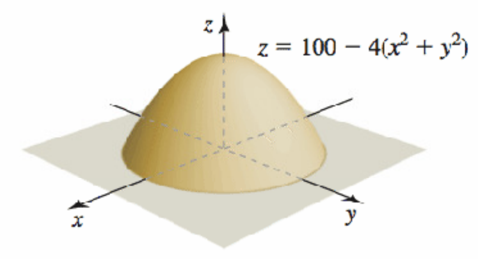
24. $\displaystyle\iint_R e^{-x^2-y^2}\,dA;\ R = \left\{(x,\ y) : x^2 + y^2 \le 9\right\}$

25-28. Island problems *The surface of an island is defined by the following functions over the region on which the function is nonnegative. Find the volume of the island.*

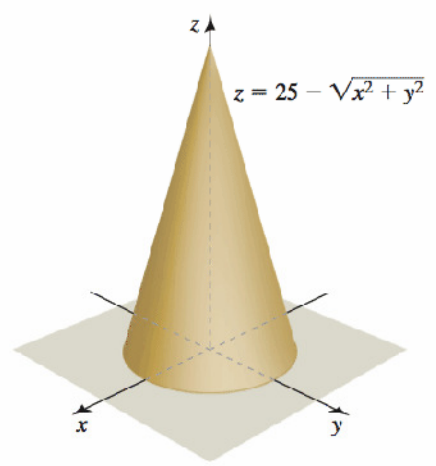
25. $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$



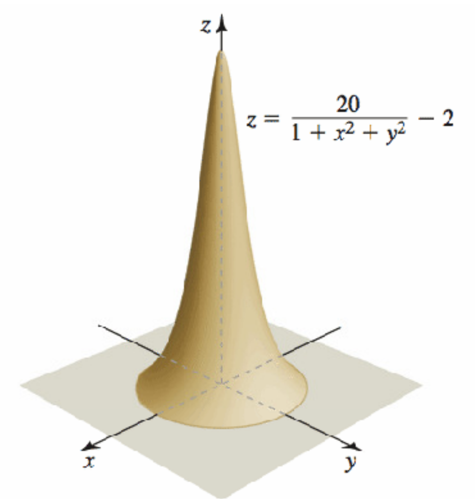
26. $z = 100 - 4\left(x^2 + y^2\right)$



27. $z = 25 - \sqrt{x^2 + y^2}$



28. $z = \dfrac{20}{1 + x^2 + y^2} - 2$



29-34. Describing general regions *Sketch the following regions R. Then express* $\displaystyle\iint_R f(r, \theta)\, dA$ *as an iterated integral over R.*

29. The region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$.

30. The region inside the leaf of the rose $r = 2\sin 2\theta$ in the first quadrant.

31. The region inside the lobe of the lemniscate $r^2 = 2\sin 2\theta$ in the first quadrant.

32. The region outside the circle $r = 2$ and inside the circle $r = 4\sin\theta$.

33. The region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in the first quadrant.

34. The region outside the circle $r = \frac{1}{2}$ and inside the cardioid $r = 1 + \cos\theta$.

35-40. Computing areas *Sketch each region and use integration to find its area.*

35. The annular region $\{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

36. The region bounded by the cardioid $r = 2(1 - \sin\theta)$

37. The region bounded by all leaves of the rose $r = 2\cos 3\theta$

38. The region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

39. The region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

40. The region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis.

41-44. Average values *Find the following average values.*

41. The average distance between points of the disk $\{(r, \theta) : 0 \le r \le a\}$ and the origin

42. The average distance between points within the cardioid $r = 1 + \cos\theta$ and the origin

43. The average distance squared between points on the unit disk $\{(r, \theta) : 0 \le r \le 1\}$ and the point $(1, 1)$

44. The average value of $1/r^2$ over the annulus $\{(r, \theta) : 2 \le r \le 4\}$

Further Explorations

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Let R be the unit circle centered at $(0, 0)$. Then, $\displaystyle\iint_R (x^2 + y^2)\, dA = \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta$.

b. The average distance between the points of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the origin is 2 (no integral needed).

c. The integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ is easier to evaluate in polar coordinates than in Cartesian coordinates.

46-51. Miscellaneous integrals *Sketch the region of integration and evaluate the following integrals, using the method of your choice.*

46. $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

47. $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx$

48. $\displaystyle\int_{-4}^4 \int_0^{\sqrt{16-y^2}} (16 - x^2 - y^2)\, dx\, dy$

49. $\displaystyle\int_0^{\pi/4} \int_0^{\sec\theta} r^3\, dr\, d\theta$

50. $\displaystyle\iint_R \frac{x-y}{x^2+y^2+1}\,dA$; R is the region bounded by the unit circle centered at the origin.

51. $\displaystyle\iint_R \frac{1}{4+\sqrt{x^2+y^2}}\,dA$; $R = \{(r,\theta): 0 \le r \le 2,\ \pi/2 \le \theta \le 3\pi/2\}$

52. Areas of circles Use integration to show that the circles $r = 2a\cos\theta$ and $r = 2a\sin\theta$ have the same area, which is πa^2.

53. Filling bowls with water Which bowl holds more water if it is filled to a depth of four units?

- The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$
- The cone $z = \sqrt{x^2+y^2}$, for $0 \le z \le 4$
- The hyperboloid $z = \sqrt{1+x^2+y^2}$, for $1 \le z \le 5$

54. Equal volumes To what height (above the bottom of the bowl) must the cone and paraboloid bowls of Exercise 53 be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units ($1 \le z \le 5$)?

55. Volume of a hyperbolic paraboloid Consider the surface $z = x^2 - y^2$.

a. Find the region in the xy-plane in polar coordinates for which $z \ge 0$.

b. Let $R = \{(r,\theta): 0 \le r \le a,\ -\pi/4 \le \theta \le \pi/4\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region R.

56. Slicing a hemispherical cake A cake is shaped like a hemisphere of radius 4 with its base on the xy-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the xy-plane and separated by an angle of ϕ.

a. Use a double integral to find the volume of the slice for $\phi = \pi/4$. Use geometry to check your answer.

b. Now suppose the cake is sliced by a plane perpendicular to the xy-plane at $x = a > 0$. Let D be the smaller of the two pieces produced. For what value of a is the volume of D equal to the volume in part (a)?

57-60. Improper integrals *Improper integrals arise in polar coordinates when the radial coordinate r becomes arbitrarily large. Under certain conditions, these integrals are treated in the usual way:*

$$\int_\alpha^\beta \int_a^\infty g(r,\theta)\,r\,dr\,d\theta = \lim_{b\to\infty}\int_\alpha^\beta \int_a^b g(r,\theta)\,r\,dr\,d\theta\,.$$

Use this technique to evaluate the following integrals.

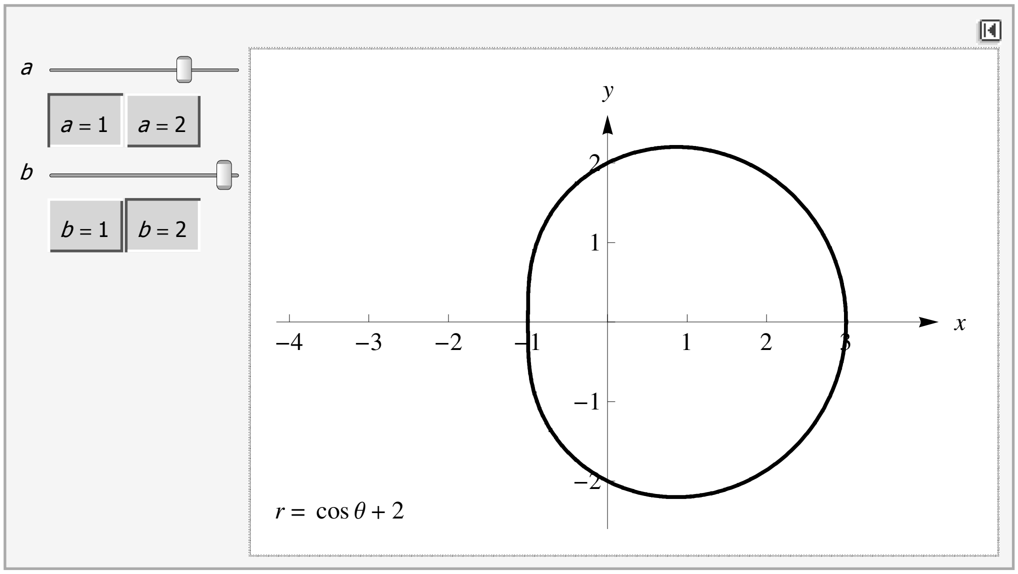
57. $\displaystyle\int_0^{\pi/2}\int_1^\infty \frac{\cos\theta}{r^3}\,r\,dr\,d\theta$

58. $\displaystyle\iint_R \frac{dA}{(x^2+y^2)^{5/2}}$; $R = \{(r,\theta): 1 \le r < \infty,\ 0 \le \theta \le 2\pi\}$

59. $\displaystyle\iint_R e^{-x^2-y^2}\,dA$; $R = \{(r,\theta): 0 \le r < \infty,\ 0 \le \theta \le \pi/2\}$

60. $\displaystyle\iint_R \frac{1}{(1+x^2+y^2)^2}\,dA$; R is the first quadrant.

61. Limaçon loops The limaçon $r = b + a\cos\theta$ has an inner loop if $b < a$ and no inner loop if $b > a$.



a. Find the area of the region bounded by the limaçon $r = 2 + \cos\theta$.

b. Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos\theta$.

c. Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos\theta$.

Applications

T **62. Mass from density data** The following table gives the density (in units of $\mathrm{g/cm^2}$) at selected points of a thin semicircular plate of radius 3. Estimate the mass of the plate and explain your method.

	$\theta = 0$	$\theta = \pi/4$	$\theta = \pi/2$	$\theta = 3\pi/4$	$\theta = \pi$
$r = 1$	2.0	2.1	2.2	2.3	2.4
$r = 2$	2.5	2.7	2.9	3.1	3.3
$r = 3$	3.2	3.4	3.5	3.6	3.7

63. A mass calculation Suppose the density of a thin plate represented by the region R is $\rho(r, \theta)$ (in units of mass per area). The mass of the plate is $\iint_R \rho(r, \theta)\, dA$. Find the mass of the thin half annulus $R = \{(r, \theta) : 1 \le r \le 4,\ 0 \le \theta \le \pi\}$ with a density $\rho(r, \theta) = 4 + r\sin\theta$.

Additional Exercises

64. Area formula In Section 10.3 it was shown that the area of a region enclosed by the polar curve $r = g(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$, where $\beta - \alpha \le 2\pi$, is $A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$. Prove this result using the area formula with double integrals.

65. Normal distribution An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$. It is evaluated in the following steps.

a. Assume that $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$, where we have chosen the variables of integration to be x and y and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$.

b. Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} x\,e^{-x^2}\,dx$, and $\int_0^{\infty} x^2\,e^{-x^2}\,dx$ (using part (a) if needed).

66. Existence of integrals For what values of p does the integral $\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA$ exist in the following cases?

a. $R = \{(r,\theta) : 1 \le r < \infty,\ 0 \le \theta \le 2\pi\}$

b. $R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

67. Integrals in strips Consider the integral $I = \iint_R \frac{1}{\left(1+x^2+y^2\right)^2}\,dA$, where $R = \{(x,\ y) : 0 \le x \le 1,\ 0 \le y \le a\}$.

a. Evaluate I for $a = 1$. (*Hint*: Use polar coordinates.)

b. Evaluate I for arbitrary $a > 0$.

c. Let $a \to \infty$ in part (b) to find I over the infinite strip $R = \{(x,\ y) : 0 \le x \le 1,\ 0 \le y < \infty\}$.

T **68. Area of an ellipse** In polar coordinates an equation of an ellipse with eccentricity $0 < e < 1$ and semimajor axis a is $r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$.

a. Write the integral that gives the area of the ellipse.

b. Show that the area of an ellipse is $\pi\,a\,b$, where $b^2 = a^2\left(1-e^2\right)$.