Chapter Preview

This culminating chapter of the book provides a beautiful, unifying conclusion to our study of calculus. Many ideas and themes that have appeared throughout the book come together in these final pages. First, we combine vector-valued functions (Chapter 11) and functions of several variables (Chapter 12) to form *vector fields*. Once vector fields have been introduced and illustrated through their many applications, we begin exploring the calculus of vector fields. Concepts such as limits and continuity carry over directly. The extension of derivatives to vector fields leads to two new operations that underlie this chapter; the *curl* and the *divergence*. When integration is extended to vector fields, we discover new versions of the Fundamental Theorem of Calculus. The chapter ends with a final look at the Fundamental Theorem of Calculus and the several related forms in which it has appeared throughout the book.

14.1 Vector Fields

It is not difficult to find everyday examples of vector fields. Imagine sitting on a beach in a breeze: focus on a point in space and consider the motion of the air at that point at a single instant of time. The motion is described by a velocity vector with three components (east-west, north-south, up-down). At another point in space at the same time, the air is moving with a different direction and speed, and a different velocity vector is associated with that point. In general, at one instant in time, every point in space has a velocity vector associated with it (Figure 14.1). This collection of velocity vectors is a vector field.

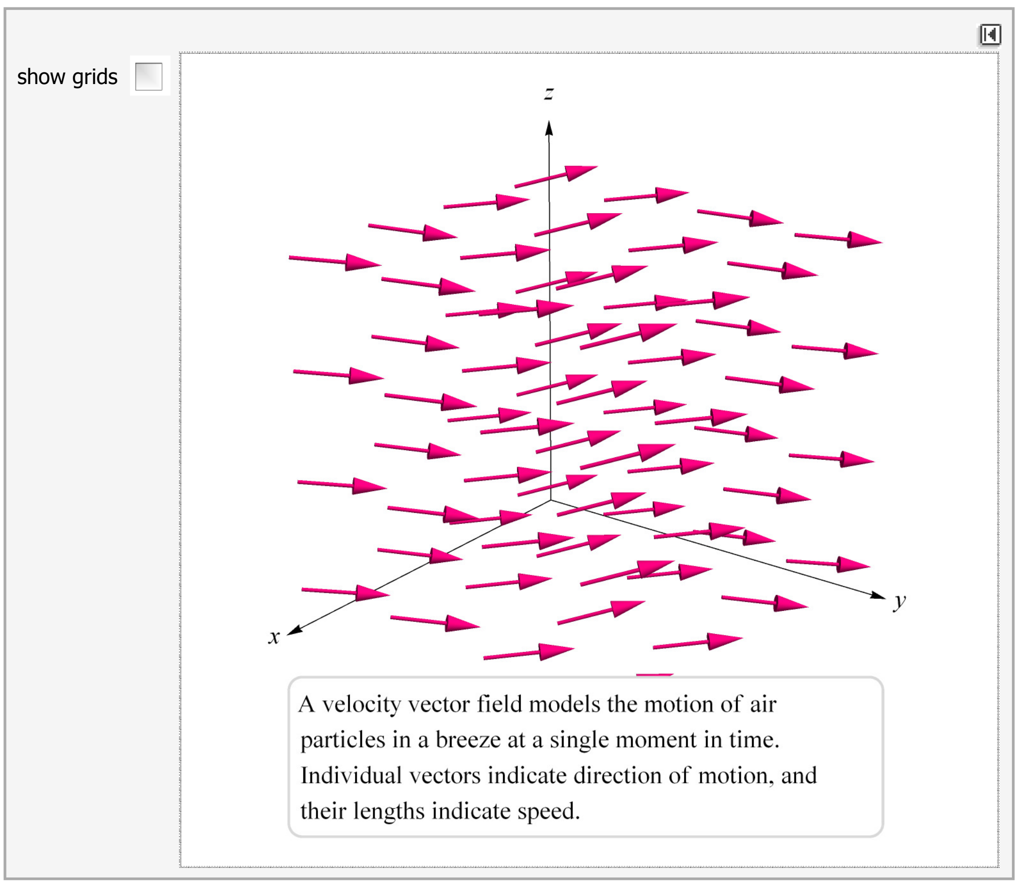


FIGURE 14.1

Other examples of vector fields include the wind patterns in a hurricane (Figure 14.2a), the flow of air around an airplane wing, and the circulation of water in a heat exchanger (Figure 14.2b). Gravitational, magnetic, and electric force fields are represented by vector fields (Figure 14.2c), as are the stresses and strains in buildings and bridges. Beyond physics and engineering, the transport of a chemical pollutant in a lake or human migration patterns can be modeled by vector fields.

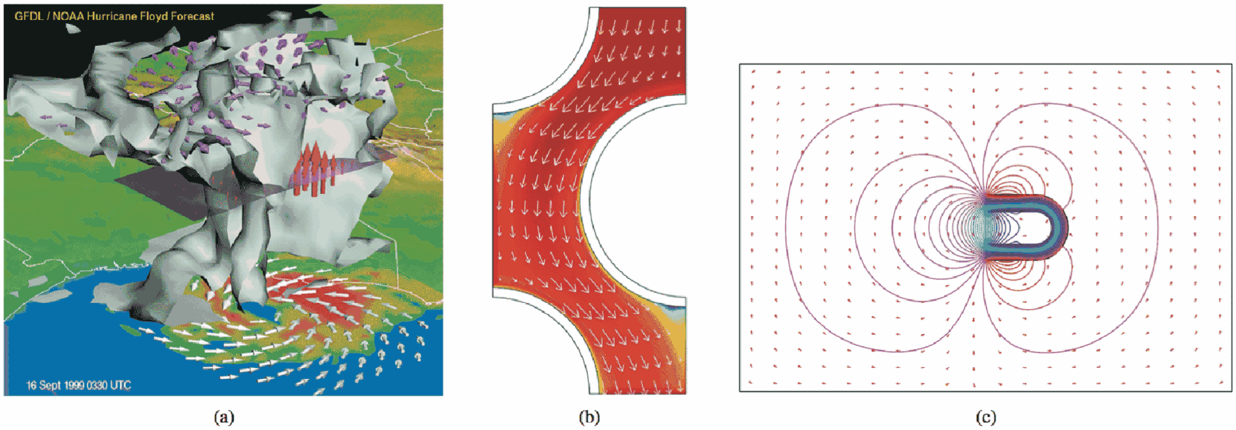


FIGURE 14.2

Vector Fields in Two Dimensions

Vector Fields in Three Dimensions

Quick Quiz

SECTION 14.1 EXERCISES

Review Questions

1. Explain how a vector field $\mathbf{F} = \langle f, g, h\rangle$ is used to describe the motion of the air in a room at one instant in time.

2. Sketch the vector field $\mathbf{F} = \langle x, y\rangle$.

3. How do you graph the vector field $\mathbf{F} = \langle f(x, y), g(x, y)\rangle$?

4. Given a function ϕ, how does the gradient of ϕ produce a vector field?

5. Interpret the gradient field of the temperature function $T = f(x, y)$.

Basic Skills

6-15. Two-dimensional vector fields *Make a sketch of the following vector fields.*

6. $\mathbf{F} = \langle 1, y\rangle$

7. $\mathbf{F} = \langle x, 0\rangle$

8. $\mathbf{F} = \langle -x, -y\rangle$

9. $\mathbf{F} = \langle x, -y\rangle$

10. $\mathbf{F} = \langle 2x, 3y\rangle$

11. $\mathbf{F} = \langle y, -x\rangle$

12. $\mathbf{F} = \langle x + y, y\rangle$

13. $\mathbf{F} = \langle x, y - x\rangle$

14. $\mathbf{F} = \langle \sin x, \sin y \rangle$

15. $\mathbf{F} = \langle e^{-x}, 0 \rangle$

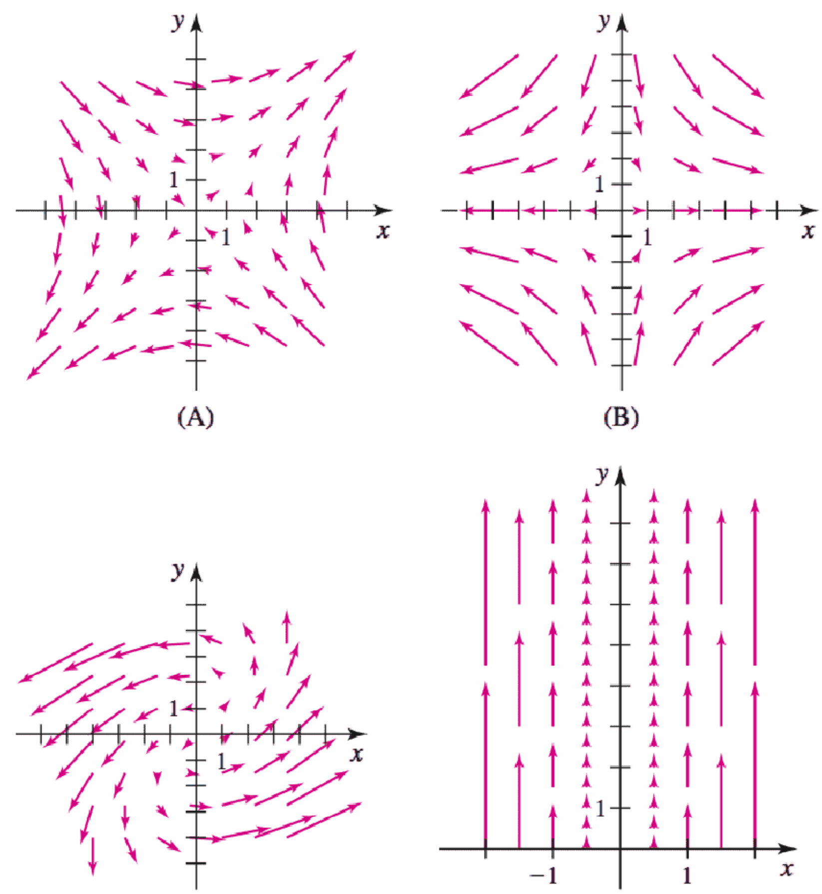
16. Matching vector fields with graphs Match vector fields a-d with graphs A-D.

a. $\mathbf{F} = \left\langle 0, x^2 \right\rangle$

b. $\mathbf{F} = \langle x - y, x \rangle$

c. $\mathbf{F} = \langle 2x, -y \rangle$

d. $\mathbf{F} = \langle y, x \rangle$



(A) (B)

(C) (D)

17-20. Normal and tangential components *Determine whether the vector field* **F** *is tangent to or normal to the curve C at points on C. A vector* **n** *normal to C is also given. Sketch C and a few representative vectors of* **F**.

17. $\mathbf{F} = \langle x, y \rangle$, where $C = \left\{(x, y) : x^2 + y^2 = 4\right\}$ and $\mathbf{n} = \langle x, y \rangle$

18. $\mathbf{F} = \langle y, -x \rangle$, where $C = \left\{(x, y) : x^2 + y^2 = 1\right\}$ and $\mathbf{n} = \langle x, y \rangle$

19. $\mathbf{F} = \langle x, y \rangle$, where $C = \{(x, y) : x = 1\}$ and $\mathbf{n} = \langle 1, 0 \rangle$

20. $\mathbf{F} = \langle y, x \rangle$, where $C = \left\{(x, y) : x^2 + y^2 = 1\right\}$ and $\mathbf{n} = \langle x, y \rangle$

T **21-24. Three-dimensional vector fields** *Sketch a few representative vectors of the following vector fields.*

21. $\mathbf{F} = \langle 1,\ 0,\ z\rangle$

22. $\mathbf{F} = \langle x,\ y,\ z\rangle$

23. $\mathbf{F} = \langle y,\ -x,\ 0\rangle$

24. $\mathbf{F} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2 + y^2 + z^2}}$

T **25-28. Gradient fields** *Find the gradient field* $\mathbf{F} = \nabla\phi$ *for the potential function* ϕ. *Sketch a few level curves of* ϕ *and a few vectors of* $\mathbf{F}$.

25. $\phi(x,\ y) = x^2 + y^2$, for $x^2 + y^2 \le 16$

26. $\phi(x,\ y) = \sqrt{x^2 + y^2}$, for $x^2 + y^2 \le 9$, $(x,\ y) \ne (0,\ 0)$

27. $\phi(x,\ y) = \sin x \sin y$, for $|x| \le \pi$, $|y| \le \pi$

28. $\phi(x,\ y) = 2\,x\,y$, for $|x| \le 2$, $|y| \le 2$

29-32. Gradient fields *Find the gradient field* $\mathbf{F} = \nabla\phi$ *for the following potential functions* ϕ.

29. $\phi(x,\ y,\ z) = \left(x^2 + y^2 + z^2\right)/2$

30. $\phi(x,\ y,\ z) = \ln\left(1 + x^2 + y^2 + z^2\right)$

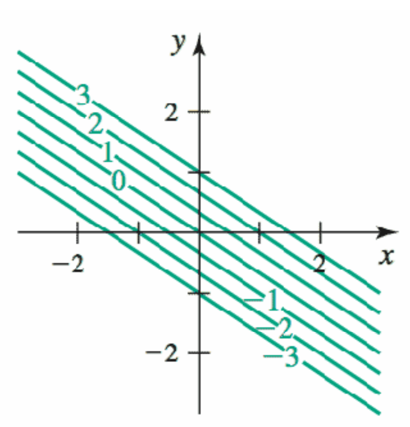
31. $\phi(x,\ y,\ z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$

32. $\phi(x,\ y,\ z) = e^{-z}\sin(x + y)$

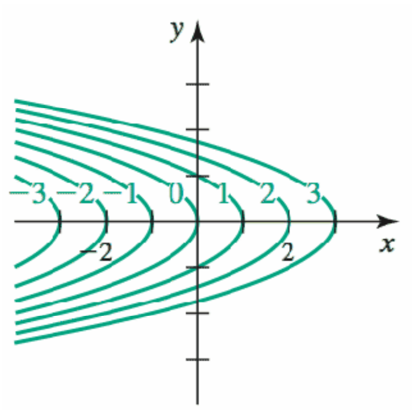
33-36. Equipotential curves *Consider the following potential functions and graphs of their equipotential curves.*

- ***a.*** *Find the associated gradient field* $\mathbf{F} = \nabla\phi$.
- ***b.*** *Show that the vector field is orthogonal to the equipotential curve at the point* $(1,\ 1)$. *Illustrate this result on the figure.*
- ***c.*** *Show that the vector field is orthogonal to the equipotential curve at all points* $(x,\ y)$.
- ***d.*** *Sketch two flow curves representing* $\mathbf{F}$ *that are everywhere orthogonal to the equipotential curves.*

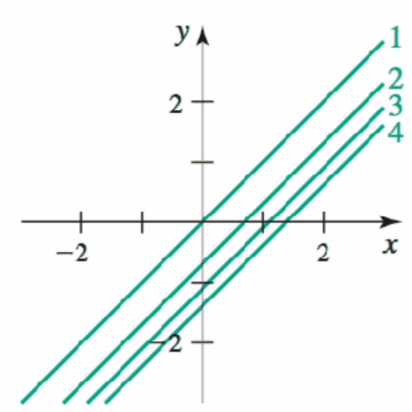
33. $\phi(x,\ y) = 2\,x + 3\,y$



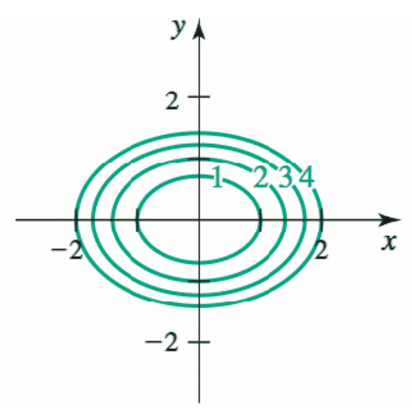
34. $\phi(x, y) = x + y^2$



35. $\phi(x, y) = e^{x-y}$



36. $\phi(x, y) = x^2 + 2y^2$



Further Explorations

37. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The vector field $\mathbf{F} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

b. The vector field $\mathbf{F} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on the unit circle.

c. The vector field $\mathbf{F} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

38-39. Vector fields on regions *Let* $S = \{(x, y) : |x| \le 1$ *and* $|y \le 1\}$ *(a square centered at the origin),* $D = \{(x, y) : |x| + |y| \le 1\}$ *(a diamond centered at the origin), and* $C = \left\{(x, y) : x^2 + y^2 \le 1\right\}$ *(a disk centered at the origin). For each vector field* **F***, draw pictures and analyze the vector field to answer the following questions.*

a. *At what points of S, D, and C does the vector field have its maximum magnitude?*

b. *At what points on the boundary of each region is the vector field directed out of the region?*

38. $\mathbf{F} = \langle x, y\rangle$

39. $\mathbf{F} = \langle -y, x\rangle$

40-43. Design your own vector field *Specify the component functions of a vector field* **F** *in* $\mathbb{R}^2$ *with the following properties. Solutions are not unique.*

40. **F** is everywhere normal to the line $x = 2$.

41. **F** is everywhere normal to the line $x = y$.

42. The flow of **F** is counterclockwise around the origin, increasing in magnitude with distance from the origin.

43. At all points except (0, 0), **F** has unit magnitude and points away from the origin along radial lines.

Applications

44. Electric field due to a point charge The electric field in the xy-plane due to a point charge at (0, 0) is a gradient field with a potential function $V(x, y) = \dfrac{k}{\sqrt{x^2 + y^2}}$, where $k > 0$ is a physical constant.

a. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

b. Show that the vectors of the electric field point in the radial direction (outward from the origin) and the radial component of **E** can be expressed as $E_r = k/r^2$, where $r = \sqrt{x^2 + y^2}$.

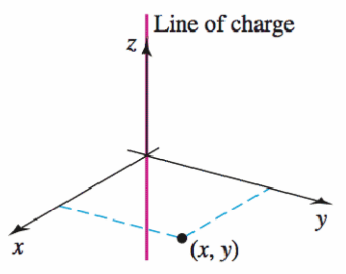
c. Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.

45. Electric field due to a line of charge The electric field in the xy-plane due to an infinite line of charge along the z-axis is a gradient field with a potential function $V(x, y) = c \ln\left(\dfrac{r_0}{\sqrt{x^2 + y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be 0 (see figure).

a. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

b. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $|\mathbf{E}| = c/r$, where $r = \sqrt{x^2 + y^2}$.

c. Show that the vector field is orthogonal to the equipotential curves at all points in the domain of V.



46. Gravitational force due to a mass The gravitational force on a point mass m due to a point mass M is a gradient field with potential $U(r) = \frac{GMm}{r}$, where G is the gravitational constant and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the masses.

a. Find the components of the gravitational force in the x-, y-, and z-directions, where $\mathbf{F}(x, y, z) = -\nabla U(x, y, z)$.

b. Show that the gravitational force points in the radial direction (outward from point mass M) and the radial component is $F(r) = \frac{GMm}{r^2}$.

c. Show that the vector field is orthogonal to the equipotential surfaces at all points in the domain of U.

Additional Exercises

47-51. Streamlines in the plane *Let* $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$ *be defined on* $\mathbb{R}^2$.

47. Explain why the flow curves or streamlines of $\mathbf{F}$ satisfy $y' = g(x, y)/f(x, y)$ and are everywhere tangent to the vector field.

T **48.** Find and graph the streamlines for the vector field $\mathbf{F} = \langle 1, x\rangle$.

T **49.** Find and graph the streamlines for the vector field $\mathbf{F} = \langle x, x\rangle$.

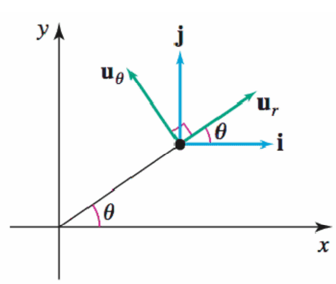
T **50.** Find and graph the streamlines for the vector field $\mathbf{F} = \langle y, x\rangle$. Note that $d/dx\left(y^2\right) = 2\,y\,y'(x)$.

T **51.** Find and graph the streamlines for the vector field $\mathbf{F} = \langle -y, x\rangle$.

52-53. Unit vectors in polar coordinates

52. Vectors in $\mathbb{R}^2$ may also be expressed in terms of polar coordinates. The standard coordinate unit vectors in polar coordinates are denoted $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (see figure). Unlike the coordinate unit vectors in Cartesian coordinates, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change their direction depending on the point (r, θ). Use the figure to show that for $r > 0$, the following relationships between the unit vectors in Cartesian and polar coordinates hold:

$$\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \qquad \mathbf{i} = \mathbf{u}_r\cos\theta - \mathbf{u}_\theta\sin\theta$$
$$\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \qquad \mathbf{j} = \mathbf{u}_r\sin\theta + \mathbf{u}_\theta\cos\theta$$



53. Verify that the relationships in Exercise 52 are consistent when $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$.

54-56. Vector fields in polar coordinates *A vector field in polar coordinates has the form* $\mathbf{F}(r, \theta) = f(r, \theta)\,\mathbf{u}_r + g(r, \theta)\,\mathbf{u}_\theta$, *where the unit vectors are defined in Exercise 52. Sketch the following vector fields and express them in Cartesian coordinates.*

54. $\mathbf{F} = \mathbf{u}_r$

55. $\mathbf{F} = \mathbf{u}_\theta$

56. $\mathbf{F} = r\mathbf{u}_\theta$

57. Write the vector field $\mathbf{F} = \langle -y, x \rangle$ in polar coordinates and sketch the field.