14.7 Stokes' Theorem

With the divergence, the curl, and surface integrals in hand, we are ready to present two of the crowning results of calculus. Fortunately, all of the heavy lifting has been done. In this section, you will see Stokes' Theorem, and in the next section we present the Divergence Theorem.

Note

Stokes' Theorem

Interpreting the Curl

Proof of Stokes' Theorem

Two Final Notes on Stokes' Theorem

Quick Quiz

SECTION 14.7 EXERCISES

Review Questions

1. Explain the meaning of the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ in Stokes' Theorem.

2. Explain the meaning of the integral $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ in Stokes' Theorem.

3. Explain the meaning of Stokes' Theorem.

4. Why does a conservative vector field produce zero circulation around a closed curve?

Basic Skills

5-10. Verifying Stokes' Theorem *Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.*

5. $\mathbf{F} = \langle y, -x, 10 \rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circle $x^2 + y^2 = 1$ in the xy-plane.

6. $\mathbf{F} = \langle 0, -x, y \rangle$; S is the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and C is the circle $x^2 + y^2 = 4$ in the xy-plane.

7. $\mathbf{F} = \langle x, y, z \rangle$; S is the paraboloid $z = 8 - x^2 - y^2$, for $0 \le z \le 8$, and C is the circle $x^2 + y^2 = 8$ in the xy-plane.

8. $\mathbf{F} = \langle 2z, -4x, 3y \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and C is the boundary of S.

9. $\mathbf{F} = \langle y - z, z - x, x - y \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$ and C is the boundary of S.

10. $\mathbf{F} = \langle -y, -x - z, y - x \rangle$; S is the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$ and C is the boundary of S.

11-16. Stokes' Theorem for evaluating line integrals *Evaluate the line integral* $\oint_C \mathbf{F}\cdot d\mathbf{r}$ *by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation.*

11. $\mathbf{F} = \langle 2y, -z, x\rangle$; C is the circle $x^2 + y^2 = 12$ in the plane $z = 0$.

12. $\mathbf{F} = \langle y, xz, -y\rangle$; C is the ellipse $x^2 + y^2/4 = 1$ in the plane $z = 1$.

13. $\mathbf{F} = \langle x^2 - z^2, y, 2xz\rangle$; C is the boundary of the plane $z = 4 - x - y$ in the first octant.

14. $\mathbf{F} = \langle x^2 - y^2, z^2 - x^2, y^2 - z^2\rangle$; C is the boundary of the square $|x| \le 1$, $|y| \le 1$ in the plane $z = 0$.

15. $\mathbf{F} = \langle y^2, -z^2, x\rangle$; C is the circle $\mathbf{r}(t) = \langle 3\cos t, 4\cos t, 5\sin t\rangle$.

16. $\mathbf{F} = \langle 2xy\sin z, x^2\sin z, x^2 y\cos z\rangle$; C is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

17-20. Stokes' Theorem for evaluating surface integrals *Evaluate the line integral in Stokes' Theorem to evaluate the surface integral* $\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$. *Assume that* $\mathbf{n}$ *is in the positive z-direction.*

17. $\mathbf{F} = \langle x, y, z\rangle$; S is the upper half of the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$.

18. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$; S is the paraboloid $x = 9 - y^2 - z^2$, for $0 \le x \le 9$ (excluding its base), and $\mathbf{r} = \langle x, y, z\rangle$.

19. $\mathbf{F} = \langle 2y, -z, x - y - z\rangle$; S is the cap of the sphere (excluding its base) $x^2 + y^2 + z^2 = 25$, for $3 \le x \le 5$.

20. $\mathbf{F} = \langle x + y, y + z, z + x\rangle$; S is the tilted disk enclosed by $\mathbf{r}(t) = \langle \cos t, 2\sin t, \sqrt{3}\cos t\rangle$.

21-24. Interpreting and graphing the curl *For the following velocity fields, compute the curl, make a sketch of the curl, and interpret the curl.*

21. $\mathbf{v} = \langle 0, 0, y\rangle$

22. $\mathbf{v} = \langle 1 - z^2, 0, 0\rangle$

23. $\mathbf{v} = \langle -2z, 0, 1\rangle$

24. $\mathbf{v} = \langle 0, -z, y\rangle$

Further Explorations

25. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** A paddle wheel with its axis in the direction $\langle 0, 1, -1\rangle$ would not spin when put in the vector field $\mathbf{F} = \langle 1, 1, 2\rangle \times \langle x, y, z\rangle$.
- **b.** Stokes' Theorem relates the flux of a vector field $\mathbf{F}$ across a surface to values of $\mathbf{F}$ on the boundary of the surface.
- **c.** A vector field of the form $\mathbf{F} = \langle a + f(x), b + g(y), c + h(z)\rangle$, where a, b, and c are constants, has zero circulation on a closed curve.
- **d.** If a vector field has zero circulation on all simple closed smooth curves C in a region D, then $\mathbf{F}$ is conservative on R.

26-29. Conservative fields *Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C.*

26. $\mathbf{F} = \langle 2x, -2y, 2z \rangle$

27. $\mathbf{F} = \nabla\left(x \sin y e^{z}\right)$

28. $\mathbf{F} = \left\langle 3x^2 y, x^3 + 2yz^2, 2y^2 z \right\rangle$

29. $\mathbf{F} = \left\langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \right\rangle$

30-34. Tilted disks *Let S be the disk enclosed by the curve* $C : \mathbf{r}(t) = \langle \cos\phi \cos t, \sin t, \sin\phi \cos t \rangle$*, for* $0 \le t \le 2\pi$*, where* $0 \le \phi \le \pi/2$ *is a fixed angle.*

30. What is the area of S (in terms of ϕ)? Find a vector normal to S.

31. What is the length of C (in terms of ϕ)?

32. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\mathbf{F} = \langle -y, x, 0 \rangle$ as a function of ϕ. For what value of ϕ is the circulation a maximum?

33. What is the circulation on C of the vector field $\mathbf{F} = \langle -y, -z, x \rangle$ as a function of ϕ? For what value of ϕ is the circulation a maximum?

34. Consider the vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a constant nonzero vector and $\mathbf{r} = \langle x, y, z \rangle$. Show that the circulation is a maximum when $\mathbf{a}$ points in the direction of the normal to S.

35. Circulation in a plane A circle C in the plane $x + y + z = 8$ has a radius of 4 and center $(2, 3, 3)$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 0, -z, 2y \rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

36. No integrals Let $\mathbf{F} = \langle 2z, z, 2y + x \rangle$ and let S be the hemisphere of radius a with its base in the xy-plane and center at the origin.

a. Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ by computing $\nabla \times \mathbf{F}$ and appealing to symmetry.

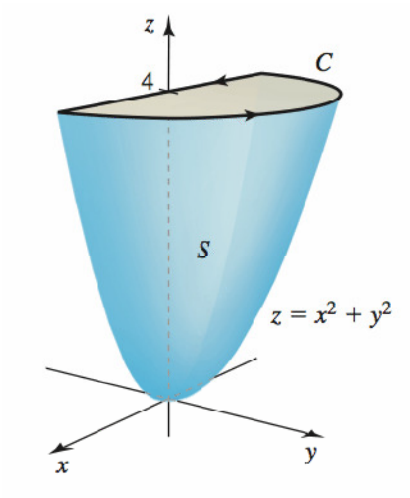
b. Evaluate the line integral using Stokes' Theorem to check part (a).

37. Compound surface and boundary Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$ (including the planar surface in the xz-plane) (see figure). Let C be the semicircle and line segment that bound the cap of S in the plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y, 2x + z, 2y + x \rangle$.

a. Describe the direction of the vectors normal to the surface.

b. Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

c. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and check for agreement with part (b).



Applications

38. Ampère's Law The French physicist André-Marie Ampère (1775-1836) discovered that an electrical current I in a wire produces a magnetic field $\mathbf{B}$. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\mathbf{J}$ as $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \mathbf{B} = \mu \mathbf{J}$.

39. Maximum surface integral Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\mathbf{F} = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ have its maximum value?

40. Area of a region in a plane Let R be a region in a plane that has a unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and boundary C. Let $\mathbf{F} = \langle b z,\ c x,\ a y \rangle$.

a. Show that $\nabla \times \mathbf{F} = \mathbf{n}$.

b. Use Stokes' Theorem to show that

$$\text{area of } R = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

c. Consider the curve C given by $\mathbf{r} = \langle 5 \sin t,\ 13 \cos t,\ 12 \sin t \rangle$, for $0 \leq t \leq 2\pi$. Prove that C lies in a plane by showing that $\mathbf{r} \times \mathbf{r}'$ is constant for all t.

d. Use part (b) to find the area of the region enclosed by C in part (c). (*Hint:* Find the unit normal vector that is consistent with the orientation of C.)

41. Choosing a more convenient surface The goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \langle yz,\ -xz,\ xy \rangle$ and S is the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1$ $(z \geq 0)$.

a. Evaluate a surface integral over a more convenient surface to find the value of A.

b. Evaluate A using a line integral.

Additional Exercises

42. Radial fields and zero circulation Consider the radial vector fields $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where p is a real number and $\mathbf{r} = \langle x,\ y,\ z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a. Evaluate a line integral to show that the field has zero circulation on C.

b. For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

43. Zero curl Consider the vector field $\mathbf{F} = \dfrac{-y}{x^2 + y^2}\,\mathbf{i} + \dfrac{x}{x^2 + y^2}\,\mathbf{j} + z\,\mathbf{k}$.

a. Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

b. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is not zero on a circle C in the xy-plane enclosing the origin.

c. Explain why Theorem 14.13 does not apply in this case.

44. Average circulation Let S be a small circular disk of radius R centered at the point P with a unit normal vector $\mathbf{n}$. Let C be the boundary of S.

a. Express the average circulation of the vector field $\mathbf{F}$ on S as a surface integral of $\nabla \times \mathbf{F}$.

b. Argue that for small R, the average circulation approaches $(\nabla \times \mathbf{F})_P \cdot \mathbf{n}$ (the component of $\nabla \times \mathbf{F}$ in the direction of $\mathbf{n}$ evaluated at P) with the approximation improving as $R \to 0$.

45. Proof of Stokes' Theorem Confirm the following step in the proof of Stokes' Theorem. If $z = s(x, y)$ and f, g, and h are functions of x, y, and z, with $M = f + h\,z_x$ and $N = g + h\,z_y$, then

$$M_y = f_y + f_z\,z_y + h\,z_{xy} + z_x\left(h_y + h_z\,z_y\right) \quad \text{and}$$
$$N_x = g_x + g_z\,z_x + h\,z_{yx} + z_y(h_x + h_z\,z_x).$$

46. Stokes' Theorem on closed surfaces Prove that if $\mathbf{F}$ satisfies the conditions of Stokes' Theorem, then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = 0$, where S is a smooth surface that encloses a region.

47. Rotated Green's Theorem Use Stokes' Theorem to write the circulation form of Green's Theorem in the yz-plane.