

13.5 Triple Integrals in Cylindrical and Spherical Coordinates

When evaluating triple integrals, you may have noticed that some regions (such as spheres, cones, and cylinders) have awkward descriptions in Cartesian coordinates. In this section we examine two other coordinate systems in \mathbb{R}^3 that are easier to use when working with certain types of regions. These coordinate systems are helpful not only for integration, but also for general problem solving.

Cylindrical Coordinates

Integration in Cylindrical Coordinates

Spherical Coordinates

Integration in Spherical Coordinates

Quick Quiz

SECTION 13.5 EXERCISES

Review Questions

1. Explain how cylindrical coordinates are used to describe a point in \mathbb{R}^3 .
2. Explain how spherical coordinates are used to describe a point in \mathbb{R}^3 .
3. Describe the set $\{(r, \theta, z) : r = 4z\}$ in cylindrical coordinates.
4. Describe the set $\{(\rho, \phi, \theta) : \phi = \pi/4\}$ in spherical coordinates.
5. Explain why $dz r dr d\theta$ is the volume of a small "box" in cylindrical coordinates.
6. Explain why $\rho^2 \sin \phi d\rho d\phi d\theta$ is the volume of a small "box" in spherical coordinates.
7. Write the integral $\int \int \int_D f(r, \theta, z) dV$ as an iterated integral where $D = \{(r, \theta, z) : G(r, \theta) \leq z \leq H(r, \theta), g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$.
8. Write the integral $\int \int \int_D f(\rho, \phi, \theta) dV$ as an iterated integral, where $D = \{(\rho, \phi, \theta) : g(\phi, \theta) \leq \rho \leq h(\phi, \theta), a \leq \phi \leq b, \alpha \leq \theta \leq \beta\}$.
9. What coordinate system is *suggested* if the integrand of a triple integral involves $x^2 + y^2$?
10. What coordinate system is *suggested* if the integrand of a triple integral involves $x^2 + y^2 + z^2$?

Basic Skills

- 11-14. Sets in cylindrical coordinates** Identify and sketch the following sets in cylindrical coordinates.
11. $\{(r, \theta, z) : 0 \leq r \leq 3, 0 \leq \theta \leq \pi/3, 1 \leq z \leq 4\}$

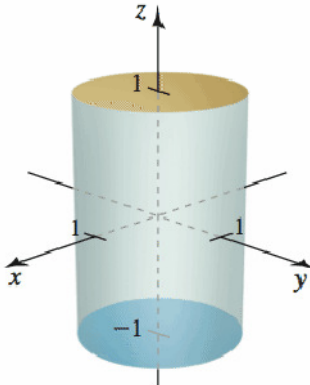
12. $\{(r, \theta, z) : 0 \leq \theta \leq \pi/2, z = 1\}$

13. $\{(r, \theta, z) : 2r \leq z \leq 4\}$

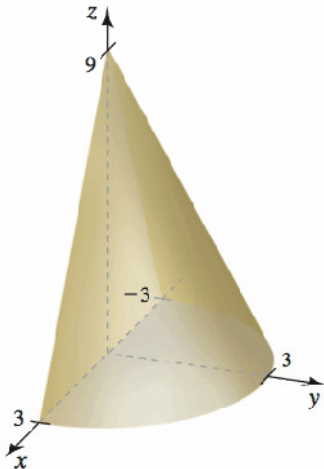
14. $\{(r, \theta, z) : 0 \leq z \leq 8 - 2r\}$

15-18. **Integrals in cylindrical coordinates** Evaluate the following integrals in cylindrical coordinates.

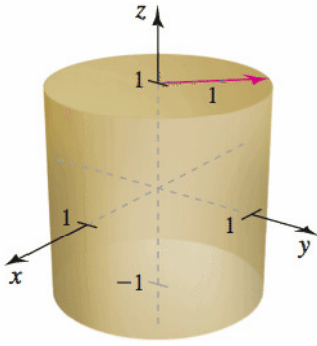
15. $\int_0^{2\pi} \int_0^1 \int_{-1}^1 dz r dr d\theta$



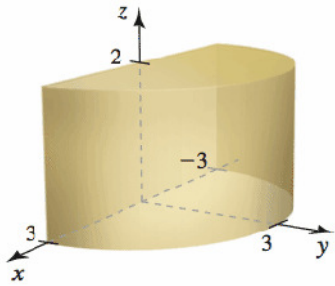
16. $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dz dx dy$



17. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^1 (x^2 + y^2)^{3/2} dz dx dy$



18.
$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2} dz dy dx$$



19-22. Integrals in cylindrical coordinates Evaluate the following integrals in cylindrical coordinates.

19.
$$\int_0^4 \int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} dy dx dz$$

20.
$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 dz dy dx$$

21.
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} (x^2+y^2)^{-1/2} dz dy dx$$

22.
$$\int_{-1}^1 \int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} (x^2+y^2)^{1/2} dx dy dz$$

23-26. Mass from density Find the mass of the following objects with the given density functions.

23. The solid cylinder $D = \{(r, \theta, z) : 0 \leq r \leq 4, 0 \leq z \leq 10\}$ with density $\rho(r, \theta, z) = 1 + z/2$

24. The solid cylinder $D = \{(r, \theta, z) : 0 \leq r \leq 3, 0 \leq z \leq 2\}$ with density $\rho(r, \theta, z) = 5e^{-r^2}$

25. The solid cone $D = \{(r, \theta, z) : 0 \leq z \leq 6 - r, 0 \leq r \leq 6\}$ with density $\rho(r, \theta, z) = 7 - z$

26. The solid paraboloid $D = \{(r, \theta, z) : 0 \leq z \leq 9 - r^2, 0 \leq r \leq 3\}$ with density $\rho(r, \theta, z) = 1 + z/9$

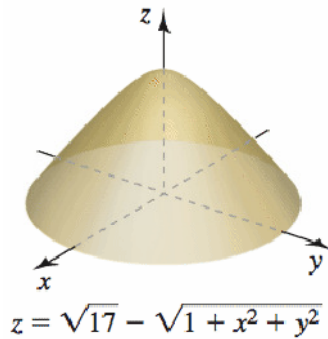
27. Which weighs more? For $0 \leq r \leq 1$, the solid bounded by the cone $z = 4 - 4r$ and the solid bounded by the paraboloid $z = 4 - 4r^2$ have the same base in the xy -plane and the same height. Which object has the greater mass if the density of both objects is $\rho(r, \theta, z) = 10 - 2z$?

28. Which weighs more? Which of the objects in Exercise 27 weighs more if the density of both objects is

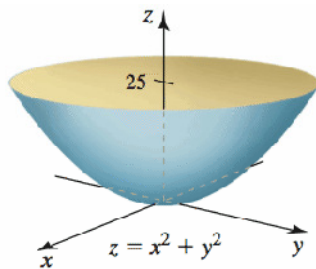
$$\rho(r, \theta, z) = \frac{8}{\pi} e^{-z}?$$

29-34. Volumes in cylindrical coordinates Use cylindrical coordinates to find the volume of the following solid regions.

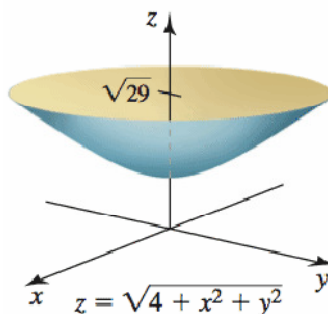
29. The region bounded by the plane $z = 0$ and the hyperboloid $z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$



30. The region bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$



31. The region bounded by the plane $z = \sqrt{29}$ and the hyperboloid $z = \sqrt{4 + x^2 + y^2}$



32. The solid cylinder whose height is 4 and whose base is the disk $\{(r, \theta) : 0 \leq r \leq 2 \cos \theta\}$

33. The region in the first octant bounded by the cylinder $r = 1$ and the plane $z = x$

34. The region bounded by the cylinders $r = 1$ and $r = 2$ and the planes $z = 4 - x - y$ and $z = 0$

35-38. Sets in spherical coordinates Identify and sketch the following sets in spherical coordinates.

35. $\{(\rho, \phi, \theta) : 1 \leq \rho \leq 3\}$

36. $\{(\rho, \phi, \theta) : \rho = 2 \csc \phi, 0 < \phi < \pi\}$

37. $\{(\rho, \phi, \theta) : \rho = 4 \cos \phi, 0 \leq \phi \leq \pi/2\}$

38. $\{(\rho, \phi, \theta) : \rho = 2 \sec \phi, 0 \leq \phi < \pi/2\}$

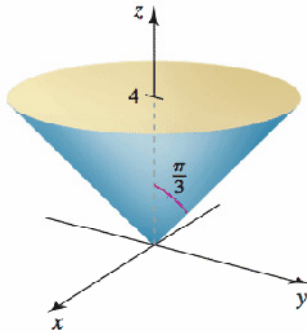
39-45. Integrals in spherical coordinates Evaluate the following integrals in spherical coordinates.

39. $\iiint_D (x^2 + y^2 + z^2)^{5/2} dV$; D is the unit ball.

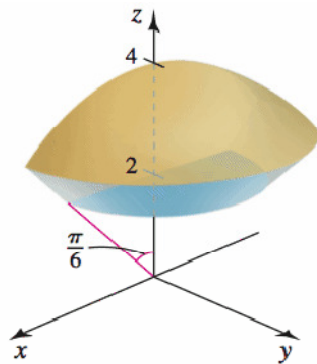
40. $\iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV$; D is the unit ball.

41. $\iiint_D \frac{1}{(x^2 + y^2 + z^2)^{3/2}} dV$; D is the region between the spheres of radius 1 and 2 centered at the origin.

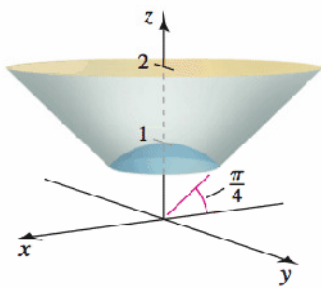
42. $\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4 \sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta$



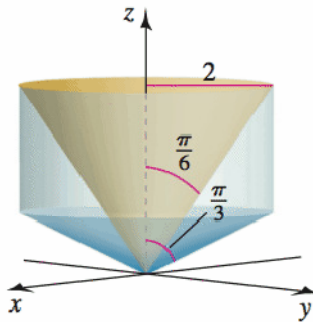
43. $\int_0^{\pi} \int_0^{\pi/6} \int_{2 \sec \phi}^4 \rho^2 \sin \phi d\rho d\phi d\theta$



44. $\int_0^{2\pi} \int_0^{\pi/4} \int_1^{2 \sec \phi} (\rho^{-3}) \rho^2 \sin \phi d\rho d\phi d\theta$

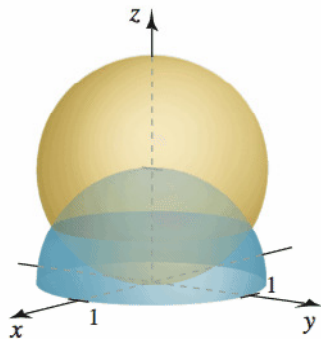


45.
$$\int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2 \csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

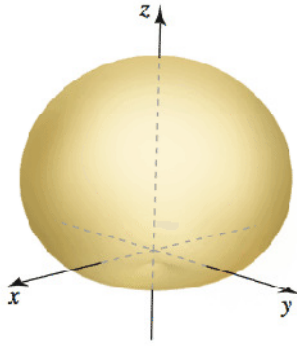


46-52. Volumes in spherical coordinates Use spherical coordinates to find the volume of the following regions.

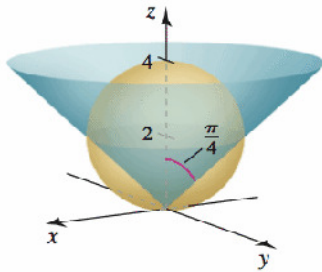
46. A ball of radius $a > 0$.
47. The region bounded by the sphere $\rho = 2 \cos \phi$ and the hemisphere $\rho = 1, z \geq 0$



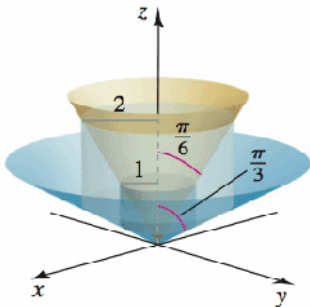
48. The cardioid of revolution $D = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 1 + \cos \phi, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$



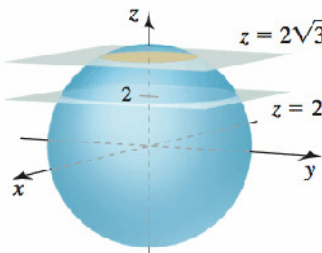
49. The region outside the cone $\phi = \pi/4$ and inside the sphere $\rho = 4 \cos \phi$



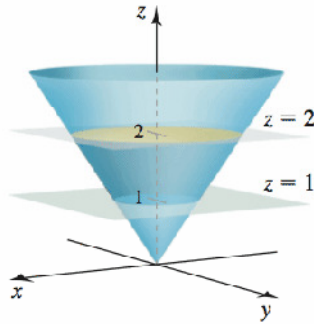
50. The region bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\phi = \pi/6$ and $\phi = \pi/3$



51. That part of the ball $\rho \leq 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



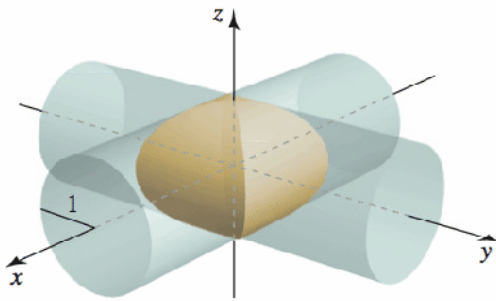
52. The region inside the solid cone $z = (x^2 + y^2)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$



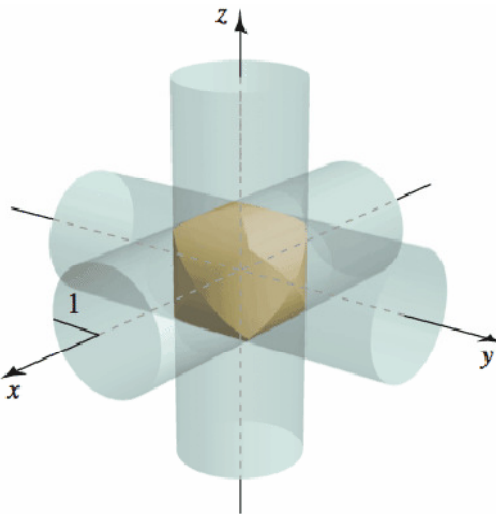
Further Explorations

- 53. Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
- Any point on the z -axis has more than one representation in both cylindrical and spherical coordinates.
 - The sets $\{(r, \theta, z) : r = z\}$ and $\{(\rho, \phi, \theta) : \phi = \pi/4\}$ are the same.
- 54. Spherical to rectangular** Convert the equation $\rho^2 = \sec 2\phi$, where $0 \leq \phi < \pi/4$, to rectangular coordinates and identify the surface.
- 55. Spherical to rectangular** Convert the equation $\rho^2 = -\sec 2\phi$, where $\pi/4 < \phi \leq \pi/2$, to rectangular coordinates and identify the surface.
- 56-59. Mass from density** Find the mass of the following objects with the given density functions.
- 56.** The ball of radius 4 centered at the origin with a density $f(\rho, \phi, \theta) = 1 + \rho$
- 57.** The ball of radius 8 centered at the origin with a density $f(\rho, \phi, \theta) = 2e^{-\rho^3}$
- 58.** The solid cone $\{(\rho, \phi, \theta) : \phi \leq \pi/3, 0 \leq z \leq 4\}$ with a density $f(\rho, \phi, \theta) = 5 - z$
- 59.** The solid cylinder $\{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1\}$ with a density of $\rho(r, z) = (2 - |z|)(4 - r)$
- 60-61. Changing order of integration** If possible, write iterated integrals in cylindrical coordinates for the following regions in the specified orders. Sketch the region of integration.
- 60.** The region outside the cylinder $r = 1$ and inside the sphere $\rho = 5$ for $z \geq 0$ in the orders $dz dr d\theta$, $dr dz d\theta$, and $d\theta dz dr$
- 61.** The region above the cone $z = r$ and below the sphere $\rho = 2$ for $z \geq 0$ in the orders $dz dr d\theta$, $dr dz d\theta$, and $d\theta dz dr$
- 62-63. Changing order of integration** If possible, write iterated integrals in spherical coordinates for the following regions in the specified orders. Sketch the region of integration. Assume that f is continuous on the region.
- 62.** $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{4 \sec \phi} f(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta$ in the orders $d\rho d\theta d\phi$ and $d\theta d\rho d\phi$
- 63.** $\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_{\csc \phi}^2 f(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta$ in the orders $d\rho d\theta d\phi$ and $d\theta d\rho d\phi$
- 64-72. Miscellaneous volumes** Choose the best coordinate system and find the volume of the following solid regions. Surfaces are specified using the coordinates that give the simplest description, but the simplest integration may be with respect to different variables.
- 64.** The region inside the sphere $\rho = 1$ and below the cone $\phi = \pi/4$, for $z \geq 0$

- 65. That part of the solid cylinder $r \leq 2$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$
- 66. That part of the ball $\rho \leq 2$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$
- 67. The region bounded by the cylinder $r = 1$, for $0 \leq z \leq x + y$
- 68. The region inside the cylinder $r = 2 \cos \theta$, for $0 \leq z \leq 4 - x$
- 69. The wedge cut from the cardioid cylinder $r = 1 + \cos \theta$ by the planes $z = 2 - x$ and $z = x - 2$
- 70. **Volume of a drilled hemisphere** Find the volume of material remaining in a hemisphere of radius 2 after a cylindrical hole of radius 1 is drilled through the center of the hemisphere perpendicular to its base.
- 71. **Two cylinders** The x - and y -axes form the axes of two right circular cylinders with radius 1 (see figure). Find the volume of the region that is common to the two cylinders.



- 72. **Three cylinders** The coordinate axes form the axes of three right circular cylinders with radius 1 (see figure). Find the volume of the region that is common to the three cylinders.



Applications

- 73. **Density distribution** A right circular cylinder with height 8 cm and radius 2 cm is filled with water. A heated filament running along its axis produces a variable density in the water given by $\rho(r) = 1 - 0.05 e^{-0.01 r^2}$ g/cm³ (ρ stands for density here, not the radial spherical coordinate). Find the mass of the water in the cylinder. Neglect the volume of the filament.

- 74. Charge distribution** A spherical cloud of electric charge has a known charge density $Q(\rho)$, where $0 \leq \rho < \infty$ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a. $Q(\rho) = \frac{2 \times 10^{-4}}{1 + \rho^3}$

b. $Q(\rho) = (2 \times 10^{-4}) e^{-0.01 \rho^3}$

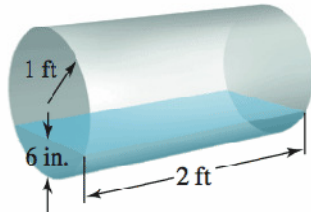
- 75. Gravitational field due to spherical shell** A point mass m is a distance d from the center of a thin spherical shell of mass M and radius R . The magnitude of the gravitational force on the point mass is given by the integral

$$F(d) = \frac{GMm}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{(d - R \cos \phi) \sin \phi}{(R^2 + d^2 - 2Rd \cos \phi)^{3/2}} d\phi d\theta,$$

where G is the gravitational constant.

- a. Use the change of variable $x = \cos \phi$ to evaluate the integral and show that if $d > R$, then $F(d) = \frac{GMm}{d^2}$, which means the force is the same as if the mass of the shell were concentrated at its center.
- b. Show that if $d < R$ (the point mass is inside the shell), then $F = 0$.

- 76. Water in a gas tank** Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 ft and a radius of 1 ft. If the water level is 6 in above the lowest part of the tank, determine how much water must be drained from the tank.

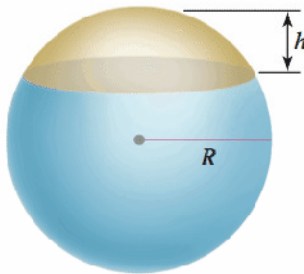


Additional Exercises

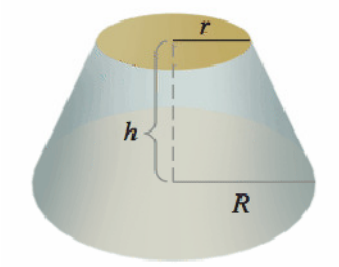
77-80. General volume formulas Use integration to find the volume of the following solids. In each case, choose a convenient coordinate system, find equations for the bounding surfaces, set up a triple integral, and evaluate the integral. Assume that a , b , c , r , R , and h are positive constants.

- 77. Cone** Find the volume of a solid right circular cone with height h and base radius r .

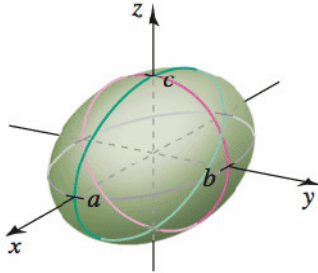
- 78. Spherical cap** Find the volume of the cap of a sphere of radius R with thickness h .



- 79. Frustum of a cone** Find the volume of a truncated solid cone of height h whose ends have radii r and R .



80. Ellipsoid Find the volume of a solid ellipsoid with axes of length $2a$, $2b$, and $2c$.



81. Intersecting spheres One sphere is centered at the origin and has a radius of R . Another sphere is centered at $(0, 0, r)$ and has a radius of r , where $r > R/2$. What is the volume of the region common to the two spheres?