14.3 Conservative Vector Fields

This is an action-packed section in which several fundamental ideas come together. At the heart of the matter are two questions:

- When can a vector field be expressed as the gradient of a potential function? A vector field with this property will be defined as a *conservative* vector field.
- What special properties do conservative vector fields have?

After some preliminary definitions, we present a test to determine whether a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ is conservative. The test is followed by a procedure to find a potential function for a conservative field. We then develop several equivalent properties shared by all conservative vector fields.

Types of Curves and Regions

Test for Conservative Vector Fields

Finding Potential Functions

Fundamental Theorem for Line Integrals and Path Independence

Line Integrals on Closed Curves

Summary of the Properties of Conservative Vector Fields

Quick Quiz

SECTION 14.3 EXERCISES

Review Questions

1. Explain with pictures what is meant by a simple curve and a closed curve.

2. Explain with pictures what is meant by a connected region and a simply connected region.

3. How do you determine whether a vector field in $\mathbb{R}^2$ is conservative (has a potential function ϕ such that $\mathbf{F} = \nabla\phi$)?

4. How do you determine whether a vector field in $\mathbb{R}^3$ is conservative?

5. Briefly describe how to find a potential function ϕ for a conservative vector field $\mathbf{F} = \langle f, g \rangle$.

6. If $\mathbf{F}$ is a conservative vector field on a region R, how do you evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a path between two points A and B in R?

7. If $\mathbf{F}$ is a conservative vector field on a region R, what is the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is a simple closed smooth oriented curve in R?

8. Give three equivalent properties of conservative vector fields.

Basic Skills

9-14. Testing for conservative vector fields *Determine whether the following vector fields are conservative on* $\mathbb{R}^2$.

9. $\mathbf{F} = \langle 1, 1 \rangle$

10. $\mathbf{F} = \langle x, y \rangle$

11. $\mathbf{F} = \langle -y, -x \rangle$

12. $\mathbf{F} = \langle -y, x + y \rangle$

13. $\mathbf{F} = \langle e^{-x} \cos y, e^{-x} \sin y \rangle$

14. $\mathbf{F} = \left\langle 2x^3 + x y^2, 2y^3 + x^2 y \right\rangle$

15-26. Finding potential functions *Determine whether the following vector fields are conservative on the specified region. If so, determine a potential function. Let R^* and D^* be open regions of* $\mathbb{R}^2$ *and* $\mathbb{R}^3$*, respectively, that do not include the origin.*

15. $\mathbf{F} = \langle x, y \rangle$ on $\mathbb{R}^2$

16. $\mathbf{F} = \langle -y, -x \rangle$ on $\mathbb{R}^2$

17. $\mathbf{F} = \left\langle x^3 - x y, x^2 / 2 + y \right\rangle$ on $\mathbb{R}^2$

18. $\mathbf{F} = \dfrac{\langle x, y \rangle}{x^2 + y^2}$ on R^*

19. $\mathbf{F} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ on R^*

20. $\mathbf{F} = \langle y, x, 1 \rangle$ on $\mathbb{R}^3$

21. $\mathbf{F} = \langle z, 1, x \rangle$ on $\mathbb{R}^3$

22. $\mathbf{F} = \langle y z, x z, x y \rangle$ on $\mathbb{R}^3$

23. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$ on $\mathbb{R}^3$

24. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$ on D^*

25. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$ on D^*

26. $\mathbf{F} = \left\langle x^3, 2y, -z^3 \right\rangle$ on $\mathbb{R}^3$

27-32. Evaluating line integrals *Evaluate the line integral* $\displaystyle\int_C \nabla \phi \cdot d\mathbf{r}$ *for the following functions ϕ and oriented curves C in two ways.*

a. *Use a parametric description of C and evaluate the integral directly.*

b. *Use the Fundamental Theorem for line integrals.*

27. $\phi(x, y) = x y$; $C : \mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \le t \le \pi$

28. $\phi(x, y) = (x^2 + y^2)/2$; $C : \mathbf{r}(t) = \langle \sin t, \cos t \rangle$, for $0 \le t \le \pi$

29. $\phi(x, y) = x + 3y$; $C : \mathbf{r}(t) = \langle 2 - t, t \rangle$, for $0 \le t \le 2$

30. $\phi(x, y, z) = x + y + z$; $C : \mathbf{r}(t) = \langle \sin t, \cos t, t/\pi \rangle$, for $0 \le t \le \pi$

31. $\phi(x, y, z) = (x^2 + y^2 + z^2)/2$; $C : \mathbf{r}(t) = \langle \cos t, \sin t, t/\pi \rangle$, for $0 \le t \le 2\pi$

32. $\phi(x, y, z) = x y + x z + y z$; $C : \mathbf{r}(t) = \langle t, 2t, 3t \rangle$, for $0 \le t \le 4$

33-38. Line integrals of vector fields on closed curves *Evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *for the following vector fields and closed oriented curves C by parameterizing C. If the integral is not zero, give an explanation.*

33. $\mathbf{F} = \langle x, y \rangle$; C is the circle of radius 4 centered at the origin oriented counterclockwise.

34. $\mathbf{F} = \langle y, x \rangle$; C is the circle of radius 8 centered at the origin oriented counterclockwise.

35. $\mathbf{F} = \langle x, y \rangle$; C is the triangle with vertices $(0, \pm 1)$ and $(1, 0)$ oriented counterclockwise.

36. $\mathbf{F} = \langle y, -x \rangle$; C is the circle of radius 3 centered at the origin oriented counterclockwise.

37. $\mathbf{F} = \langle x, y, z \rangle$; $C : \mathbf{r}(t) = \langle \cos t, \sin t, 2 \rangle$, for $0 \le t \le 2\pi$

38. $\mathbf{F} = \langle y - z, z - x, x - y \rangle$; $C : \mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \le t \le 2\pi$

Further Explorations

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\mathbf{F} = \langle -y, x \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

b. If $\mathbf{F} = \langle x, -y \rangle$ and C is the circle of radius 4 centered at $(1, 0)$ oriented counterclockwise, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

c. A constant vector field is conservative on $\mathbb{R}^2$.

d. The vector field $\mathbf{F} = \langle f(x), g(y) \rangle$ is conservative on $\mathbb{R}^2$.

40-43. Line integrals *Evaluate the following line integrals using a method of your choice.*

40. $\int_C \nabla(1 + x^2 y z) \cdot d\mathbf{r}$, where C is the helix $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$, for $0 \le t \le 4\pi$

41. $\int_C \nabla(e^{-x} \cos y) \cdot d\mathbf{r}$, where C is the line from $(0, 0)$ to $(\ln 2, 2\pi)$

42. $\oint_C e^{-x}(\cos y\, dx + \sin y\, dy)$, where C is the square with vertices $(\pm 1, \pm 1)$ oriented counterclockwise

43. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2xy + z^2, x^2, 2xz\rangle$ and C is the circle $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle$, for $0 \le t \le 2\pi$.

44. Closed curve integrals Evaluate $\oint_C ds$, $\oint_C dx$, and $\oint_C dy$, where C is the unit circle oriented counterclockwise.

45-48. Work in force fields *Find the work required to move an object in the following force fields along a straight line between the given points. Check to see if the force is conservative.*

45. $\mathbf{F} = \langle x, 2\rangle$ from $A(0, 0)$ to $B(2, 4)$

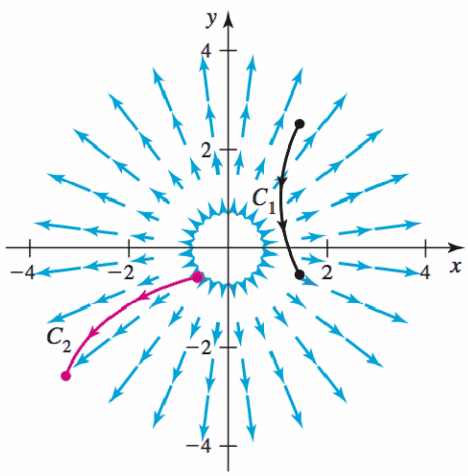
46. $\mathbf{F} = \langle x, y\rangle$ from $A(1, 1)$ to $B(3, -6)$

47. $\mathbf{F} = \langle x, y, z\rangle$ from $A(1, 2, 1)$ to $B(2, 4, 6)$

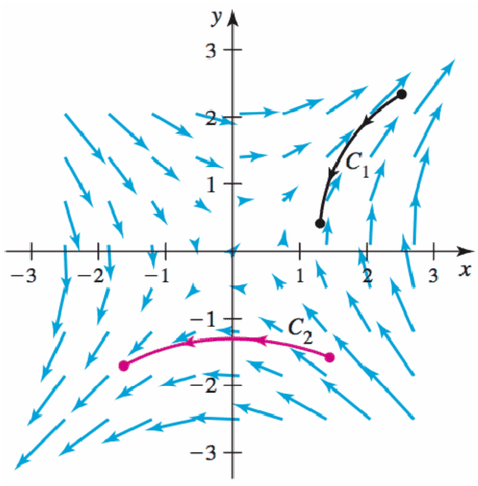
48. $\mathbf{F} = e^{x+y}\langle 1, 1, z\rangle$ from $A(0, 0, 0)$ to $B(-1, 2, -4)$

49-50. Work from graphs *Determine whether* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *along the paths* C_1 *and* C_2 *shown in the following vector fields is positive or negative. Explain your reasoning.*

49.



50.



Applications

51. Work by a constant force Evaluate a line integral to show that the work done in moving an object from point A to point B in the presence of a constant force $\mathbf{F} = \langle a, b, c\rangle$ is $\mathbf{F}\cdot\overrightarrow{AB}$.

52. Conservation of energy Suppose an object with mass m moves in a conservative force field given by $\mathbf{F} = -\nabla\phi$, where ϕ is a potential function in a region R. The motion of the object is governed by Newton's Second Law of Motion, $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{a}$ is the acceleration. Suppose the object moves (either in the plane or in space) from point A to point B in R.

a. Show that the equation of motion is $m\dfrac{d\mathbf{v}}{dt} = -\nabla\phi$.

b. Show that $\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \dfrac{1}{2}\dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v})$.

c. Take the dot product of both sides of the equation in part (a) with $\mathbf{v}(t) = \mathbf{r}'(t)$ and integrate along a curve between A and B. Use part (b) and the fact that $\mathbf{F}$ is conservative to show that the total energy (kinetic plus potential) $\dfrac{1}{2}m|\mathbf{v}|^2 + \phi$ is the same at A and B. Conclude that because A and B are arbitrary, energy is conserved in R.

53. Gravitational potential The gravitational force between two point masses M and m is

$$\mathbf{F} = GMm\frac{\mathbf{r}}{|\mathbf{r}|^3} = GMm\frac{\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}},$$

where G is the gravitational constant.

a. Verify that this force field is conservative on any region excluding the origin.

b. Find a potential function ϕ for this force field such that $\mathbf{F} = -\nabla\phi$.

c. Suppose the object with mass m is moved from a point A to a point B, where A is a distance r_1 from M and B is a distance r_2 from M. Show that the work done in moving the object is $GMm\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$.

d. Does the work depend on the path between A and B? Explain.

Additional Exercises

54. Radial fields in $\mathbb{R}^3$ are conservative Prove that the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number, is conservative on any region not containing the origin. For what values of p is $\mathbf{F}$ conservative on a region that contains the origin?

55. Rotation fields are usually not conservative

a. Prove that the rotation field $\mathbf{F} = \dfrac{\langle -y, x \rangle}{|\mathbf{r}|^p}$, where $\mathbf{r} = \langle x, y \rangle$ is not conservative for $p \neq 2$.

b. For $p = 2$, show that $\mathbf{F}$ is conservative on any region not containing the origin.

c. Find a potential function for $\mathbf{F}$ when $p = 2$.

56. Linear and quadratic vector fields

a. For what values of a, b, c, and d is the field $\mathbf{F} = \langle a\,x + b\,y, c\,x + d\,y \rangle$ conservative?

b. For what values of a, b, and c is the field $\mathbf{F} = \left\langle a\,x^2 - b\,y^2, c\,x\,y \right\rangle$ conservative?

57. Alternative construction of potential functions in $\mathbb{R}^2$ Assume that the vector field $\mathbf{F}$ is conservative on $\mathbb{R}^2$, so that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Use the following procedure to construct a potential function ϕ for the vector field $\mathbf{F} = \langle f, g \rangle = \langle 2\,x - y, -x + 2\,y \rangle$.

a. Let A be $(0, 0)$ and let B be an arbitrary point (x, y). Define $\phi(x, y)$ to be the work required to move an object from A to B, where $\phi(A) = 0$. Let C_1 be the path from A to $(x, 0)$ to B and let C_2 be the path from A to $(0, y)$ to B. Draw a picture.

b. Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} f\,dx + g\,dy$ and conclude that $\phi(x, y) = x^2 - x\,y + y^2$.

c. Verify that the same potential function is obtained by evaluating the line integral over C_2.

58-61. Alternative construction of potential functions *Use the procedure in Exercise 57 to construct potential functions for the following fields.*

58. $\mathbf{F} = \langle -y, -x \rangle$

59. $\mathbf{F} = \langle x, y \rangle$

60. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r} = \langle x, y \rangle$

61. $\mathbf{F} = \left\langle 2\,x^3 + x\,y^2, 2\,y^3 + x^2\,y \right\rangle$