13.7 Change of Variables in Multiple Integrals

Converting double integrals from rectangular coordinates to polar coordinates (Section 13.3) and converting triple integrals from rectangular coordinates to cylindrical or spherical coordinates (Section 13.5) are examples of a general procedure known as a *change of variables*. The idea is not new: The Substitution Rule introduced in Chapter 5 with single-variable integrals is also an example of a change of variables. The aim of this section is to show how to apply a general change of variables to double and triple integrals.

Recap of Change of Variables

Transformations in the Plane

Change of Variables in Triple Integrals

Stategies for Choosing New Variables

Quick Quiz

SECTION 13.7 EXERCISES

Review Questions

1. If S is the unit square in the first quadrant of the uv-plane, describe the image of the transformation $T: x = 2u,\ y = 2v$.

2. Explain how to compute the Jacobian of the transformation $T: x = g(u, v),\ y = h(u, v)$.

3. Using the transformation $T: x = u + v,\ y = u - v$, the image of the unit square $S = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 1\}$ is a region R in the xy-plane. Explain how to change variables in the integral $\iint_R f(x, y)\, dA$ to find a new integral over S.

4. If S is the unit cube in the first octant of uvw-space with one vertext at the origin, describe the image of the transformation $T: x = u/2,\ y = v/2,\ z = w/2$.

Basic Skills

5-12. Transforming a square *Let $S = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the uv-plane. Find the image of S in the xy-plane under the following transformations.*

5. $T: x = 2u,\ y = v/2$

6. $T: x = -u,\ y = -v$

7. $T: x = (u + v)/2,\ \ y = (u - v)/2$

8. $T: x = 2u + v,\ \ y = 2u$

9. $T: x = u^2 - v^2,\ \ y = 2uv$

10. $T: x = 2uv,\ \ y = u^2 - v^2$

11. $T: x = u\cos(\pi v),\ \ y = u\sin(\pi v)$

12. $T: x = v\sin(\pi u),\ \ y = v\cos(\pi u)$

13-16. Images of regions *Find the image R in the xy-plane of the region S using the given transformation T. Sketch both R and S.*

13. $S = \{(u, v) : v \le 1 - u,\ u \ge 0,\ v \ge 0\};\ \ T : x = u,\ y = v^2$

14. $S = \left\{(u, v) : u^2 + v^2 \le 1\right\};\ \ T : x = 2u,\ y = 4v$

15. $S = \{(u, v) : 1 \le u \le 3,\ 2 \le v \le 4\};\ \ T : x = u/v,\ y = v$

16. $S = \{(u, v) : 2 \le u \le 3,\ 3 \le v \le 6\};\ \ T : x = u,\ y = v/u$

17-22. Computing Jacobians *Compute the Jacobian $J(u, v)$ for the following transformations.*

17. $T : x = 3u,\ y = -3v$

18. $T : x = 4v,\ y = -2u$

19. $T : x = 2uv,\ \ y = u^2 - v^2$

20. $T : x = u\cos(\pi v),\ \ y = u\sin(\pi v)$

21. $T : x = (u + v)\Big/\sqrt{2},\ \ y = (u - v)\Big/\sqrt{2}$

22. $T : x = u/v,\ y = v$

23-26. Solve and compute Jacobians *Solve the following relations for x and y, and compute the Jacobian $J(u, v)$.*

23. $u = x + y,\ v = 2x - y$

24. $u = xy,\ \ v = x$

25. $u = 2x - 3y,\ v = y - x$

26. $u = x + 4y,\ v = 3x + 2y$

27-30. Double integrals—transformation given *To evaluate the following integrals carry out these steps.*

- ***a.*** *Sketch the original region of integration R in the xy-plane and the new region S in the uv-plane using the given change of variables.*
- ***b.*** *Find the limits of integration for the new integral with respect to u and v.*
- ***c.*** *Compute the Jacobian.*
- ***d.*** *Change variables and evaluate the new integral.*

27. $\displaystyle\iint_R xy\,dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$; use $x = u + v,\ y = u - v$.

28. $\displaystyle\iint_R x^2 y\,dA$, where $R = \{(x, y) : 0 \le x \le 2,\ x \le y \le x + 4\}$; use $x = 2u,\ y = 4v + 2u$.

29. $\displaystyle\iint_R x^2 \sqrt{x + 2y}\ dA$, where $R = \{(x, y) : 0 \le x \le 2,\ -x/2 \le y \le 1 - x\}$; use $x = 2u,\ y = v - u$.

30. $\iint_R x\,y\,dA$, where R is bounded by the ellipse $9\,x^2 + 4\,y^2 = 36$; use $x = 2\,u,\ y = 3\,v$.

31-36. Double integrals—your choice of transformation *Evaluate the following integrals using a change of variables of your choice. Sketch the original and new regions of integration, R and S.*

31. $\int_0^1 \int_y^{y+2} \sqrt{x - y}\ dx\,dy$

32. $\iint_R \sqrt{y^2 - x^2}\ dA$, where R is the diamond bounded by $y - x = 0,\ y - x = 2,\ y + x = 0$, and $y + x = 2$

33. $\iint_R \left(\dfrac{y - x}{y + 2\,x + 1}\right)^4 dA$, where R is the parallelogram bounded by $y - x = 1,\ y - x = 2,\ y + 2\,x = 0$, and $y + 2\,x = 4$

34. $\iint_R e^{x\,y}\,dA$, where R is the region bounded by the hyperbolas $x\,y = 1$ and $x\,y = 4$, and the lines $y/x = 1$ and $y/x = 3$

35. $\iint_R x\,y\,dA$, where R is the region bounded by the hyperbolas $x\,y = 1$ and $x\,y = 4$, and the lines $y = 1$ and $y = 3$

36. $\iint_R (x - y)\sqrt{x - 2\,y}\ dA$, where R is the triangular region bounded by $y = 0,\ x - 2\,y = 0$, and $x - y = 1$

37-40. Jacobians in three variables *Evaluate the Jacobians $J(u,\ v,\ w)$ for the following transformations.*

37. $x = v + w,\ y = u + w,\ z = u + v$

38. $x = u + v - w,\ y = u - v + w,\ z = -u + v + w$

39. $x = v\,w,\ y = u\,w,\ z = u^2 - v^2$

40. $u = x - y,\ v = x - z,\ w = y + z$ (Solve for x, y, and z first).

41-44. Triple integrals *Use a change of variables to evaluate the following integrals.*

41. $\iiint_D x\,y\,dV$; D is bounded by the planes $y - x = 0,\ y - x = 2,\ z - y = 0,\ z - y = 1,\ z = 0,\ z = 3$.

42. $\iiint_D dV$; D is bounded by the planes $y - 2\,x = 0,\ y - 2\,x = 1,\ z - 3\,y = 0,\ z - 3\,y = 1,\ z - 4\,x = 0,\ z - 4\,x = 3$.

43. $\iiint_D z\,dV$; D is bounded by the paraboloid $z = 16 - x^2 - 4\,y^2$ and the xy-plane. Use $x = 4\,u\cos v,\ y = 2\,u\sin v,\ z = w$.

44. $\iiint_D dV$; D is bounded by the upper half of the ellipsoid $x^2/9 + y^2/4 + z^2 = 1$ and the xy-plane. Use $x = 3\,u,\ y = 2\,v,$ $z = w$.

Further Explorations

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the transformation $T: x = g(u, v),\ y = h(u, v)$ is linear in u and v, then the Jacobian is a constant.

b. The transformation $x = a\,u + b\,v,\ y = c\,u + d\,v$ generally maps triangular regions to triangular regions.

c. The transformation $x = 2\,v,\ y = -2\,u$ maps circles to circles.

46. Cylindrical coordinates Evaluate the Jacobian for the transformation from cylindrical coordinates (r, θ, Z) to rectangular coorrdinates $(x, y, z): x = r\cos\theta,\ y = r\sin\theta,\ z = Z$. Show that $J(r, \theta, Z) = r$.

47. Spherical coordinates Evaluate the Jacobian for the transformation from spherical to rectangular coordinates: $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Show that $J(\rho, \phi, \theta) = \rho^2\sin\phi$.

48-52. Ellipse problems *Let R be the region bounded by the ellipse* $x^2/a^2 + y^2/b^2 = 1$, *where* $a > 0$ *and* $b > 0$ *are real numbers. Let T be the transformation* $x = a\,u,\ y = b\,v$.

48. Find the area of R.

49. Evaluate $\displaystyle\iint_R |x\,y|\,dA$.

50. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

51. Find the average square of the distance between points of R and the origin.

52. Find the average distance between points in the upper half of R and the x-axis.

53-56. Ellipsoid problems *Let D be the region bounded by the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, *where* $a > 0$, $b > 0$, *and* $c > 0$ *are real numbers. Let T be the transformation* $x = a\,u,\ y = b\,v,\ z = c\,w$.

53. Find the volume of D.

54. Evaluate $\displaystyle\iiint_D |x\,y\,z|\,dA$.

55. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

56. Find the average square of the distance between points of D and the origin.

57. Parabolic coordinates Let T be the transformation $x = u^2 - v^2,\ y = 2\,u\,v$.

a. Show that the lines $u = a$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices on the positive x-axis.

b. Show that the lines $v = b$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices on the negative x-axis.

c. Evaluate $J(u, v)$.

d. Use a change of variables to find the area of the region bounded by $x = 4 - y^2/16$ and $x = y^2/4 - 1$.

e. Use a change of variables to find the area of the curved rectangle above the x-axis bounded by $x = 4 - y^2/16$, $x = 9 - y^2/36$, $x = y^2/4 - 1$, and $x = y^2/64 - 16$.

f. Describe the effect of the transformation $x = 2\,u\,v,\ y = u^2 - v^2$ on horizontal and vertical lines in the uv-plane.

Applications

58. Shear transformations in $\mathbf{R}^2$ The transformation T in $\mathbb{R}^2$ given by $x = au + bv$, $y = cv$, where a, b, and c are positive real numbers, is a *shear transformation*. Let S be the unit square $\{(u, v) : 0 \le u \le 1,\ 0 \le v \le 1\}$. Let $R = T(S)$ be the image of S.

- **a.** Explain with pictures the effect of T on S.
- **b.** Compute the Jacobian of T.
- **c.** Find the area of R and compare it to the area of S (which is 1).
- **d.** Assuming a constant density, find the center of mass of R (in terms of a, b, and c) and compare it to the center of mass of S (which is $\left(\frac{1}{2}, \frac{1}{2}\right)$).
- **e.** Find an analogous transformation that gives a shear in the y-direction.

59. Shear transformations in $\mathbf{R}^3$ The transformation T in $\mathbb{R}^3$ given by

$$x = au + bv + cw, \quad y = dv + ew, \quad z = w,$$

where a, b, c, d, and e are positive real numbers, is one of many possible shear transformations in $\mathbb{R}^3$. Let S be the unit cube $\{(u, v, w) : 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1\}$. Let $D = T(S)$ be the image of S.

- **a.** Explain with pictures and words the effect of T on S.
- **b.** Compute the Jacobian of T.
- **c.** Find the volume of D and compare it to the volume of S (which is 1).
- **d.** Assuming a constant density, find the center of mass of D and compare it to the center of mass of S (which is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$).

Additional Exercises

60. Linear transformations Consider the linear transformation T in $\mathbb{R}^2$ given by $x = au + bv$, $y = cu + dv$, where a, b, c, and d are real numbers, with $ad \ne bc$.

- **a.** Find the Jacobian of T.
- **b.** Let S be the square in the uv-plane with vertices (0, 0), (1, 0), (0, 1), and (1, 1), and let $R = T(S)$. Show that area$(R) = |J(u, v)|$.
- **c.** Let l be the line segment joining the points P and Q in the uv-plane. Show that $T(l)$ (the image of l under T) is the line segment joining $T(P)$ and $T(Q)$ in the xy-plane. (*Hint:* Use vectors.)
- **d.** Show that if S is a parallelogram in the uv-plane and $R = T(S)$, then area$(R) = |J(u, v)|$ area(S). (*Hint:* Without loss of generality, assume the vertices of S area (0, 0), $(A, 0)$, (B, C), and $(A + B, C)$, where A, B, and C are positive, and use vectors.)

61. Meaning of the Jacobian The Jacobian is a magnification (or reduction) factor that relates the area of a small region near the point (u, v) to the area of the image of that region near the point (x, y).

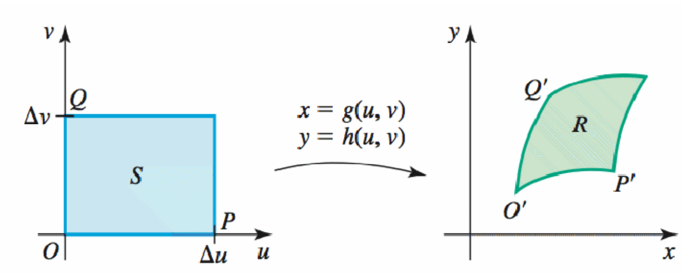
- **a.** Suppose S is a rectangle in the uv-plane with vertices $O(0, 0)$, $P(\Delta u, 0)$, $(\Delta u, \Delta v)$, and $Q(0, \Delta v)$ (see figure). The image of S under the transformation $x = g(u, v)$, $y = h(u, v)$ is a region R in the xy-plane. Let O', P' and Q' be the images of O, P, and Q, respectively, in the xy-plane, where O', P', and Q' do not all lie on the same line. Explain why the coordinates of O', P', and Q' are $(g(0, 0), h(0, 0))$, $(g(\Delta u, 0), h(\Delta u, 0))$ and $(g(0, \Delta v), h(0, \Delta v))$, respectively.
- **b.** Use a Taylor series in both variables to show that

$$g(\Delta u, 0) \approx g(0, 0) + g_u(0, 0)\,\Delta u$$
$$g(0, \Delta v) \approx g(0, 0) + g_v(0, 0)\,\Delta v$$
$$h(\Delta u, 0) \approx h(0, 0) + h_u(0, 0)\,\Delta u$$
$$h(0, \Delta v) \approx h(0, 0) + h_v(0, 0)\,\Delta v$$

where $g_u(0, 0)$ is $\dfrac{\partial g}{\partial u} = \dfrac{\partial x}{\partial u}$ evaluated at (0, 0), with similar meanings for g_v, h_u, and h_v.

c. Consider the vectors $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$ and the parallelogram, two of whose sides are $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$. Use the cross product to show that the area of the parallelogram is approximately $|J(u, v)|\,\Delta u\,\Delta v$.

d. Explain why the ratio of the area of R to the area of S is approximately $|J(u, v)|$.



62. Open and closed boxes Consider the region R bounded by three pairs of parallel planes: $ax + by = 0$, $ax + by = 1$, $cx + dz = 0$, $cx + dz = 1$, $ey + fz = 0$, $ey + fz = 1$, where a, b, c, d, e, and f are real numbers. For the purposes of evaluating triple integrals, when do these six planes bound a finite region? Carry out the following steps.

a. Find three vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ each of which is normal to one of the three pairs of planes.

b. Show that the three normal vectors lie in a plane if their triple scalar product $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ is zero.

c. Show that the three normal vectors lie in a plane if $ade + bcf = 0$.

d. Assuming $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ lie in a plane P, find a vector $\mathbf{N}$ that is normal to P. Explain why a line in the direction of $\mathbf{N}$ does not intersect any of the six planes, and thus the six planes do not form a bounded region.

e. Consider the change of variables $u = ax + by$, $v = cx + dz$, $w = ey + fz$. Show that

$$J(x, y, z) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = -ade - bcf\,.$$

What is the value of the Jacobian if R is unbounded?