14.6 Surface Integrals

We have studied integrals on intervals, on regions in the plane, on solid regions in space, and along curves in space. One situation is still unexplored. Suppose a sphere has a known temperature distribution; perhaps it is cold near the poles and warm near the equator. How do you find the average temperature over the entire sphere? In analogy with other average value calculations, we should expect to "add up" the temperature values over the sphere and divide by the surface area of the sphere. Because the temperature varies continuously over the sphere, adding up means integrating. How do you integrate a function over a surface? This question leads to *surface integrals*.

It helps to keep curves, arc length, and line integrals in mind as we discuss surfaces, surface area, and surface integrals. What we discover about surfaces parallels what we already know about curves—all "lifted" up one dimension.

Parameterized Surfaces

Surface Integrals of Scalar-Valued Functions

Surface Integrals of Vector Fields

Quick Quiz

SECTION 14.6 EXERCISES

Review Questions

1. Give a parametric description for a cylinder with radius a and height h, including the intervals for the parameters.

2. Give a parametric description for a cone with radius a and height h, including the intervals for the parameters.

3. Give a parametric description for a sphere with radius a, including the intervals for the parameters.

4. Explain how to compute the surface integral of a scalar-valued function f over a cone using an explicit description of the cone.

5. Explain how to compute the surface integral of a scalar-valued function f over a sphere using a parametric description of the sphere.

6. Explain how to compute a flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over a cone using an explicit description and a given orientation of the cone.

7. Explain how to compute a surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over a sphere using a parametric description of the sphere and a given orientation.

8. Explain what it means for a surface to be orientable.

9. Describe the usual orientation of a closed surface such as a sphere.

10. Why is the upward flux of a vertical vector field $\mathbf{F} = \langle 0, 0, 1\rangle$ across a surface equal to the area of the projection of the surface in the xy-plane?

Basic Skills

11-16. Parametric descriptions *Give a parametric description of the form* $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ *for the following surfaces. The descriptions are not unique.*

11. The plane $2x - 4y + 3z = 16$

12. The cap of the sphere $x^2 + y^2 + z^2 = 16$, for $4/\sqrt{2} \le z \le 4$

13. The frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 8$

14. The hyperboloid $z^2 = 1 + x^2 + y^2$, for $1 \le z \le 10$

15. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

16. The cylinder $y^2 + z^2 = 36$, for $0 \le x \le 9$

17-20. Identify the surface *Describe the surface with the given parametric representation.*

17. $\mathbf{r}(u, v) = \langle u, v, 2u + 3v - 1\rangle$, for $1 \le u \le 3$, $2 \le v \le 4$

18. $\mathbf{r}(u, v) = \langle u, u + v, 2 - u - v\rangle$, for $0 \le u \le 2$, $0 \le v \le 2$

19. $\mathbf{r}(u, v) = \langle v\cos u, v\sin u, 4v\rangle$, for $0 \le u \le \pi$, $0 \le v \le 3$

20. $\mathbf{r}(u, v) = \langle v, 6\cos u, 6\sin u\rangle$, for $0 \le u \le 2\pi$, $0 \le v \le 2$

21-26. Surface area using a parametric description *Find the area of the following surfaces using a parametric description of the surface.*

21. The half cylinder $\{(r, \theta, z) : r = 4, 0 \le \theta \le \pi, 0 \le z \le 7\}$

22. The plane $z = 3 - x - 3y$ in the first octant

23. The plane $z = 10 - x - y$ above the square $|x| \le 2$, $|y| \le 2$

24. The hemisphere $x^2 + y^2 + z^2 = 100$, for $z \ge 0$

25. A cone with base radius r and height h, where r and h are positive constants

26. The cap of the sphere $x^2 + y^2 + z^2 = 4$, for $1 \le z \le 2$

27-30. Surface integrals using a parametric description *Evaluate the surface integral* $\iint_S f(x, y, z)\, dS$ *using a parametric description of the surface.*

27. $f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36$, for $z \ge 0$

28. $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9$, $0 \le z \le 3$

29. $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1$, $0 \le y \le 3$

30. $f(\rho, \phi, \theta) = \cos\phi$, where S is the part of the unit sphere in the first octant

31-34. Surface area using an explicit description *Find the area of the following surfaces using an explicit description of the surface.*

31. The cone $z^2 = 4\left(x^2 + y^2\right)$, for $0 \le z \le 4$

32. The paraboloid $z = 2\left(x^2 + y^2\right)$, for $0 \le z \le 8$

33. The trough $z = x^2$, for $-2 \le x \le 2$, $0 \le y \le 4$

34. The part of the hyperbolic paraboloid $z = x^2 - y^2$ above the sector $R = \{(r, \theta) : 0 \le r \le 4, -\pi/4 \le \theta \le \pi/4\}$

35-38. Surface integrals using an explicit description *Evaluate the surface integral* $\iint_S f(x, y, z)\, dS$ *using an explicit representation of the surface.*

35. $f(x, y, z) = x\,y$; S is the plane $z = 2 - x - y$ in the first octant.

36. $f(x, y, z) = x^2 + y^2$; S is the paraboloid $z = x^2 + y^2$ for $0 \le z \le 4$.

37. $f(x, y, z) = 25 - x^2 - y^2$; S is the hemisphere centered at the origin with radius 5, for $z \ge 0$.

38. $f(x, y, z) = e^z$; S is the plane $z = 8 - x - 2\,y$ in the first octant.

39-42. Average values

39. Find the average temperature on that part of the plane $3\,x + 4\,y + z = 6$ over the square $|x| \le 1$, $|y| \le 1$, where the temperature is given by $T(x, y, z) = e^{-z}$.

40. Find the average squared distance between the origin and the points on the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$.

41. Find the average value of the function $f(x, y, z) = x\,y\,z$ on the unit sphere in the first octant.

42. Find the average value of the temperature function $T(x, y, z) = 100 - 25\,z$ on the cone $z^2 = x^2 + y^2$, for $0 \le z \le 2$.

43-48. Surface integrals of vector fields *Find the flux of the following vector fields across the given surface with the specified orientation. You may use either an explicit or parametric description of the surface.*

43. $\mathbf{F} = \langle 0, 0, -1\rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point in the positive z-direction.

44. $\mathbf{F} = \langle x, y, z\rangle$ across the slanted face of the tetrahedron $z = 10 - 2\,x - 5\,y$ in the first octant; normal vectors point in the positive z-direction.

45. $\mathbf{F} = \langle x, y, z\rangle$ across the slanted surface of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$; normal vectors point in the positive z-direction.

46. $\mathbf{F} = \langle e^{-y}, 2\,z, x\,y\rangle$ across the curved sides of the surface $S = \{(x, y, z) : z = \cos y, |y| \le \pi, 0 \le x \le 4\}$, where normal vectors point upward.

47. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$ across the sphere of radius a centered at the origin, where $\mathbf{r} = \langle x, y, z\rangle$; the normal vectors point outward.

48. $\mathbf{F} = \langle -y, x, 1\rangle$ across the cylinder $y = x^2$ for $0 \le x \le 1$, $0 \le z \le 4$; normal vectors point in the positive y-direction.

Further Explorations

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = 10\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, 10)\, dx\, dy$.

b. If the surface S is given by $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, z = x\}$, then $\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy$.

c. The surface $\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$, for $0 \le u \le \pi$, $0 \le v \le 2$ is the same as the surface $\mathbf{r} = \langle \sqrt{v} \cos 2u, \sqrt{v} \sin 2u, v \rangle$, for $0 \le u \le \pi/2$, $0 \le v \le 4$.

d. Given the standard parameterization of a sphere, the normal vectors $\mathbf{t}_u \times \mathbf{t}_v$ are outward normal vectors.

50-53. Miscellaneous surface integrals *Evaluate the following integrals using the method of your choice. Assume normal vectors point either outward or in the positive z-direction.*

50. $\iint_S \nabla \ln r \cdot \mathbf{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, where $r = |\langle x, y, z \rangle|$

51. $\iint_S |\mathbf{r}|\, dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, where $\mathbf{r} = \langle x, y, z \rangle$

52. $\iint_S x\, y\, z\, dS$, where S is that part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 4$

53. $\iint_S \dfrac{\langle x, 0, z \rangle}{\sqrt{x^2 + z^2}} \cdot \mathbf{n}\, dS$, where S is the cylinder $x^2 + z^2 = a^2$, $|y| \le 2$

54. Cone and sphere The cone $z^2 = x^2 + y^2$, for $z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a. Find the surface area of the sphere below C, for $z \ge 0$.

b. Find the surface area of the sphere above C.

c. Find the surface area of the cone below C, for $z \ge 0$.

55. Cylinder and sphere Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x - 1)^2 + y^2 = 1$, for $z \ge 0$.

a. Find the surface area of the cylinder inside the sphere.

b. Find the surface area of the sphere inside the cylinder.

56. Flux on a tetrahedron Find the upward flux of the field $\mathbf{F} = \langle x, y, z \rangle$ across the plane $x/a + y/b + z/c = 1$ in the first octant. Show that the flux equals c times the area of the base of the region. Interpret the result physically.

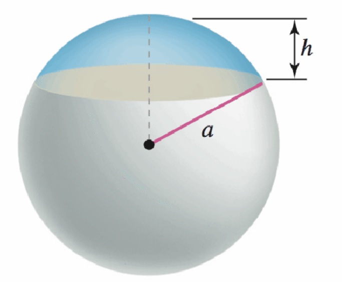
57. Flux across a cone Consider the field $\mathbf{F} = \langle x, y, z \rangle$ and the cone $z^2 = \left(x^2 + y^2\right)/a^2$, for $0 \le z \le 1$.

a. Show that when $a = 1$, the outward flux across the cone is zero. Interpret the result.

b. Find the outward flux (away from the z-axis), for any $a > 0$. Interpret the result.

58. Surface area formula for cones Find the general formula for the surface area of a cone with height h and base radius a (excluding the base).

59. Surface area formula for spherical cap A sphere of radius a is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane (see figure). Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h.



60. Radial fields and spheres Consider the radial field $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where $\mathbf{r} = \langle x, y, z \rangle$ and p is a real number. Let S be the sphere of radius a centered at the origin. Show that the outward flux of $\mathbf{F}$ across the sphere is $4\pi / a^{p-3}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

Applications

61-63. Heat flux *The heat flow vector field for conducting objects is* $\mathbf{F} = -k\,\nabla T$, *where* $T(x, y, z)$ *is the temperature in the object and* $k > 0$ *is a constant that depends on the material. Compute the outward flux of* $\mathbf{F}$ *across the following surfaces* S *for the given temperature distributions. Assume* $k = 1$.

61. $T(x, y, z) = 100\,e^{-x-y}$; S consists of the faces of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$.

62. $T(x, y, z) = 100\,e^{-x^2-y^2-z^2}$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

63. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is the sphere $x^2 + y^2 + z^2 = a^2$.

64. Flux across a cylinder Let S be the cylinder $x^2 + y^2 = a^2$ for $-L \le z \le L$.

a. Find the outward flux of the field $\mathbf{F} = \langle x, y, 0 \rangle$ across S.

b. Find the outward flux of the field $\mathbf{F} = \dfrac{\langle x, y, 0 \rangle}{(x^2 + y^2)^{p/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$ across S, where $|\mathbf{r}|$ is the distance from the z-axis and p is a real number.

c. In part (b), for what values of p is the flux finite as $a \to \infty$ (with L fixed)?

d. In part (b), for what values of p is the flux finite as $L \to \infty$ (with a fixed)?

65. Flux across concentric spheres Consider the radial fields $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number. Let S consist of the spheres A and B centered at the origin with radii $0 < a < b$, respectively. The total outward flux across S consists of the *outward* flux across the outer sphere B minus the *inward* flux across the inner sphere A.

a. Find the total flux across S with $p = 0$. Interpret the result.

b. Show that for $p = 3$ (an inverse square law), the flux across S is independent of a and b.

66-69. Mass and center of mass *Let S be a surface that represents a thin shell with density ρ. The moments about the coordinate planes (see Section 13.6) are* $M_{yz} = \iint_S x\,\rho(x, y, z)\,dS$, $M_{xz} = \iint_S y\,\rho(x, y, z)\,dS$, $M_{xy} = \iint_S z\,\rho(x, y, z)\,dS$. *The coordinates of the center of mass of the shell are* $\bar{x} = \dfrac{M_{yz}}{m}$, $\bar{y} = \dfrac{M_{xz}}{m}$, $\bar{z} = \dfrac{M_{xy}}{m}$, *where m is the mass of the shell. Find the mass and center of mass of the following shells. Use symmetry whenever possible.*

66. The constant-density hemispherical shell $x^2 + y^2 + z^2 = a^2$, $z \geq 0$

67. The constant-density cone with radius a, height h, and base in the xy-plane

68. The constant-density half cylinder $x^2 + z^2 = a^2$, $-h/2 \leq y \leq h/2$, $z \geq 0$

69. The cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq 2$, with density $\rho(x, y, z) = 1 + z$

Additional Exercises

70. Outward normal to a sphere Show that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2 \sin u$ for a sphere of radius a defined parametrically by $\mathbf{r}(u, v) = \langle a \sin u \cos v,\ a \sin u \sin v,\ a \cos u\rangle$, where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

71. Special case of surface integrals of scalar-valued functions Suppose that a surface S is defined as $z = g(x, y)$ on a region R. Show that $\mathbf{t}_x \times \mathbf{t}_y = \left\langle -z_x, -z_y, 1\right\rangle$ and that $\iint_S f(x, y, z)\,dS = \iint_R f(x, y, z)\sqrt{{z_x}^2 + {z_y}^2 + 1}\,dA$.

72. Surfaces of revolution Let $y = f(x)$ be a curve in the xy-plane with $f(x) \neq 0$, for $a \leq x \leq b$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis.

a. Show that S is described parametrically by $\mathbf{r}(u, v) = \langle u,\ f(u)\cos v,\ f(u)\sin v\rangle$, for $a \leq u \leq b$, $0 \leq v \leq 2\pi$.

b. Find an integral that gives the surface area of S.

c. Apply the result of part (b) to the surface generated with $f(x) = x^3$, for $1 \leq x \leq 2$.

d. Apply the result of part (b) to the surface generated with $f(x) = \left(25 - x^2\right)^{1/2}$, for $3 \leq x \leq 4$.

73. Rain on roofs Let $z = s(x, y)$ define a surface over a region R in the xy-plane, where $z \geq 0$ on R. Show that the downward flux of the vertical vector field $\mathbf{F} = \langle 0, 0, -1\rangle$ across S equals the area of R. Interpret the result physically.

74. Surface area of a torus

a. Show that a torus with radii $R > r$ (see figure) may be described parametrically by $r(u, v) = \langle (R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r \sin u\rangle$, for $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$.

b. Show that the surface area of the torus is $4\pi^2 R r$.