14.2 Line Integrals

With integrals of a single variable, we integrate over intervals in $\mathbb{R}^1$(the real line). With double and triple integrals, we integrate over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. *Line integrals* (which really should be called *curve integrals*) are another class of integrals that play an important role in vector calculus. They are used to integrate either scalar-valued functions or vector fields along curves.

Suppose a thin, circular plate has a known temperature distribution and you must compute the average temperature along the edge of the plate. The required calculation involves integrating the temperature function over the *curved* boundary of the plate. Similarly, to calculate the amount of work needed to put a satellite into orbit, we integrate the gravitational force (a vector field) along the curved path of the satellite. Both these calculations require line integrals. As you will see, line integrals take several different forms. It is the goal of this section to distinguish these various forms and show how and when each form should be used.

Scalar Line Integrals in the Plane

Line Integrals in $\mathbb{R}^3$

Line Integrals of Vector Fields

Circulation and Flux of a Vector Field

Quick Quiz

SECTION 14.2 EXERCISES

Review Questions

1. Explain how a line integral differs from a single-variable integral $\int_a^b f(x)\,dx$.

2. Explain how to evaluate the line integral $\int_C f\,ds$, where C is parameterized by a parameter other than arc length.

3. If a curve C is given by $\mathbf{r}(t) = \langle t, t^2 \rangle$, what is $|\mathbf{r}'(t)|$?

4. Given a vector field $\mathbf{F}$ and a parameterized curve C, explain how to evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$.

5. Explain how $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ can be written in the alternate form $\int_a^b (f\,x'(t) + g\,y'(t) + h\,z'(t))\,dt$.

6. Given a vector field $\mathbf{F}$ and a closed smooth oriented curve C, what is the meaning of the circulation of $\mathbf{F}$ on C?

7. Explain how to calculate the circulation of a vector field on a closed smooth oriented curve.

8. Given a two-dimensional vector field $\mathbf{F}$ and a smooth oriented curve C, what is the meaning of the flux of $\mathbf{F}$ across C?

9. Explain how to calculate the flux of a two-dimensional vector field across a smooth oriented curve C.

10. Sketch the oriented quarter circle from (1, 0) to (0, 1) and supply a parameterization for the curve. Draw the unit normal vector (as defined in the text) at several points on the curve.

Basic Skills

11-14. Scalar line integrals with arc length as parameter *Evaluate the following line integrals.*

11. $\int_C x\, y\, ds$; C is the unit circle $\mathbf{r}(s) = \langle \cos s,\ \sin s \rangle$, for $0 \le s \le 2\pi$.

12. $\int_C (x + y)\, ds$; C is the circle of radius 1 centered at (0, 0).

13. $\int_C \left(x^2 - 2\, y^2\right) ds$; C is the line $\mathbf{r}(s) = \left\langle s \big/ \sqrt{2}\,,\ s \big/ \sqrt{2} \right\rangle$, for $0 \le s \le 4$.

14. $\int_C x^2\, y\, ds$; C is the line $\mathbf{r}(s) = \left\langle s \big/ \sqrt{2}\,,\ 1 - s \big/ \sqrt{2} \right\rangle$, for $0 \le s \le 4$.

15-20. Scalar line integrals in the plane

a. *Find a parametric description for C in the form* $\mathbf{r}(t) = \langle x(t),\ y(t) \rangle$, *if it is not given.*

b. *Evaluate* $|\mathbf{r}\,'(t)|$.

c. *Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.*

15. $\int_C \left(x^2 + y^2\right) ds$; C is the circle of radius 4 centered at (0, 0).

16. $\int_C \left(x^2 + y^2\right) ds$; C is the line segment from (0, 0) to (5, 5).

17. $\int_C \frac{x}{x^2 + y^2}\, ds$; C is the line segment from (1, 1) to (10, 10).

18. $\int_C (x y)^{1/3}\, ds$; C is the curve $y = x^2$, for $0 \le x \le 1$.

19. $\int_C (x - y)\, ds$; C is the upper half of an ellipse, $x = 2 \cos t$, $y = 4 \sin t$, for $0 \le t \le \pi$.

20. $\int_C (2\, x - 3\, y)\, ds$; C is the line segment from $(-1, 0)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(1, 0)$.

21-24. Average values *Find the average value of the following functions on the given curves.*

21. $f(x,\ y) = x + 2\, y$ on the line segment from (1, 1) to (2, 5)

22. $f(x,\ y) = x^2 + 4\, y^2$ on the circle of radius 9 centered at the origin

23. $f(x,\ y) = 4\, x^3 - 3\, y$ on the curve $x = y^3$, for $-1 \le y \le 1$

24. $f(x, y) = x\, e^{y}$ on the circle of radius 1 centered at the origin

25-30. Scalar line integrals in $\mathbf{R}^3$ *Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.*

25. $\displaystyle\int_C (x + y + z)\, ds$; C is the circle $\mathbf{r}(t) = \langle 2\cos t, 0, 2\sin t\rangle$, for $0 \le t \le 2\pi$.

26. $\displaystyle\int_C (x - y + 2z)\, ds$; C is the circle $\mathbf{r}(t) = \langle 1, 3\cos t, 3\sin t\rangle$, for $0 \le t \le 2\pi$.

27. $\displaystyle\int_C x\,y\,z\, ds$; C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$.

28. $\displaystyle\int_C \frac{x\,y}{z}\, ds$; C is the line segment from $(1, 4, 1)$ to $(3, 6, 3)$.

29. $\displaystyle\int_C (y - z)\, ds$; C is the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$, for $0 \le t \le 2\pi$.

30. $\displaystyle\int_C x\, e^{y\,z}\, ds$; C is $\mathbf{r}(t) = \langle t, 2t, -4t\rangle$, for $1 \le t \le 2$.

31-32. Length of curves *Use a scalar line integral to find the length of the following curves.*

31. $\mathbf{r}(t) = \langle 20\sin t/4, 20\cos t/4, t/2\rangle$, for $0 \le t \le 2$

32. $\mathbf{r}(t) = \langle 30\sin t, 40\sin t, 50\cos t\rangle$, for $0 \le t \le 2\pi$

33-38. Line integrals of vector fields in the plane *Given the following vector fields and oriented curves C, evaluate* $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\, ds$.

33. $\mathbf{F} = \langle x, y\rangle$ on the parabola $\mathbf{r}(t) = \left\langle 4t, t^2\right\rangle$, for $0 \le t \le 1$

34. $\mathbf{F} = \langle -y, x\rangle$ on the semicircle $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, for $0 \le t \le \pi$

35. $\mathbf{F} = \langle y, x\rangle$ on the line segment from $(1, 1)$ to $(5, 10)$

36. $\mathbf{F} = \dfrac{\langle x, y\rangle}{\left(x^2 + y^2\right)^{3/2}}$ on the line segment from $(2, 2)$ to $(10, 10)$

37. $\mathbf{F} = \dfrac{\langle x, y\rangle}{\left(x^2 + y^2\right)^{3/2}}$ on the curve $\mathbf{r}(t) = \left\langle t^2, 3t^2\right\rangle$, for $1 \le t \le 2$

38. $\mathbf{F} = \dfrac{\langle x, y\rangle}{x^2 + y^2}$ on the line $\mathbf{r}(t) = \langle t, 4t\rangle$, for $1 \le t \le 10$

39-42. Work integrals *Given the force field* **F**, *find the work required to move an object on the given oriented curve.*

39. $\mathbf{F} = \langle y, -x \rangle$ on the path consisting of the line segment from (1, 2) to (0, 0) followed by the line segment from (0, 0) to (0, 4)

40. $\mathbf{F} = \langle x, y \rangle$ on the path consisting of the line segment from (−1, 0) to (0, 8) followed by the line segment from (0, 8) to (2, 8)

41. $\mathbf{F} = \langle y, x \rangle$ on the parabola $y = 2x^2$ from (0, 0) to (2, 8)

42. $\mathbf{F} = \langle y, -x \rangle$ on the line $y = 10 - 2x$ from (1, 8) to (3, 4)

43-46. Work integrals in $\mathbb{R}^3$ *Given the force field* **F**, *find the work required to move an object on the given oriented curve.*

43. $\mathbf{F} = \langle x, y, z \rangle$ on the tilted ellipse $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 4\cos t \rangle$, for $0 \le t \le 2\pi$

44. $\mathbf{F} = \langle -y, x, z \rangle$ on the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t/2\pi \rangle$, for $0 \le t \le 2\pi$

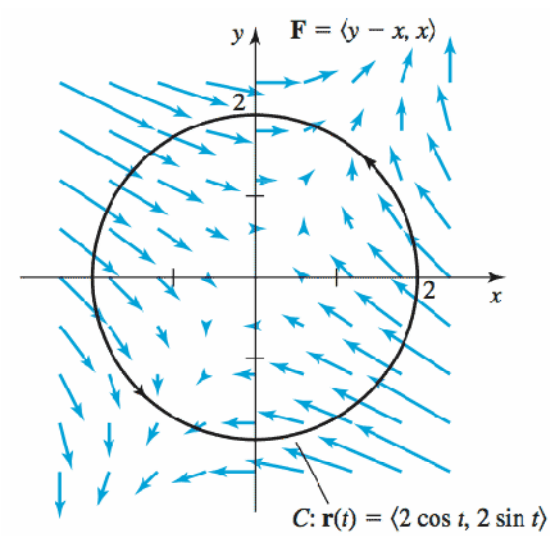
45. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from (1, 1, 1) to (10, 10, 10)

46. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$ on the line segment from (1, 1, 1) to (8, 4, 2)

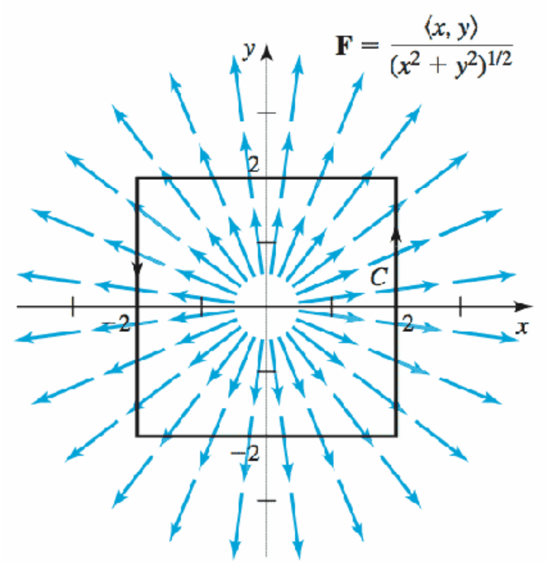
47-48. Circulation *Consider the following vector fields* **F** *and closed oriented curves C in the plane (see figures).*

- ***a.*** *Based on the picture, make a conjecture about whether the circulation of* **F** *on C is positive, negative, or zero.*
- ***b.*** *Compute the circulation and interpret the result.*

47. $\mathbf{F} = \langle y - x, x \rangle$; $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, for $0 \le t \le 2\pi$



48. $\mathbf{F} = \dfrac{\langle x, y \rangle}{\left(x^2 + y^2\right)^{1/2}}$; C is the boundary of the square with vertices $(\pm 2, \pm 2)$, traversed counterclockwise.



49-50. Flux *Consider the vector fields and curves in Exercises 47-48.*

a. *Based on the picture make a conjecture about whether the outward flux of* **F** *across C is positive, negative, or zero.*

b. *Compute the flux for the vector fields and curves.*

49. **F** and C given in Exercise 47

50. **F** and C given in Exercise 48

Further Explorations

51. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If a curve has a parametric description $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$, where t is the arc length, then $|\mathbf{r}'(t)| = 1$.

b. The vector field $\mathbf{F} = \langle y,\ x\rangle$ has both zero circulation along and zero flux across the unit circle centered at the origin.

c. If at all points of a path, a force acts in a direction orthogonal to the path, then no work is done in moving an object along the path.

d. The flux of a vector field across a curve in $\mathbb{R}^2$ can be computed using a line integral.

52. Flying into a headwind An airplane flies in the xz-plane, where x increases in the eastward direction and $z \ge 0$ represents vertical distance above the ground. A wind blows horizontally out of the west, producing a force $\mathbf{F} = \langle 150,\ 0\rangle$. On which path between the points (100, 0) and (−100, 0) is the most work done overcoming the wind:

a. The straight line $\mathbf{r}(t) = \langle x(t),\ z(t)\rangle = \langle -t,\ 0\rangle$, for $-100 \le t \le 100$ or

b. The arc of a circle $\mathbf{r}(t) = \langle 100\cos t,\ 100\sin t\rangle$, for $0 \le t \le \pi$?

53. Flying into a headwind

a. How does the result of Exercise 52 change if the force due to the wind is $\mathbf{F} = \langle 141,\ 50\rangle$ (approximately the same magnitude, but different direction)?

b. How does the result of Exercise 52 change if the force due to the wind is $\mathbf{F} = \langle 141,\ -50\rangle$ (approximately the same magnitude, but different direction)?

54. Changing orientation Let $f(x,\ y) = x + 2y$ and let C be the unit circle.

a. Find a parameterization of C with a counterclockwise orientation and evaluate $\displaystyle\int_C f\,ds$.

b. Find a parameterization of C with a clockwise orientation and evaluate $\int_C f\,ds$.

c. Compare the results of (a) and (b).

55. Changing orientation Let $f(x, y) = x$ and let C be the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

a. Find a parameterization of C in the direction from O to P. Evaluate $\int_C f\,ds$.

b. Find a parameterization of C in the direction from P to O. Evaluate $\int_C f\,ds$.

c. Compare the results of (a) and (b).

56-57. Zero circulation fields

56. For what values of b and c does the vector field $\mathbf{F} = \langle b\,y,\ c\,x\rangle$ have zero circulation on the unit circle centered at the origin and oriented counterclockwise?

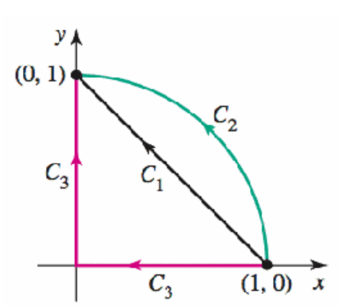
57. Consider the vector field $\mathbf{F} = \langle a\,x + b\,y,\ c\,x + d\,y\rangle$. Show that $\mathbf{F}$ has zero circulation on any circle centered at the origin and oriented counterclockwise, for any a, b, c, and d, provided $b = c$.

58-59. Zero flux fields

58. For what values of a and d does the vector field $\mathbf{F} = \langle a\,x,\ d\,y\rangle$ have zero flux across the unit circle centered at the origin and oriented counterclockwise?

59. Consider the vector field $\mathbf{F} = \langle a\,x + b\,y,\ c\,x + d\,y\rangle$. Show that $\mathbf{F}$ has zero flux across any circle centered at the origin and oriented counterclockwise for any a, b, c, and d, provided $a = -d$.

60. Work in a rotation field Consider the rotation field $\mathbf{F} = \langle -y,\ x\rangle$ and the three paths shown in the figure. Compute the work done on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is independent of the path, where C is a path from $(1, 0)$ to $(0, 1)$?



61. Work in a hyperbolic field Consider the hyperbolic force field $\mathbf{F} = \langle y,\ x\rangle$ (the streamlines are hyperbolas) and the three paths shown in the figure for Exercise 60. Compute the work done on each of the three paths. Does it appear that the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is independent of the path, where C is a path from $(1, 0)$ to $(0, 1)$?

Applications

62-63. Mass and density *A thin wire represented by the smooth curve C with a density ρ (units of mass per length) has a mass* $M = \int_C \rho\, ds$. *Find the mass of the following wires with the given density.*

62. $C:\ \mathbf{r}(\theta) = \langle \cos\theta,\ \sin\theta \rangle$, for $0 \le \theta \le \pi$; $\rho(\theta) = 2\,\theta/\pi + 1$

T **63.** $C:\ \{(x,\ y) : y = 2\,x^2,\ 0 \le x \le 3\}$; $\rho(x,\ y) = 1 + x\,y$

64. Heat flux in a plate A square plate $R = \{(x,\ y) :\ 0 \le x \le 1,\ 0 \le y \le 1\}$ has a temperature distribution $T(x,\ y) = 100 - 50\,x - 25\,y$.

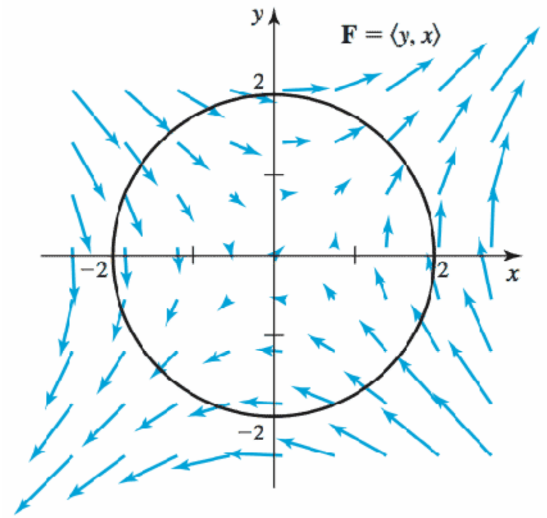
- **a.** Sketch two level curves of the temperature in the plate.
- **b.** Find the gradient of the temperature $\nabla T(x,\ y)$.
- **c.** Assume that the flow of heat is determined by the vector field $\mathbf{F} = -\nabla T(x,\ y)$. Compute $\mathbf{F}$.
- **d.** Find the outward heat flux across the boundary $\{(x,\ y) : x = 1,\ 0 \le y \le 1\}$.
- **e.** Find the outward heat flux across the boundary $\{(x,\ y) : 0 \le x \le 1,\ y = 1\}$.

65. Inverse force fields Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x,\ y,\ z \rangle}{|\mathbf{r}|^p}$, where $p > 1$ (the inverse square law corresponds to $p = 3$). Let C be the line from $(1,\ 1,\ 1)$ to $(a,\ a,\ a)$, where $a > 1$, given by $\mathbf{r}(t) = \langle t,\ t,\ t \rangle$, for $1 \le t \le a$.

- **a.** Find the work done in moving an object along C with $p = 2$.
- **b.** If $a \to \infty$ in part (a), is the work finite?
- **c.** Find the work done in moving an object moving along C with $p = 4$.
- **d.** If $a \to \infty$ in part (c), is the work finite?
- **e.** Find the work done in moving an object moving along C for any $p > 1$.
- **f.** If $a \to \infty$ in part (e), for what values of p is the work finite?

66. Flux across curves in a flow field Consider the flow field $\mathbf{F} = \langle y,\ x \rangle$ shown in the figure.

- **a.** Compute the outward flux across the quarter circle $C:\ \mathbf{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$, for $0 \le t \le \pi/2$.
- **b.** Compute the outward flux across the quarter circle $C:\ \mathbf{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$, for $\pi/2 \le t \le \pi$.
- **c.** Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part (a).
- **d.** Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part (b).
- **e.** What is the outward flux across the full circle?



Additional Exercises

67-68. Looking ahead: area from line integrals *The area of a region R in the plane, whose boundary is the curve C, may be computed using line integrals with the formula*

$$\text{area of } R = \int_C x\,dy = -\int_C y\,dx.$$

These ideas reappear later in the chapter.

67. Let R be the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, (a, b) and let C be the boundary of R oriented counterclockwise. Compute the area of R using the formula $A = \int_C x\,dy$.

68. Let $R = \{(r, \theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ be the disk of radius a centered at the origin and let C be the boundary of R oriented counterclockwise. Compute the area of R using the formula $A = -\int_C y\,dx$.