14.5 Divergence and Curl

Green's Theorem sets the stage for the final act in our exploration of calculus. The last four sections of the book have the following goal: to lift both forms of Green's Theorem out of the plane ($\mathbb{R}^2$) and into space ($\mathbb{R}^3$). It is done as follows.

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. In an analogous manner, we will see that *Stokes' Theorem* (Section 14.7) relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface bounded by that curve.
- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the *Divergence Theorem* (Section 14.8) relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by that surface.

In order to make these extensions, we need a few more tools.

- The two-dimensional divergence and two-dimensional curl must be extended to three dimensions (this section).
- The idea of a *surface integral* must be introduced (Section 14.6).

The Divergence

The Curl

Working with Divergence and Curl

Summary of Properties of Conservative Vector Fields

Quick Quiz

SECTION 14.5 EXERCISES

Review Questions

1. Explain how to compute the divergence of the vector field $\mathbf{F} = \langle f, g, h \rangle$.

2. Interpret the divergence of a vector field.

3. What does it mean if the divergence of a vector field is zero throughout a region?

4. Explain how to compute the curl of the vector field $\mathbf{F} = \langle f, g, h \rangle$.

5. Interpret the curl of a general rotation vector field.

6. What does it mean if the curl of a vector field is zero throughout a region?

7. What is the value of $\nabla \cdot (\nabla \times \mathbf{F})$?

8. What is the value of $\nabla \times \nabla u$?

Basic Skills

9-16. Divergence of vectors fields *Find the divergence of the following vector fields.*

9. $\mathbf{F} = \langle 2x, 4y, -3z \rangle$

10. $\mathbf{F} = \langle -2\,y,\ 3\,x,\ z \rangle$

11. $\mathbf{F} = \langle 12\,x,\ -6\,y,\ -6\,z \rangle$

12. $\mathbf{F} = \left\langle x^2\,y\,z,\ -x\,y^2\,z,\ -x\,y\,z^2 \right\rangle$

13. $\mathbf{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

14. $\mathbf{F} = \langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \rangle$

15. $\mathbf{F} = \dfrac{\langle x,\ y,\ z \rangle}{1 + x^2 + y^2}$

16. $\mathbf{F} = \langle y z \sin x,\ \ x z \cos y,\ x y \cos z \rangle$

17-20. Divergence of radial fields *Calculate the divergence of the following radial fields. Express the result in terms of the position vector* **r** *and its length* |**r**|. *Check for agreement with Theorem 14.8.*

17. $\mathbf{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2 + y^2 + z^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

18. $\mathbf{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

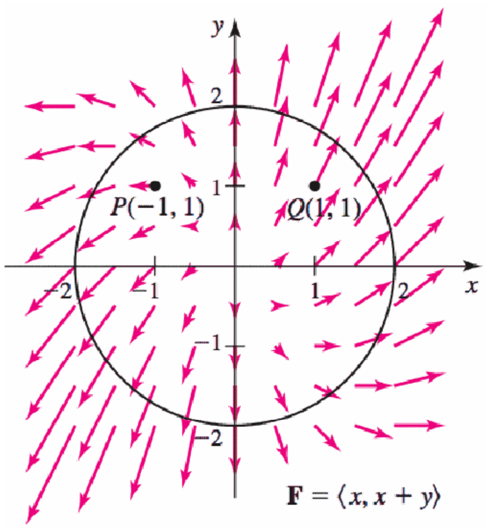
19. $\mathbf{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^2} = \dfrac{\mathbf{r}}{|\mathbf{r}|^4}$

20. $\mathbf{F} = \langle x,\ y,\ z \rangle \left(x^2 + y^2 + z^2\right) = \mathbf{r}\,|\mathbf{r}|^2$

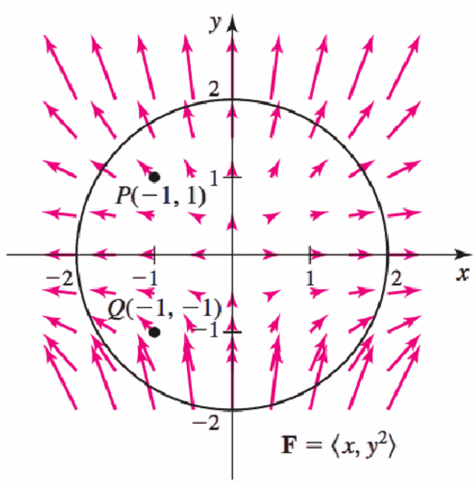
21-22. Divergence and flux from graphs *Consider the following vector fields, the circle C, and two points P and Q.*

- ***a.*** *Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q? Justify your answer.*
- ***b.*** *Compute the divergence and confirm your conjecture in part (a).*
- ***c.*** *On what part of C is the flux outward? Inward?*
- ***d.*** *Is the net outward flux across C positive or negative?*

21. $\mathbf{F} = \langle x,\ x + y \rangle$



22. $\mathbf{F} = \langle x,\ y^2 \rangle$



23-26. Curl of a rotational field *Consider the following vector fields, where* $\mathbf{r} = \langle x,\ y,\ z \rangle$.

a. *Compute the curl of the field and verify that it has the same direction as the axis of rotation.*

b. *Compute the magnitude of the curl of the field.*

23. $\mathbf{F} = \langle 1,\ 0,\ 0 \rangle \times \mathbf{r}$

24. $\mathbf{F} = \langle 1,\ -1,\ 0 \rangle \times \mathbf{r}$

25. $\mathbf{F} = \langle 1,\ -1,\ 1 \rangle \times \mathbf{r}$

26. $\mathbf{F} = \langle 1,\ -2,\ -3 \rangle \times \mathbf{r}$

27-34. Curl of a vector field *Compute the curl of the following vector fields.*

27. $\mathbf{F} = \langle x^2 - y^2,\ xy,\ z \rangle$

28. $\mathbf{F} = \langle 0,\ z^2 - y^2,\ -yz \rangle$

29. $\mathbf{F} = \langle x^2 - z^2, 1, 2xz \rangle$

30. $\mathbf{F} = \mathbf{r} = \langle x, y, z \rangle$

31. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

32. $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{1/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$

33. $\mathbf{F} = \langle z^2 \sin y, x z^2 \cos y, 2xz \sin y \rangle$

34. $\mathbf{F} = \left\langle 3xz^3 e^{y^2}, 2xz^3 e^{y^2}, 3xz^2 e^{y^2} \right\rangle$

35-38. Derivative rules *Prove the following identities. Use Theorem 14.11 (Product Rule) whenever possible.*

35. $\nabla\left(\dfrac{1}{|\mathbf{r}|^3}\right) = \dfrac{-3\,\mathbf{r}}{|\mathbf{r}|^5}$ (used in Example 5)

36. $\nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = \dfrac{-2\,\mathbf{r}}{|\mathbf{r}|^4}$

37. $\nabla \cdot \nabla\left(\dfrac{1}{|\mathbf{r}|^2}\right) = \dfrac{2}{|\mathbf{r}|^4}$ (use Exercise 36)

38. $\nabla(\ln |\mathbf{r}|) = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$

Further Explorations

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. For a function f of a single variable, if $f'(x) = 0$ for all x in the domain, then f is a constant function. If $\nabla \cdot \mathbf{F} = 0$ for all points in the domain, then $\mathbf{F}$ is constant.

b. If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is constant.

c. A vector field consisting of parallel vectors has zero curl.

d. A vector field consisting of parallel vectors has zero divergence.

e. curl $\mathbf{F}$ is orthogonal to $\mathbf{F}$.

40. Another derivative combination Let $\mathbf{F} = \langle f, g, h \rangle$ and let u be a differentiable scalar-valued function.

a. Take the dot product of $\mathbf{F}$ and the del operator; then apply the result to u to show that

$$(\mathbf{F} \cdot \nabla)\, u = \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}\right) u = f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} + h \frac{\partial u}{\partial z}$$

b. Evaluate $(\mathbf{F} \cdot \nabla)(x y^2 z^3)$ at $(1, 1, 1)$, where $\mathbf{F} = \langle 1, 1, 1 \rangle$.

41. Does it make sense? Are the following expressions defined? If so, state whether the result is a scalar or a vector. Assume $\mathbf{F}$ is a sufficiently differentiable vector field and ϕ is a sufficiently differentiable scalar-valued function.

a. $\nabla \cdot \phi$

b. $\nabla \mathbf{F}$

c. $\nabla \cdot \nabla \phi$

d. $\nabla (\nabla \cdot \phi)$

e. $\nabla (\nabla \times \phi)$

f. $\nabla \cdot (\nabla \cdot \mathbf{F})$

g. $\nabla \times \nabla \phi$

h. $\nabla \times (\nabla \cdot \mathbf{F})$

i. $\nabla \times (\nabla \times \mathbf{F})$

42. Zero divergence of the rotation field Show that the general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z \rangle$, has zero divergence.

43. Curl of the rotation field For the general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z \rangle$, show that curl $\mathbf{F} = 2\,\mathbf{a}$.

44. Inward to outward Find the exact points on the circle $x^2 + y^2 = 4$ at which the field $\mathbf{F} = \langle f, g \rangle = \langle x^2, y \rangle$ switches from pointing inward to outward on the circle, or vice versa.

45. Maximum divergence Within the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$, where does div $\mathbf{F}$ have the greatest magnitude when $\mathbf{F} = \langle x^2 - y^2, x y^2 z, 2 xz \rangle$?

46. Maximum curl Let $\mathbf{F} = \langle z, 0, -y \rangle$.

a. What is the component of curl $\mathbf{F}$ in the direction $\mathbf{n} = \langle 1, 0, 0 \rangle$?

b. What is the component of curl $\mathbf{F}$ in the direction $\mathbf{n} = \langle 1, -1, 1 \rangle$?

c. In what direction $\mathbf{n}$ is $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ a maximum?

47. Zero component of the curl For what vectors $\mathbf{n}$ is $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 0$ when $\mathbf{F} = \langle y, -2z, -x \rangle$?

48-49. Find a vector field *Find a vector field* $\mathbf{F}$ *with the given curl. In each case, is the vector field you found unique?*

48. curl $\mathbf{F} = \langle 0, 1, 0 \rangle$

49. curl $\mathbf{F} = \langle 0, z, -y \rangle$

50. Curl and angular speed Consider the rotational velocity field $\mathbf{v} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a nonzero constant vector and $\mathbf{r} = \langle x, y, z \rangle$. Use the fact that an object moving in a circular path of radius R with speed $|\mathbf{v}|$ has an angular speed of $\omega = |\mathbf{v}| / R$.

a. Sketch a position vector $\mathbf{a}$, which is the axis of rotation for the vector field, and a position vector $\mathbf{r}$ of a point P in $\mathbb{R}^3$. Let θ be the angle between the two vectors. Show that the perpendicular distance from P to the axis of rotation is $R = |\mathbf{r}| \sin \theta$.

b. Show that the speed of a particle in the velocity field is $|\mathbf{a} \times \mathbf{r}|$ and that the angular speed of the object is $|\mathbf{a}|$.

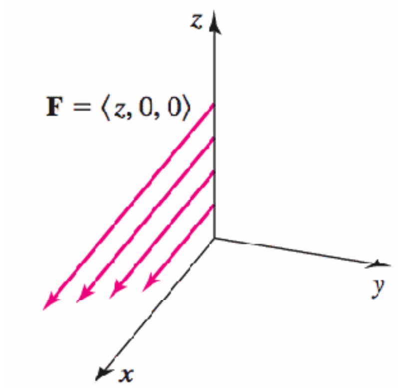
c. Conclude that $\omega = \frac{1}{2} |\nabla \times \mathbf{v}|$.

51. Paddle wheel in a vector field Let $\mathbf{F} = \langle z, 0, 0 \rangle$ and let $\mathbf{n}$ be a unit vector aligned with the axis of a paddle wheel located on the x-axis (see figure).

a. If the paddle wheel is oriented with $\mathbf{n} = \langle 1, 0, 0 \rangle$, in what direction (if any) does the wheel spin?

b. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 1, 0 \rangle$, in what direction (if any) does the wheel spin?

c. If the paddle wheel is oriented with $\mathbf{n} = \langle 0, 0, 1 \rangle$, in what direction (if any) does the wheel spin?



52. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle -2y, 2z, 0 \rangle$.

a. If a paddle wheel is placed in the xy-plane with its axis normal to this plane, what is its angular speed?

b. If a paddle wheel is placed in the xz-plane with its axis normal to this plane, what is its angular speed?

c. If a paddle wheel is placed in the yz-plane with its axis normal to this plane, what is its angular speed?

53. Angular speed Consider the rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

Applications

54-56. Heat flux *Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\mathbf{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature* decrease. *The divergence of the heat flow vector is $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ (the Laplacian of T). Compute the heat flow vector field and its divergence for the following temperature distributions.*

54. $T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

55. $T(x, y, z) = 100\, e^{-x^2+y^2+z^2}$

56. $T(x, y, z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

57. Gravitational potential The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\phi = G M m / |\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of the mass m and G is the gravitational constant.

a. Compute the gravitational force field $\mathbf{F} = -\nabla\phi$.

b. Show that the field is irrotational; that is $\nabla \times \mathbf{F} = \mathbf{0}$.

58. Electric potential The potential function for the force field due to a charge q at the origin is $\phi = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{|\mathbf{r}|}$, where $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of a point in the field and ϵ_0 is the permittivity of free space.

a. Compute the force field $\mathbf{F} = -\nabla\phi$.

b. Show that the field is irrotational; that is $\nabla \times \mathbf{F} = \mathbf{0}$.

59. Navier-Stokes equation The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the motion of water in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right) = -\nabla\, p + \mu\,(\nabla\cdot\nabla)\,\mathbf{V}.$$

In this notation $\mathbf{V} = \langle u,\ v,\ w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation. (See Exercise 40 for an interpretation of the operations.)

T **60. Stream function and vorticity** The rotation of a three-dimensional velocity field $\mathbf{V} = \langle u,\ v,\ w\rangle$ is measured by the **vorticity** $\boldsymbol{\omega} = \nabla \times \mathbf{V}$. If $\boldsymbol{\omega} = \mathbf{0}$ at all points in the domain, the flow is irrotational.

- **a.** Which of the following velocity fields is irrotational: $\mathbf{V} = \langle 2,\ -3\,y,\ 5\,z\rangle$ or $\mathbf{V} = \langle y,\ x - z,\ -y\rangle$?
- **b.** Recall that for a two-dimensional source-free flow $\mathbf{V} = \langle u,\ v,\ 0\rangle$, a stream function $\psi(x,\ y)$ may be defined such that $u = \psi_y$ and $v = -\psi_x$. For such a two-dimensional flow, let $\zeta = \mathbf{k}\cdot\nabla \times \mathbf{V}$ be the $\mathbf{k}$-component of the vorticity. Show that $\nabla^2\psi = \nabla\cdot\nabla\psi = -\zeta$.
- **c.** Consider the stream function $\psi(x,\ y) = \sin x \sin y$ on the square region $R = \{(x,\ y) : 0 \le x \le \pi,\ 0 \le y \le \pi\}$. Find the velocity components u and v; then sketch the velocity field.
- **d.** For the stream function in part (c) find the vorticity function ζ as defined in part (b). Plot several level curves of the vorticity function. Where on R is it a maximum? A minimum?

61. Maxwell's equation One of Maxwell's equations for electromagnetic waves (also called Ampere's Law) is $\nabla \times \mathbf{B} = C\,\dfrac{\partial \mathbf{E}}{\partial t}$, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and C is a constant.

- **a.** Show that the fields

$$\mathbf{E}(z,\ t) = A\sin(kz - \omega t)\,\mathbf{i} \qquad \mathbf{B}(z,\ t) = A\sin(kz - \omega t)\,\mathbf{j}$$

  satisfy the equation for constants A, k, and ω, provided $\omega = k/C$.
- **b.** Make a rough sketch showing the directions of $\mathbf{E}$ and $\mathbf{B}$.

Additional Exercises

62. Splitting a vector field Express the vector field $\mathbf{F} = \langle x\,y,\ 0,\ 0\rangle$ in the form $\mathbf{V} + \mathbf{W}$, where $\nabla\cdot\mathbf{V} = 0$ and $\nabla \times \mathbf{W} = \mathbf{0}$.

63. Properties of div and curl Prove the following properties of the divergence and curl. Assume $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields and c is a real number.

- **a.** $\nabla\cdot(\mathbf{F} + \mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$
- **b.** $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
- **c.** $\nabla\cdot(c\,\mathbf{F}) = c\,(\nabla\cdot\mathbf{F})$
- **d.** $\nabla \times (c\,\mathbf{F}) = c\,(\nabla \times \mathbf{F})$

64. Equal curls If two functions of one variable, f and g, have the property that $f' = g'$, then f and g differ by a constant. Prove or disprove: If $\mathbf{F}$ and $\mathbf{G}$ are nonconstant vector fields in $\mathbb{R}^2$ with curl $\mathbf{F}$ = curl $\mathbf{G}$ and div $\mathbf{F}$ = div $\mathbf{G}$ at all points of $\mathbb{R}^2$, then $\mathbf{F}$ and $\mathbf{G}$ differ by a constant vector.

65-70. Identities *Prove the following identities. Assume that ϕ is a differentiable scalar-valued function and* **F** *and* **G** *are differentiable vector fields, all defined on a region of* $\mathbb{R}^3$.

65. $\nabla\cdot(\phi\,\mathbf{F}) = \nabla\phi\cdot\mathbf{F} + \phi\,\nabla\cdot\mathbf{F}$ (Product Rule)

66. $\nabla \times (\phi\,\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi\,\nabla \times \mathbf{F}$ (Product Rule)

67. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

68. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\,\mathbf{F} - \mathbf{G}\,(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\,\mathbf{G} + \mathbf{F}\,(\nabla \cdot \mathbf{G})$

69. $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\,\mathbf{F} + (\mathbf{F} \cdot \nabla)\,\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$

70. $\nabla \times (\nabla \times \mathbf{F}) = \nabla\,(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\,\mathbf{F}$

71. Divergence of radial fields Prove that for a real number p, with $\mathbf{r} = \langle x, y, z \rangle$, $\nabla \cdot \dfrac{\langle x, y, z \rangle}{|\mathbf{r}|^p} = \dfrac{3 - p}{|\mathbf{r}|^p}$.

72. Gradients and radial fields Prove that for a real number p, with $\mathbf{r} = \langle x, y, z \rangle$, $\nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{-p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}$.

73. Divergence of gradient fields Prove that for a real number p, with $r = \langle x, y, z \rangle$, $\nabla \cdot \nabla\left(\dfrac{1}{|\mathbf{r}|^p}\right) = \dfrac{p(p-1)}{|\mathbf{r}|^{p+2}}$.