13.4 Triple Integrals

At this point, you may be able to see the pattern that is developing with respect to integration. In Chapter 5, we introduced integrals of single-variable functions. In the first three sections of this chapter, we moved up one dimension to double integrals of two-variable functions. In this section we take one more step and investigate triple integrals of three-variable functions. There is no end to the progression of multiple integrals. It is possible to define integrals with respect to any number of variables. For example, problems in statistics and statistical mechanics involve integration over regions of many dimensions.

Triple Integrals in Rectangular Coordinates

Changing the Order of Integration

Average Value of a Function of Three Variables

Quick Quiz

SECTION 13.4 EXERCISES

Review Questions

1. Sketch the region $D = \{(x, y, z) : x^2 + y^2 \le 4, 0 \le z \le 4\}$.

2. Write an iterated integral for $\iiint_D f(x, y, z)\, dV$, where D is the box $\{(x, y, z) : 0 \le x \le 3, 0 \le y \le 6, 0 \le z \le 4\}$.

3. Write an iterated integral for $\iiint_D f(x, y, z)\, dV$, where D is a sphere of radius 9 centered at (0, 0, 0). Use the order $dz\,dy\,dx$.

4. Sketch the region of integration for the integral $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x, y, z)\, dx\, dy\, dz$.

5. Write the integral for the Exercise 4 in the order $dy\,dx\,dz$.

6. Write an integral for the average value of $f(x, y, z) = xyz$ over the region bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane (assuming the volume of the region is known).

Basic Skills

7-14. Integrals over boxes *Evaluate the following integrals. A sketch of the region of integration may be useful.*

7. $\int_{-2}^{2} \int_3^6 \int_0^2 dx\, dy\, dz$

8. $\int_{-1}^{1} \int_{-1}^{2} \int_0^1 6xyz\, dy\, dx\, dz$

9. $\int_{-2}^{2} \int_1^2 \int_1^e \frac{xy^2}{z}\, dz\, dx\, dy$

10. $\int_1^{\ln 8}\int_0^{\ln 4}\int_0^{\ln 2} e^{-x-y-2z}\,dx\,dy\,dz$

11. $\int_0^{\pi/2}\int_0^{1}\int_0^{\pi/2} \sin \pi x \cos y \sin 2z\,dy\,dx\,dz$

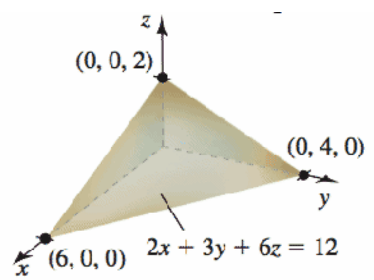
12. $\int_0^{2}\int_1^{2}\int_0^{1} y z e^{x}\,dx\,dz\,dy$

13. $\iiint_D (xy + xz + yz)\,dV;\ D = \{(x, y, z) : -1 \le x \le 1,\ -2 \le y \le 2,\ -3 \le z \le 3\}$

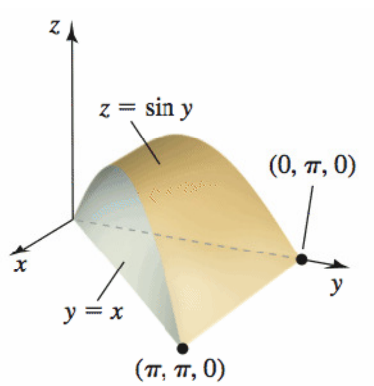
14. $\iiint_D x y z e^{-x^2-y^2}\,dV;\ D = \left\{(x, y, z) : 0 \le x \le \sqrt{\ln 2},\ 0 \le y \le \sqrt{\ln 4},\ 0 \le z \le 1\right\}$

15-24. Volumes of solids *Find the volume of the following solids using triple integrals.*

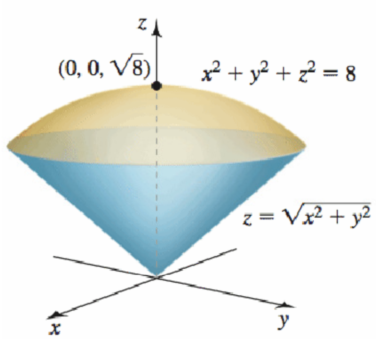
15. The region in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



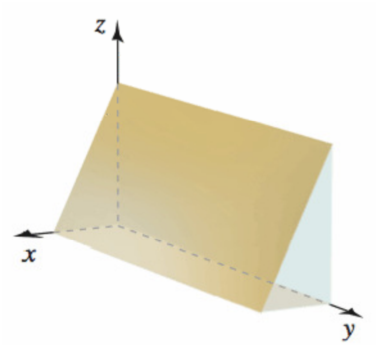
16. The region in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



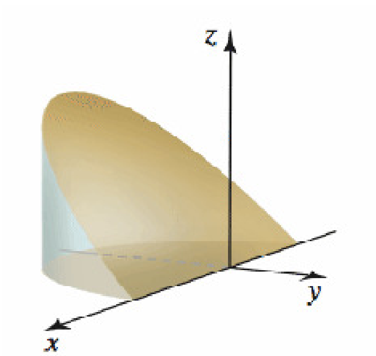
17. The region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$



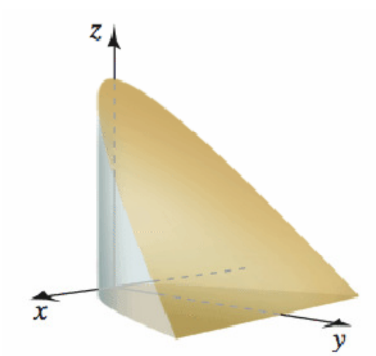
18. The prism in the first octant bounded by $z = 2 - 4x$ and $y = 8$



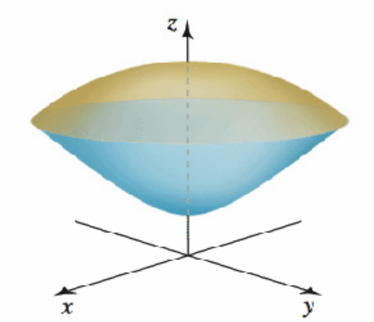
19. The wedge above the xy-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



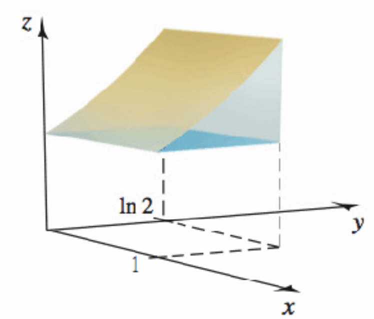
20. The region bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



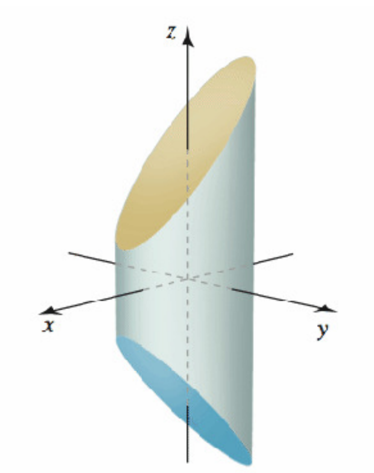
21. The region between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



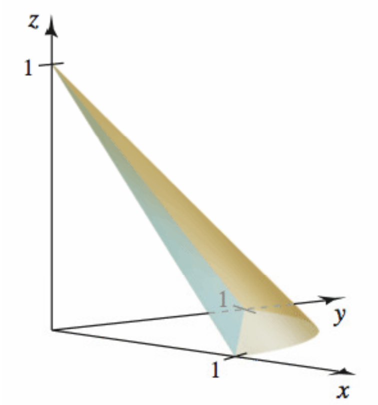
22. The region bounded by the surfaces $z = e^{y}$ and $z = 1$ over the rectangle $\{(x, y) : 0 \le x \le 1, 0 \le y \le \ln 2\}$



23. The wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



24. The region in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



25-34. Triple integrals *Evaluate the following integrals.*

25. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

26. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\,dz\,dy\,dx$

27. $\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-(x^2/4)-y^2} dz\,dx\,dy$

28. $\int_1^6 \int_0^{4-2y/3} \int_0^{12-2y-3z} \frac{1}{y}\,dx\,dz\,dy$

29. $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy\,dx\,dz$

30. $\int_0^{\pi} \int_0^{\pi} \int_0^{\sin x} \sin y\,dz\,dx\,dy$

31. $\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln(2y)} e^{x+y^2-z}\,dx\,dy\,dz$

32. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\,dz\,dy\,dx$

33. $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\,dz\,dx\,dy$

34. $\int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy\,dz\,dx\,dy$

35-38. Changing the order of integration *Rewrite the following integrals using the indicated order of integration and then evaluate the resulting integral.*

35. $\int_0^5 \int_{-1}^0 \int_0^{4x+4} dy\,dx\,dz$ in the order $dz\,dx\,dy$

36. $\int_0^1 \int_{-2}^{2} \int_0^{\sqrt{4-y^2}} dz\, dy\, dx$ in the order $dy\, dz\, dx$

37. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy\, dz\, dx$ in the order $dz\, dy\, dx$

38. $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy\, dz\, dx$ in the order $dx\, dy\, dz$

39-44. Average value *Find the following average values.*

39. The average temperature in the box $D = \{x, y, z) : 0 \le x \le \ln 2, 0 \le y \le \ln 4, 0 \le z \le \ln 8\}$ with a temperature distribution of $T(x, y, z) = 100\, e^{-x-y-z}$

40. The average value of $f(x, y, z) = 6\, x\, y\, z$ over the points inside the hemisphere of radius 4 centered at the origin with its base in the xy-plane

41. The average of the *squared* distance between the origin and points in the solid cylinder $D = \left\{(x, y, z) : x^2 + y^2 \le 4, 0 \le z \le 2\right\}$

42. The average of the *squared* distance between the origin and points in the solid paraboloid $D = \left\{(x, y, z) : 0 \le z \le 4 - x^2 - y^2\right\}$

43. The average z-coordinate of points in a hemisphere of radius 4 centered at the origin with its base in the xy-plane

44. The average of the *squared* distance between the z-axis and points in the conical region $D = \left\{(x, y, z) : 2\sqrt{x^2 + y^2} \le z \le 8\right\}$

Further Explorations

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. An iterated integral of a function over the box $D = \{(x, y, z) : 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$ can be expressed in eight different ways.

b. One possible iterated integral of f over the prism $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 3x - 3, 0 \le z \le 5\}$ is $\int_0^{3x-3} \int_0^1 \int_0^5 f(x, y, z)\, dz\, dx\, dy.$

c. The region $D = \left\{(x, y, z) : 0 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}, 0 \le z \le \sqrt{1 - x^2}\right\}$ is a sphere.

46. Changing the order of integration Use another order of integration to evaluate $\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\, dy\, dx\, dz.$

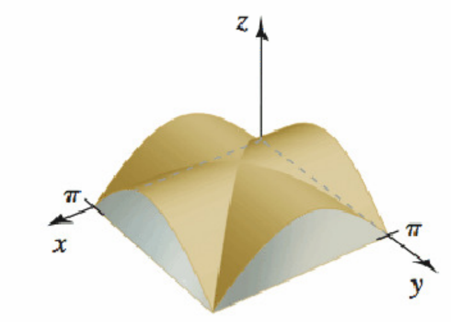
47-51. Miscellaneous volumes *Use a triple integral to compute the volume of the following regions.*

47. The parallelepiped (slanted box) with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1), (1, 1, 1), (0, 2, 1), (1, 2, 1) (Use integration and find the best order of integration.)

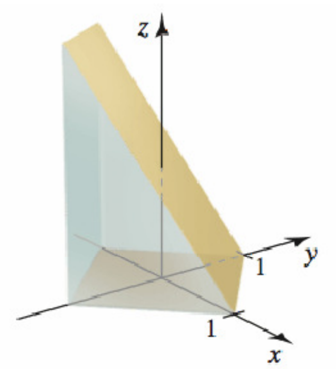
48. The larger of two solids formed when the parallelepiped (slanted box) with vertices (0, 0, 0), (2, 0, 0), (0, 2, 0), (2, 2, 0), (0, 1, 1), (2, 1, 1), (0, 3, 1), (2, 3, 1) is sliced by the plane $y = 2$.

49. The pyramid with vertices (0, 0, 0), (2, 0, 0), (2, 2, 0), (0, 2, 0), (0, 0, 4)

50. The region common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le \pi\}$ (The figure shows the cylinders, but not the common region.)



51. The wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



52. Partitioning a cube Consider the region $D_1 = \{(x, y, z) : 0 \le x \le y \le z \le 1\}$.

a. Find the volume of D_1.

b. Let $D_{2,\ldots,} D_6$ be the "cousins" of D_1 formed by rearranging x, y, and z in the inequality $0 \le x \le y \le z \le 1$. Show that the volumes of $D_1, \ldots, D_6$ are equal.

c. Show that the union of $D_1, \ldots, D_6$ is a unit cube.

Applications

53. Comparing two masses Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the z-direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid.

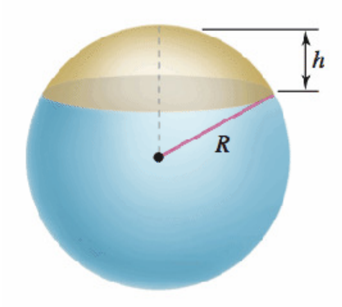
54. Dividing the cheese Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$, and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find a with $0 < a < 4$ such that slicing the wedge with the plane $y = a$ divides the wedge into two equal pieces.

55-59. General volume formulas *Find equations for the bounding surfaces, set up a volume integral, and evaluate the integral to obtain a volume formula for each region. Assume that a, b, c, r, R, and h are positive constants.*

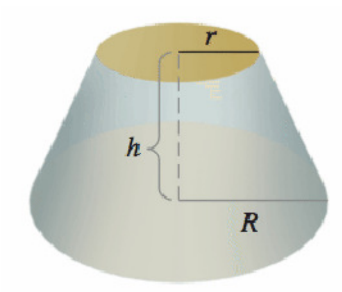
55. Cone Find the volume of a right circular cone with height h and base radius r.

56. Tetrahedron Find the volume of a tetrahedron whose vertices are located at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

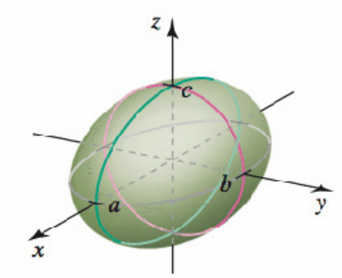
57. Spherical cap Find the volume of the cap of a sphere of radius R with height h.



58. Frustum of a cone Find the volume of a truncated cone of height h whose ends have radii r and R.



59. Ellipsoid Find the volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$.



60. Exponential distribution The occurrence of random events (such as phone calls or e-mail messages) is often idealized using an exponential distribution. If λ is the average rate of occurrence of such an event, assumed to be constant over time, then the average time between occurrences is λ^{-1} (for example, if phone calls arrive at a rate of $\lambda = 2/\text{min}$, then the mean time between phone calls is $\lambda^{-1} = \frac{1}{2}$ min). The exponential distribution is given by $f(t) = \lambda e^{-\lambda t}$, for $0 \le t < \infty$.

a. Suppose you work at a customer service desk and phone calls arrive at an average rate of $\lambda_1 = 0.8/\text{min}$ (meaning the average time between phone calls is $1/0.8 = 1.25$ min). The probability that a phone call arrives during the interval $[0, T]$ is $p(T) = \int_0^T \lambda_1 e^{-\lambda_1 t}\, dt$. Find the probability that a phone call arrives during the first 45 s (0.75 min) that you work at the desk.

b. Now suppose that walk-in customers also arrive at your desk at an average rate of $\lambda_2 = 0.1/\text{min}$. The probability that a phone call *and* a customer arrive during the interval $[0, T]$ is $p(T) = \int_0^T \int_0^T \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 s}\, dt\, ds$. Find the probability that a phone call and a customer arrive during the first 45 s that you work at the desk.

c. E-mail messages also arrive at your desk at an average rate of $\lambda_3 = 0.05/\text{min}$. The probability that a phone call *and* a customer *and* an e-mail message arrive during the interval $[0, T]$ is
$p(T) = \int_0^T \int_0^T \int_0^T \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 s} \lambda_3 e^{-\lambda_3 u}\, dt\, ds\, du$. Find the probability that a phone call and a customer and an e-mail message arrive during the first 45 s that you work at the desk.

Additional Exercises

61. Hypervolume Find the volume of the four-dimensional pyramid bounded by $w + x + y + z + 1 = 0$ and the coordinate planes $w = 0,\ x = 0,\ y = 0,\ z = 0$.

62. An identity (Putnam Exam 1941) Let f be a continuous function on $[0, 1]$. Prove that

$$\int_0^1 \int_x^1 \int_x^y f(x)\, f(y)\, f(z)\, dz\, dy\, dx = \frac{1}{6}\left(\int_0^1 f(x)\, dx\right)^3.$$