13.6 Integrals for Mass Calculations

Intuition says that a thin circular disk (like a DVD without a hole) should balance on a pencil placed at the center of the disk (Figure 13.62). If, however, you were given a thin plate with an irregular shape, then at what point does it balance? This question asks about the *center of mass* of a thin object (thin enough that it can be treated as a two-dimensional plane region). Similarly, given a solid object with an irregular shape and variable density, where is the point at which all of the mass of the object would be located if it were treated as a point mass? In this section we use integration to compute the center of mass of one-, two- and three-dimensional objects.

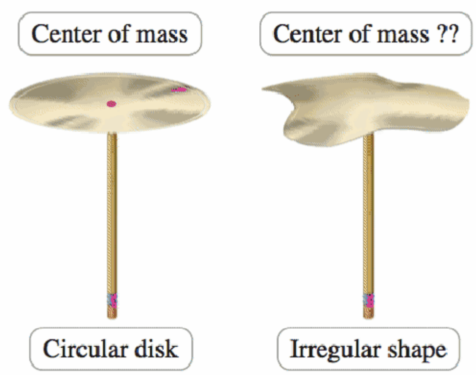


FIGURE 13.62

Sets of Individual Objects

Continuous Objects in One Dimension

Two-Dimensional Objects

Three-Dimensional Objects

Quick Quiz

SECTION 13.6 EXERCISES

Review Questions

1. Explain how to find the balance point for two people on opposite ends of a (massless) plank that rests on a pivot.

2. If a thin 1-m cylindrical rod has a density of $\rho = 1$ g/cm for its left half and a density of $\rho = 2$ g/cm for its right half, what is its mass and where is its center of mass?

3. Explain how to find the center of mass of a thin plate with a variable density.

4. In the integral for the moment M_x of a thin plate, why does y appear in the integrand?

5. Explain how to find the center of mass of a three-dimensional object with a variable density.

6. In the integral for the moment M_{xz} of a solid with respect to the xz-plane, why does y appear in the integrand?

Basic Skills

7-8. Individual masses on a line *Sketch the following systems on a number line and find the location of the center of mass.*

7. $m_1 = 10$ kg located at $x = 3$ m; $m_2 = 3$ kg located at $x = -1$ m

8. $m_1 = 8$ kg located at $x = 2$ m; $m_2 = 4$ kg located at $x = -4$ m; $m_3 = 1$ kg located at $x = 0$ m

9-14. One-dimensional objects *Find the mass and center of mass of the thin rods with the following density functions.*

9. $\rho(x) = 1 + \sin x$, for $0 \le x \le \pi$

10. $\rho(x) = 1 + x^3$, for $0 \le x \le 1$

11. $\rho(x) = 2 - x^2 / 16$, for $0 \le x \le 4$

12. $\rho(x) = 2 + \cos x$, for $0 \le x \le \pi$

13. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } 2 < x \le 4 \end{cases}$

14. $\rho(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x(2 - x) & \text{if } 1 < x \le 2 \end{cases}$

15-20. Centroid calculations *Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.*

15. The region bounded by $y = \sin x$ and $y = 1 - \sin x$ between $x = \pi/4$ and $x = 3\pi/4$

16. The region in the first quadrant bounded by $x^2 + y^2 = 16$

17. The region bounded by $y = 1 - |x|$ and the x-axis

18. The region bounded by $y = e^x$, $y = e^{-x}$, $x = 0$, and $x = \ln 2$

19. The region bounded by $y = \ln x$, the x-axis, and $x = e$

20. The region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, for $y \ge 0$

21-26. Variable-density plates *Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region.*

21. $R = \{(x, y) : 0 \le x \le 4, 0 \le y \le 2\}$; $\rho(x, y) = 1 + x/2$

22. $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 5\}$; $\rho(x, y) = 2e^{-y/2}$

23. The triangular plate in the first quadrant bounded by $x + y = 4$ with $\rho(x, y) = 1 + x + y$

24. The upper half ($y \ge 0$) of the disk bounded by the circle $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + y/2$

25. The upper half ($y \ge 0$) of the disk bounded by the ellipse $x^2 + 9y^2 = 9$ with $\rho(x, y) = 1 + y$

26. The quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + x^2 + y^2$

27-32. Center of mass of constant-density solids *Find the center of mass of the following solids, assuming a constant density. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.*

27. The upper half of the ball $x^2 + y^2 + z^2 \le 16$ (for $z \ge 0$)

28. The region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 25$

29. The tetrahedron in the first octant bounded by $z = 1 - x - y$ and the coordinate planes

30. The region bounded by the cone $z = 16 - r$ and the plane $z = 0$

31. The sliced solid cylinder bounded by $x^2 + y^2 = 1$, $z = 0$, and $y + z = 1$

32. The region bounded by the upper half ($z \ge 0$) of the ellipsoid $4x^2 + 4y^2 + z^2 = 16$

33-38. Variable-density solids *Find the coordinates of the center of mass of the following solids with variable density.*

33. $R = \{(x, y, z) : 0 \le x \le 4,\ 0 \le y \le 1,\ 0 \le z \le 1\}$; $\rho(x, y, z) = 1 + x/2$

34. The region bounded by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$ with $\rho(x, y, z) = 5 - z$

35. The region bounded by the upper half of the sphere $\rho = 16$ and $z = 0$ with density $f(\rho, \phi, \theta) = 1 + \rho/4$

36. The interior of the cube in the first octant formed by the planes $x = 1$, $y = 1$, $z = 1$, with $\rho(x, y, z) = 2 + x + y + z$

37. The interior of the prism formed by $z = x$, $x = 1$, $y = 4$, and the coordinate planes with $\rho(x, y, z) = 2 + y$

38. The region bounded by the cone $z = 9 - r$ and $z = 0$ with $\rho(r, \theta, z) = 1 + z$

Further Explorations

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. A thin plate of constant density that is symmetric about the x-axis has a center of mass with an x-coordinate of zero.

b. A thin plate of constant density that is symmetric about both the x-axis and the y-axis has its center of mass at the origin.

c. The center of mass of a thin plate must lie on the plate.

d. The center of mass of a connected solid region (all in one piece) must lie within the region.

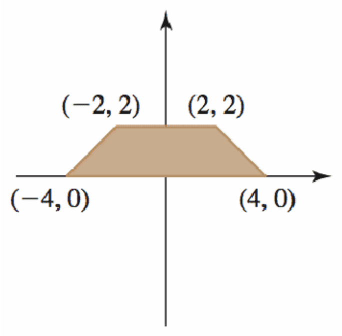
40. Limiting center of mass A thin rod of length L has a linear density given by $\rho(x) = 2e^{-x/3}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

41. Limiting center of mass A thin rod of length L has a linear density given by $\rho(x) = \dfrac{10}{1 + x^2}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

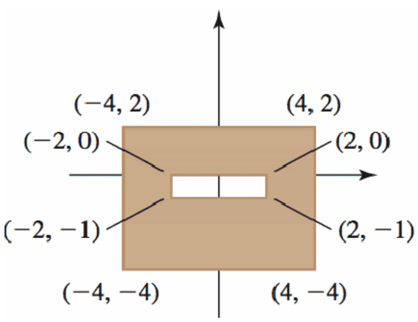
42. Limiting center of mass A thin plate is bounded by the graphs of $y = e^{-x}$, $y = -e^{-x}$, $x = 0$, and $x = L$. Find its center of mass. How does the center of mass change as $L \to \infty$?

43-44. Two-dimensional plates *Find the mass and center of mass of the thin constant-density plates shown in the figure.*

43.



44.



45-50. Centroids *Use polar coordinates to find the centroid of the following constant-density plane regions.*

45. The semicircular disk $R = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le \pi\}$

46. The quarter-circular disk $R = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le \pi/2\}$

47. The region bounded by the cardioid $r = 1 + \cos\theta$

48. The region bounded by the cardioid $r = 3 - 3\cos\theta$

49. The region bounded by one leaf of the rose $r = \sin 2\theta$ for $0 \le \theta \le \pi/2$

50. The region bounded by the limaçon $r = 2 + \cos\theta$

51. Semicircular wire A thin (one-dimensional) wire of constant density is bent into the shape of a semicircle of radius r. Find the location of its center of mass.

52. Parabolic region A thin plate of constant density occupies the region between the parabola $y = a x^2$ and the horizontal line $y = b$, where $a > 0$ and $b > 0$. Show that the center of mass is $\left(0, \dfrac{3b}{5}\right)$, independent of a.

53. Circular crescent Find the center of mass of the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the lines $x = a$ and $y = a$, where $a > 0$.

54-59. Centers of mass for general objects *Consider the following two- and three-dimensional regions with variable dimensions. Specify the surfaces and curves that bound the region, choose a convenient coordinate system, and compute the center of mass assuming constant density. All parameters are positive real numbers.*

54. A solid rectangular box has sides of length a, b, and c. Where is the center of mass relative to the faces of the box?

55. A solid cone has a base with a radius of r and a height of h. How far from the base is the center of mass?

56. A solid is enclosed by a hemisphere of radius a. How far from the base is the center of mass?

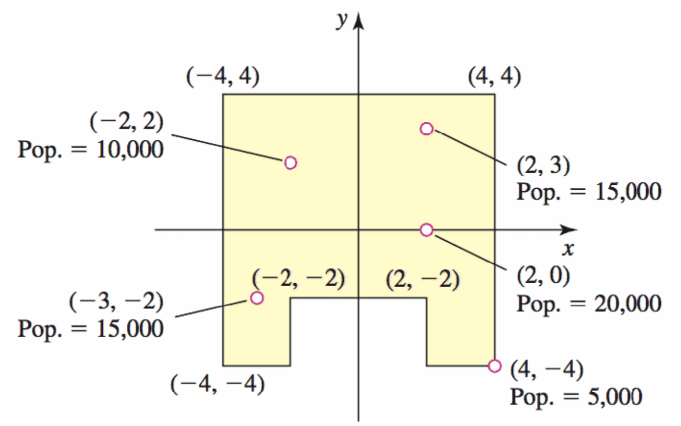
57. A region is enclosed by an isosceles triangle with two sides of length s and a base of length b. How far from the base is the center of mass?

58. A tetrahedron is bounded by the coordinate planes and the plane $x/a + y/a + z/a = 1$. What are the coordinates of the center of mass?

59. A solid is enclosed by the upper half of an ellipsoid with a circular base of radius r and a height of a. How far from the base is the center of mass?

Applications

60. Geographic vs. population center Geographers measure the *geographical center* of a country (which is the centroid) and the *population center* of a country (which is the center of mass computed with the population density). A hypothetical country is shown in the figure with the location and population of five towns. Assuming no one lives outside the towns, find the geographical center of the country and the population center of the country.



61. Center of mass on the edge Consider the thin constant-density plate $\{(r, \theta) : a \le r \le 1, 0 \le \theta \le \pi\}$ bounded by two semicircles and the x-axis.

a. Find and graph the y-coordinate of the center of mass of the plate as a function of a.

b. For what value of a is the center of mass on the edge of the plate?

62. Center of mass on the edge Consider the constant-density solid $\{(\rho, \phi, \theta) : 0 < a \le \rho \le 1, 0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi\}$ bounded by two hemispheres and the xy-plane.

a. Find and graph the z-coordinate of the center of mass of the plate as a function of a.

b. For what value of a is the center of mass on the edge of the solid?

63. Draining a soda can A cylindrical soda can has a radius of 4 cm and a height of 12 cm. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 cm above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1 \text{ g}/\text{cm}^3$ and the density of air is $0.001 \text{ g}/\text{cm}^3$.

Additional Exercises

64. Triangle medians A triangular region has a base that connects the vertices $(0, 0)$ and $(b, 0)$, and a third vertex at (a, h), where $a > 0$, $b > 0$, and $h > 0$.

a. Show that the centroid of the triangle is $\left(\frac{a+b}{3}, \frac{h}{3}\right)$.

b. Recall that the three medians of a triangle extend from each vertex to the midpoint of the opposite side. Knowing that the medians of a triangle intersect in a point M and that each median bisects the triangle, conclude that the centroid of the triangle is M.

65. The golden earring A disk of radius r is removed from a larger disk of radius R to form an earring (see figure). Assume the earring is a thin plate of uniform density.

a. Find the center of mass of the earring in terms of r and R. (*Hint:* Place the origin of a coordinate system either at the center of the large disk or at Q; either way, the earring is symmetric about the x-axis.)

b. Show that the ratio R/r such that the center of mass lies at the point P (on the edge of the inner disk) is the golden mean $\left(1+\sqrt{5}\right)/2 \approx 1.618$.

(*Source:* P. Glaister, "Golden Earrings," *Mathematical Gazette* 80 (1996): 224-225.)