14.4 Green' s Theorem

The preceding section gave a version of the Fundamental Theorem of Calculus that applies to line integrals. In this and the remaining sections of the book, you will see additional extensions of the Fundamental Theorem that apply to regions in $\mathbb{R}^2$ and $\mathbb{R}^3$. All these fundamental theorems share a common feature.

Part 2 of the Fundamental Theorem of Calculus (Chapter 5) says

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a),$$

which relates the integral of $\frac{df}{dx}$ on an interval $[a, b]$ to the values of f on the boundary of $[a, b]$. The Fundamental Theorem for line integrals says

$$\int_C \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A),$$

which relates the integral of $\nabla\phi$ on a smooth oriented curve C to the boundary values of ϕ. (The boundary consists of the two endpoints A and B.)

The subject of this section is Green's Theorem, which is another step in this progression. It relates the double integral of derivatives of a function over a region in $\mathbb{R}^2$ to function values on the boundary of that region.

Circulation Form of Green's Theorem

Flux Form of Green's Theorem

Circulation and Flux on More General Regions

Stream Functions

Proof of Green's Theorem on Special Regions

Quick Quiz

SECTION 14.4 EXERCISES

Review Questions

1. Explain why the two forms of Green's Theorem are analogs of the Fundamental Theorem of Calculus.

2. Referring to both forms of Green's Theorem, match each idea in Column 1 to an idea in Column 2:

Line integral for flux	Double integral of the curl
Line integral for circulation	Double integral of the divergence

3. Compute the two-dimensional curl of $\mathbf{F} = \langle 4x^3 y, x y^2 + x^4 \rangle$.

4. Compute the two-dimensional divergence of $\mathbf{F} = \langle 4x^3 y,\ x y^2 + x^4 \rangle$.

5. How do you use a line integral to compute the area of a plane region?

6. Why does a two-dimensional vector field with zero curl on a region have zero circulation on a closed curve that bounds the region?

7. Why does a two-dimensional vector field with zero divergence on a region have zero flux across a closed curve that bounds the region?

8. Sketch a two-dimensional vector field that has zero curl everywhere in the plane.

9. Sketch a two-dimensional vector field that has zero divergence everywhere in the plane.

10. Discuss one of the parallels between a conservative vector field and a source-free vector field.

Basic Skills

11-16. Green's Theorem, circulation form *Consider the following regions R and vector fields* **F**.

- ***a.*** *Compute the two-dimensional curl of the vector field.*
- ***b.*** *Evaluate both integrals in Green's Theorem and check for consistency.*
- ***c.*** *State whether the vector field is conservative.*

11. $\mathbf{F} = \langle x,\ y \rangle$; $R = \{(x,\ y) : x^2 + y^2 \le 2\}$

12. $\mathbf{F} = \langle y,\ x \rangle$; R is the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$.

13. $\mathbf{F} = \langle 2y,\ -2x \rangle$; R is the region bounded by $y = \sin x$ and $y = 0$, for $0 \le x \le \pi$.

14. $\mathbf{F} = \langle -3y,\ 3x \rangle$; R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, $(0,\ 2)$.

15. $\mathbf{F} = \langle 2xy,\ x^2 - y^2 \rangle$; R is the region bounded by $y = x(2 - x)$ and $y = 0$.

16. $\mathbf{F} = \langle 0,\ x^2 + y^2 \rangle$; $R = \{(x,\ y) : x^2 + y^2 \le 1\}$.

17-22. Area of regions *Use a line integral on the boundary to find the area of the following regions.*

17. A disk of radius 5

18. A region bounded by an ellipse with semimajor and semiminor axes of length 6 and 4, respectively.

19. $\{(x,\ y) : x^2 + y^2 \le 16\}$

20. $\{(x,\ y) : x^2/25 + y^2/9 \le 1\}$

21. The region bounded by the parabolas $\mathbf{r}(t) = \langle t,\ 2t^2 \rangle$ and $\mathbf{r}(t) = \langle t,\ 12 - t^2 \rangle$, for $-2 \le t \le 2$

22. The region bounded by the curve $\mathbf{r}(t) = \langle t(1 - t^2),\ 1 - t^2 \rangle$, for $-1 \le t \le 1$ (*Hint:* Plot the curve.)

23-28. Green's Theorem, flux form *Consider the following regions R and vector fields* **F**.

- ***a.*** *Compute the two-dimensional divergence of the vector field.*
- ***b.*** *Evaluate both integrals in Green's Theorem and check for consistency.*
- ***c.*** *State whether the vector field is source-free.*

23. $\mathbf{F} = \langle x,\ y\rangle$; $R = \left\{(x,\ y) : x^2 + y^2 \le 4\right\}$

24. $\mathbf{F} = \langle y,\ -x\rangle$; R is the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$.

25. $\mathbf{F} = \langle y,\ -3\,x\rangle$; R is the region bounded by $y = 4 - x^2$ and $y = 0$.

26. $\mathbf{F} = \langle -3\,y,\ 3\,x\rangle$; R is the triangle with vertices $(0,\ 0)$, $(3,\ 0)$, $(0,\ 1)$.

27. $\mathbf{F} = \left\langle 2\,x\,y,\ x^2 - y^2\right\rangle$; R is the region bounded by $y = x\,(2 - x)$ and $y = 0$.

28. $\mathbf{F} = \left\langle x^2 + y^2,\ 0\right\rangle$; $R = \left\{(x,\ y) : x^2 + y^2 \le 1\right\}$.

29-34. Line integrals *Use Green's Theorem to evaluate the following line integrals. Unless stated otherwise, assume all curves are oriented counterclockwise.*

29. $\oint_C \left(2\,x + e^{y^2}\right) d\,y - \left(4\,y^2 + e^{x^2}\right) d\,x$, where C is the boundary of the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$

30. $\int_C (2\,x - 3\,y)\,d\,y - (3\,x + 4\,y)\,d\,x$, where C is the unit circle

31. $\int_C f\,d\,y - g\,d\,x$, where $\langle f,\ g\rangle = \langle 0,\ x\,y\rangle$ and C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 4)$

32. $\oint_C f\,d\,y - g\,d\,x$, where $\langle f,\ g\rangle = \left\langle x^2,\ 2\,y^2\right\rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$ oriented *clockwise*

33. The circulation line integral of $\mathbf{F} = \left\langle 2\,x\,y^2 + x,\ 4\,x^3 + y\right\rangle$, where C is the boundary of $\{(x,\ y) : 0 \le y \le \sin x,\ 0 \le x \le \pi\}$

34. The flux line integral of $\mathbf{F} = \langle e^{x-y},\ e^{y-x}\rangle$, where C is the boundary of $\{(x,\ y) : 0 \le y \le x,\ 0 \le x \le 1\}$

35-38. General regions *For the following vector fields, compute (a) the circulation on and (b) the outward flux across the boundary of the given region. Assume boundary curves are oriented counterclockwise.*

35. $\mathbf{F} = \langle x,\ y\rangle$; R is the half annulus $\{(r,\ \theta);\ 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

36. $\mathbf{F} = \langle -y,\ x\rangle$; R is the annulus $\{(r,\ \theta) : 1 \le r \le 3,\ 0 \le \theta \le 2\,\pi\}$.

37. $\mathbf{F} = \langle 2\,x + y,\ x - 4\,y\rangle$; R is the quarter annulus $\{(r,\ \theta) : 1 \le r \le 4,\ 0 \le \theta \le \pi/2\}$.

38. $\mathbf{F} = \langle x - y,\ -x + 2\,y\rangle$; R is the parallelogram $\{(x,\ y) : 1 - x \le y \le 3 - x,\ 0 \le x \le 1\}$.

Further Explorations

39. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The work required to move an object around a closed curve C in the presence of a vector force field is the circulation of the vector field on the curve.

b. If a vector field has zero divergence throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is zero.

c. If the two-dimensional curl of a vector field is positive throughout a region (on which the conditions of Green's Theorem are met), then the circulation on the boundary of that region is positive (assuming counterclockwise orientation).

40-43. Circulation and flux *For the following vector fields, compute (a) the circulation on and (b) the outward flux across the boundary of the given region. Assume boundary curves have counterclockwise orientation.*

40. $\mathbf{F} = \left\langle \ln\left(x^2 + y^2\right), \tan^{-1}\left(\frac{y}{x}\right)\right\rangle$, where R is the annulus $\{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

41. $\mathbf{F} = \nabla\left(\sqrt{x^2 + y^2}\right)$, where R is the half annulus $\{(r, \theta) : 1 \le r \le 3,\ 0 \le \theta \le \pi\}$

42. $\mathbf{F} = \langle y \cos x, -\sin x\rangle$, where R is the square $\{(x, y) : 0 \le x \le \pi/2,\ 0 \le y \le \pi/2\}$

43. $\mathbf{F} = \left\langle x + y^2, x^2 - y\right\rangle$, where $R = \left\{(x, y) : 3y^2 \le x \le 36 - y^2\right\}$

44-45. Special line integrals *Prove the following identities, where C is a simple closed smooth oriented curve.*

44. $\oint_C dx = \oint_C dy = 0$

45. $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C

46. Double integral to line integral Use the flux form of Green's Theorem to evaluate $\iint_R \left(2xy + 4y^3\right) dA$, where R is the triangle with vertices (0, 0), (1, 0), and (0, 1).

47. Area line integral Show that the value of

$$\oint_C x y^2\,dx + \left(x^2 y + 2x\right) dy$$

depends only on the area of the region enclosed by C.

48. Area line integral In terms of the parameters a and b, how is the value of $\oint_C a y\,dx + b x\,dy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

49-52. Stream function *Recall that if the vector field* $\mathbf{F} = \langle f, g\rangle$ *is source-free (zero divergence), then a stream function* ψ *exists such that* $f = \psi_y$ *and* $g = -\psi_x$.

a. *Verify that the given vector field has zero divergence.*

b. *Integrate the relations* $f = \psi_y$ *and* $g = -\psi_x$ *to find a stream function for the field.*

49. $\mathbf{F} = \langle 4, 2\rangle$

50. $\mathbf{F} = \left\langle y^2, x^2\right\rangle$

51. $\mathbf{F} = \langle -e^{-x} \sin y, e^{-x} \cos y\rangle$

52. $\mathbf{F} = \left\langle x^2, -2xy\right\rangle$

Applications

53-56. Ideal flow *A two-dimensional vector field describes ideal flow if it has both zero curl and zero divergence on a simply connected region.*

a. *Verify that the curl and divergence of the given field is zero.*

b. *Find a potential function ϕ and a stream function ψ for the field.*

c. *Verify that ϕ and ψ satisfy Laplace's equation $\phi_{xx} + \phi_{yy} = \psi_{xx} + \psi_{yy} = 0$.*

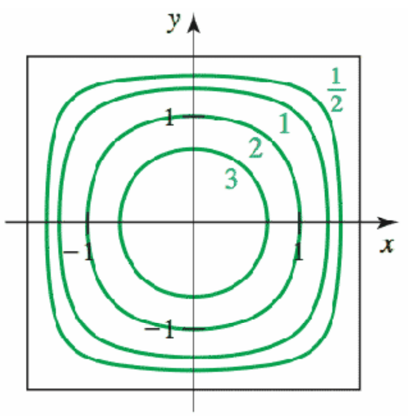
53. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

54. $\mathbf{F} = \left\langle x^3 - 3xy^2, y^3 - 3x^2 y \right\rangle$

55. $\mathbf{F} = \left\langle \tan^{-1}(y/x), \dfrac{1}{2} \ln\left(x^2 + y^2\right) \right\rangle$

56. $\mathbf{F} = \dfrac{\langle x, y \rangle}{x^2 + y^2}$

57. Flow in an ocean basin An idealized two-dimensional ocean is modeled by the square region $R = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ with boundary C. Consider the stream function $\psi(x, y) = 4 \cos x \cos y$ defined on R (see figure.)



a. The horizontal (east-west) component of the velocity is $u = \psi_y$ and the vertical (north-south) component of the velocity is $v = -\psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.

b. Is the velocity field source-free? Explain.

c. Is the velocity field irrotational? Explain.

d. Let C be the boundary of R. Find the total outward flux across C.

e. Find the circulation around C assuming counterclockwise orientation.

Additional Exercises

58. Green's Theorem as a Fundamental Theorem of Calculus Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0, \dfrac{f(x)}{c} \right\rangle$ and $R = \{(x, y) : a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus, $\displaystyle\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$.

59. Green's Theorem as a Fundamental Theorem of Calculus Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c}, 0 \right\rangle$ and $R = \{(x, y) : a \le x \le b, 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus, $\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$.

60. What's wrong? Consider the rotation field $\mathbf{F} = \frac{\langle -y, x \rangle}{x^2 + y^2}$.

- **a.** Verify that the two-dimensional curl of $\mathbf{F}$ is zero, which suggests that the double integral in the circulation form of Green's Theorem is zero.
- **b.** Use a line integral to verify that the circulation on the unit circle of the vector field is 2π.
- **c.** Explain why the results of parts (a) and (b) do not agree.

61. What's wrong? Consider the radial field $\mathbf{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$.

- **a.** Verify that the divergence of $\mathbf{F}$ is zero, which suggests that the double integral in the flux form of Green's Theorem is zero.
- **b.** Use a line integral to verify that the outward flux across the unit circle of the vector field is 2π.
- **c.** Explain why the results of parts (a) and (b) do not agree.

62. Conditions for Green's Theorem Consider the radial field $\mathbf{F} = \langle f, g \rangle = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

- **a.** Explain why the conditions of Green's Theorem do not apply to $\mathbf{F}$ on a region that includes the origin.
- **b.** Let R be the unit disk centered at the origin and compute $\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$.
- **c.** Evaluate the line integral in the flux form of Green's Theorem on the boundary of R.
- **d.** Do the results of parts (b) and (c) agree? Explain.

63. Flux integrals Assume the vector field $\mathbf{F} = \langle f, g \rangle$ is source-free (zero divergence) with stream function ψ. Let C be any smooth simple curve from A to the distinct point B. Show that the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ is independent of path; that is, $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$.

64. Streamlines are tangent to the vector field Assume that the vector field $\mathbf{F} = \langle f, g \rangle$ is related to the stream function ψ by $\psi_y = f$ and $\psi_x = -g$ on a region R. Prove that at all points of R, the vector field is tangent to the streamlines (the level curves of the stream function).

65. Streamlines and equipotential lines Assume that on $\mathbb{R}^2$ the vector field $\mathbf{F} = \langle f, g \rangle$ has a potential function ϕ such that $f = \phi_x$ and $g = \phi_y$, and it has a stream function ψ such that $f = \psi_y$ and $g = -\psi_x$. Show that the equipotential curves (level curves of ϕ) and the streamlines (level curves of ψ) are everywhere orthogonal.

66. Channel flow The flow in a long shallow channel is modeled by the velocity field $\mathbf{F} = \langle 0, 1 - x^2 \rangle$, where $R = \{(x, y) : |x| \le 1 \text{ and } |y| < \infty\}$.

- **a.** Sketch R and several streamlines of $\mathbf{F}$.
- **b.** Evaluate the curl of $\mathbf{F}$ on the lines $x = 0$, $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = 1$.

c. Compute the circulation on the boundary of the region $R = \{(x,\ y) : |x| \le 1,\ 0 \le y \le 1\}$.

d. How do you explain the fact that the curl of **F** is nonzero at points of R, but the circulation is zero?