14.8 Divergence Theorem

Vector fields can represent electric or magnetic fields, air velocities in hurricanes, or blood flow in an artery. These and other vector phenomena suggest movement of a “substance.” A frequent question concerns the amount of a substance that flows across a surface—for example, the amount of water that passes across the membrane of a cell per unit time. Such flux calculations may be done using flux integrals as in Section 14.6. The Divergence Theorem offers an alternative method. In effect, it says that instead of integrating the flow in and out of a region across its boundary, you may also add up all the sources (or sinks) of the flow throughout the region.

Note

Divergence Theorem

Proof of the Divergence Theorem

Divergence Theorem for Hollow Regions

Gauss' Law

A Final Perspective

Quick Quiz

SECTION 14.8 EXERCISES

Review Questions

1. Explain the meaning of the surface integral in the Divergence Theorem.

2. Explain the meaning of the volume integral in the Divergence Theorem.

3. Explain the meaning of the Divergence Theorem.

4. What is the net outward flux of the rotation field $\mathbf{F} = \langle 2z + y, -x, -2x \rangle$ across the surface that encloses any region?

5. What is the net outward flux of the radial field $\mathbf{F} = \langle x, y, z \rangle$ across the sphere of radius 2 centered at the origin?

6. What is the divergence of an inverse square vector field?

7. Suppose $\operatorname{div} \mathbf{F} = 0$ in a region enclosed by two concentric spheres. What is the relationship between the outward fluxes across the two spheres?

8. If $\operatorname{div} \mathbf{F} > 0$ in a region enclosed by a small cube, is the net flux of the field into or out of the cube?

Basic Skills

9-12. Verifying the Divergence Theorem *Evaluate both integrals of the Divergence Theorem for the following vector fields and regions. Check for agreement.*

9. $\mathbf{F} = \langle 2x, 3y, 4z \rangle$; $D = \{(x, y, z) : x^2 + y^2 + z^2 \le 4\}$

10. $\mathbf{F} = \langle -x, -y, -z \rangle$; $D = \{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$

11. $\mathbf{F} = \langle z - y,\ x,\ -x\rangle$; $D = \left\{(x,\ y,\ z) : x^2/4 + y^2/8 + z^2/12 \le 1\right\}$

12. $\mathbf{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; $D = \{(x,\ y,\ z) : |x| \le 1,\ |y| \le 2,\ |z| \le 3\}$

13-16. Rotation fields

13. Find the net outward flux of the field $\mathbf{F} = \langle 2\,z - y,\ x,\ -2\,x\rangle$ across the sphere of radius 1 centered at the origin.

14. Find the net outward flux of the field $\mathbf{F} = \langle z - y,\ x - z,\ y - x\rangle$ across the boundary of the cube $\{(x,\ y,\ z) : |x| \le 1,\ |y| \le 1,\ |z| \le 1\}$.

15. Find the net outward flux of the field $\mathbf{F} = \langle b\,z - c\,y,\ c\,x - a\,z,\ a\,y - b\,x\rangle$ across any smooth closed surface in $\mathbb{R}^3$, where a, b, and c are constants.

16. Find the net outward flux of $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ across any smooth closed surface in $\mathbb{R}^3$, where $\mathbf{a}$ is a constant nonzero vector and $\mathbf{r} = \langle x,\ y,\ z\rangle$.

17-24. Computing flux *Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface S.*

17. $\mathbf{F} = \langle x,\ -2\,y,\ 3\,z\rangle$; S is the sphere $\left\{(x,\ y,\ z) : x^2 + y^2 + z^2 = 6\right\}$.

18. $\mathbf{F} = \left\langle x^2,\ 2\,xz,\ y^2\right\rangle$; S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

19. $\mathbf{F} = \langle x,\ 2\,y,\ z\rangle$; S is the boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$.

20. $\mathbf{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; S is the sphere $\left\{(x,\ y,\ z) : x^2 + y^2 + z^2 = 25\right\}$.

21. $\mathbf{F} = \left\langle y - 2\,x,\ x^3 - y,\ y^2 - z\right\rangle$; S is the sphere $\left\{(x,\ y,\ z) : x^2 + y^2 + z^2 = 4\right\}$.

22. $\mathbf{F} = \langle y + z,\ x + z,\ x + y\rangle$; S consists of the faces of the cube $\{(x,\ y,\ z) : |x| \le 1,\ |y| \le 1,\ |z| \le 1\}$.

23. $\mathbf{F} = \langle x,\ y,\ z\rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the xy-plane.

24. $\mathbf{F} = \langle x,\ y,\ z\rangle$; S is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$.

25-30. Divergence Theorem for more general regions *Use the Divergence Theorem to compute the net outward flux of the following vector fields across the boundary of the given regions D.*

25. $\mathbf{F} = \langle z - x,\ x - y,\ 2\,y - z\rangle$; D is the region between the spheres of radius 2 and 4 centered at the origin.

26. $\mathbf{F} = \mathbf{r}\,|\mathbf{r}| = \langle x,\ y,\ z\rangle \sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

27. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between the spheres of radius 1 and 2 centered at the origin.

28. $\mathbf{F} = \langle z - y,\ x - z,\ 2\,y - x\rangle$; D is the region between two cubes: $\{(x,\ y,\ z) : 1 \le |x| \le 3,\ 1 \le |y| \le 3,\ 1 \le |z| \le 3\}$.

29. $\mathbf{F} = \left\langle x^2,\ -y^2,\ z^2\right\rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$.

30. $\mathbf{F} = \langle x,\ 2\,y,\ 3\,z\rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, for $0 \le z \le 8$.

Further Explorations

31. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\nabla \cdot \mathbf{F} = 0$ at all points of a region D, then $\mathbf{F} \cdot \mathbf{n} = 0$ at all points of the boundary of D.

b. If $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$ on all closed surfaces in $\mathbb{R}^3$, then $\mathbf{F}$ is constant.

c. If $|\mathbf{F}| < 1$, then $\left|\int\int\int_D \nabla \cdot \mathbf{F}\, dV\right|$ is less than the area of the surface of D.

32. Flux across a sphere Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere of radius a centered at the origin. Compute the outward flux of $\mathbf{F}$ across S using the representation $z = \pm\sqrt{a^2 - x^2 - y^2}$ for the sphere (either symmetry or two surfaces must be used).

33-35. Flux integrals *Compute the outward flux of the following vector fields across the given surfaces S. You should decide which integral of the Divergence Theorem to use.*

33. $\mathbf{F} = \left\langle x^2 e^y \cos z, -4 x e^y \cos z, 2 x e^y \sin z \right\rangle$; S is the boundary of the ellipsoid $x^2/4 + y^2 + z^2 = 1$.

34. $\mathbf{F} = \langle -y z, x z, 1 \rangle$; S is the boundary of the ellipsoid $x^2/4 + y^2/4 + z^2 = 1$.

35. $\mathbf{F} = \langle x \sin y, -\cos y, z \sin y \rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \pi/2$, $z = 0$, and $z = x$.

36. Radial fields Consider the radial vector field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{p/2}}$. Let S be the sphere of radius a centered at the origin.

a. Use a surface integral to show that the outward flux of $\mathbf{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\mathbf{r}/|\mathbf{r}|$.

b. For what values of p does $\mathbf{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact (Theorem 14.8) that $\nabla \cdot \mathbf{F} = \dfrac{3-p}{|\mathbf{r}|^p}$ to compute the flux across S using the Divergence Theorem.

37. Singular radial field Consider the radial field $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}}$.

a. Evaluate a surface integral to show that $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b. Note that the first partial derivatives of the components of $\mathbf{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless the flux across the sphere as computed in part (a) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate div $\mathbf{F}$ over the region between two spheres of radius a and $0 < \epsilon < a$. Then let $\epsilon \to 0^+$ to obtain the flux computed in part (a).

38. Logarithmic potential Consider the potential function $\phi(x, y, z) = \dfrac{1}{2} \ln\left(x^2 + y^2 + z^2\right) = \ln|\mathbf{r}|$, where $\mathbf{r} = \langle x, y, z \rangle$.

a. Show that the gradient field associated with ϕ is $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2} = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$.

b. Show that $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi a$, where S is the surface of a sphere of radius a centered at the origin.

c. Compute div $\mathbf{F}$.

d. Note that $\mathbf{F}$ is undefined at the origin, so the Divergence Theorem does not apply directly. Evaluate the volume integral as described in Exercise 37.

Applications

39. Gauss' Law for electric fields The electric field due to a point charge Q is $\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$, where $\mathbf{r} = \langle x, y, z\rangle$, and ϵ_0 is a constant.

a. Show that the flux of the field across a sphere of radius a centered at the origin is $\iint_S \mathbf{E}\cdot\mathbf{n}\,dS = \frac{Q}{\epsilon_0}$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of charge within a region D. Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,dS = \frac{1}{\epsilon_0}\iiint_D q(x, y, z)\,dV.$$

d. Assuming $\mathbf{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (c) that $\nabla\cdot\mathbf{E} = \frac{q}{\epsilon_0}$.

e. Because the electric force is conservative, it has a potential function ϕ. From part (d) conclude that $\nabla^2\phi = \nabla\cdot\nabla\phi = \frac{q}{\epsilon_0}$.

40. Gauss' Law for gravitation The gravitational force due to a point mass M is proportional to $\mathbf{F} = GM\,\mathbf{r}/|\mathbf{r}|^3$, where $\mathbf{r} = \langle x, y, z\rangle$ and G is the gravitational constant.

a. Show that the flux of the force field across a sphere of radius a centered at the origin is $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi GM$.

b. Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c. Suppose there is a distribution of mass within a region D containing the origin. Let $\rho(x, y, z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi G\iiint_D \rho(x, y, z)\,dV.$$

d. Assuming $\mathbf{F}$ satisfies the conditions of the Divergence Theorem, conclude from part (c) that $\nabla\cdot\mathbf{F} = 4\pi G\rho$.

e. Because the gravitational force is conservative, it has a potential function ϕ. From part (d) conclude that $\nabla^2\phi = 4\pi G\rho$.

41-45. Heat transfer *Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector* $\mathbf{F}$ *at a point is proportional to the negative gradient of the temperature; that is,* $\mathbf{F} = -k\nabla T$*, which means that heat energy flows from hot regions to cold regions. The constant k is called the conductivity, which has metric units of* J/m–s–K *or* W/m–K. *A temperature function for a region D is given. Find the net outward heat flux* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -k\iint_S \nabla T\cdot\mathbf{n}\,dS$ *across the boundary S of D.*

In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume that $k = 1$.

41. $T(x, y, z) = 100 + x + 2y + z$; $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

42. $T(x, y, z) = 100 + x^2 + y^2 + z^2$; $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

43. $T(x, y, z) = 100 + e^{-z}$; $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

44. $T(x, y, z) = 100 + x^2 + y^2 + z^2$; D is the unit sphere centered at the origin.

45. $T(x, y, z) = 100\, e^{-x^2 - y^2 - z^2}$; D is the sphere of radius a centered at the origin.

Additional Exercises

46. Inverse square fields are special Let $\mathbf{F}$ be a radial field $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$, where p is a real number and $\mathbf{r} = \langle x, y, z\rangle$. With $p = 3$, $\mathbf{F}$ is an inverse square field.

a. Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$.

b. Explain the observation in part (a) by finding the flux of $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^p$ across the boundaries of a spherical box $\{(\rho, \phi, \theta) : a \le \rho \le b,\ \phi_0 \le \phi \le \phi_1,\ \theta_1 \le \theta \le \theta_2\}$ for various values of p.

47. A beautiful flux integral Consider the potential function $\phi(x, y, z) = G(\rho)$, where G is any twice differentiable function and $\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, G depends only on the distance from the origin.

a. Show that the gradient vector field associated with ϕ is $\mathbf{F} = \nabla\phi = G'(\rho)\,\dfrac{\mathbf{r}}{\rho}$, where $\mathbf{r} = \langle x, y, z\rangle$ and $\rho = |\mathbf{r}|$.

b. Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S. Show that the flux of $\mathbf{F}$ across S is $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = 4\pi a^2\, G'(a)$.

c. Show that $\nabla\cdot\mathbf{F} = \nabla\cdot\nabla\phi = \dfrac{2\,G'(\rho)}{\rho} + G''(\rho)$.

d. Use part (c) to show that the flux across S (as given in part (b)) is also obtained by the volume integral $\displaystyle\iiint_D \nabla\cdot\mathbf{F}\, dV$. (*Hint:* use spherical coordinates and integrate by parts.)

48. Integration by parts (Gauss' formula) Recall the Product Rule of Theorem 14.11: $\nabla\cdot(u\,\mathbf{F}) = u\,\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla u$.

a. Integrate both sides of this identity over a solid region D with a closed boundary S and use the Divergence Theorem to prove an integration by parts rule:

$$\iiint_D u\,\nabla\cdot\mathbf{F}\, dV = \iint_S u\,\mathbf{F}\cdot\mathbf{n}\, dS - \iiint_D \mathbf{F}\cdot\nabla u\, dV.$$

b. Explain the correspondence between this rule and the integration by parts rule for single-variable functions.

c. Use integration by parts to evaluate $\displaystyle\iiint_D \left(x^2 y + y^2 z + z^2 x\right) dV$, where D is the cube in the first octant cut by the planes $x = 1$, $y = 1$, and $z = 1$.

49. Green's Formula Write Gauss' Formula of Exercise 48 in two dimensions—that is, where $\mathbf{F} = \langle f, g\rangle$, D is a plane region R and C is the boundary of R. Show that the result is Green's Formula:

$$\iint_R u\left(f_x + g_y\right) dA = \oint_C u\,(\mathbf{F}\cdot\mathbf{n})\, ds - \iint_R \left(f\, u_x + g\, u_y\right) dA.$$

Show that with $u = 1$, one form of Green's *Theorem* appears. Which form of Green's Theorem is it?

50. Green's First Identity Prove Green's First Identity for twice differentiable scalar-valued functions u and v defined on a region D:

$$\iiint\limits_D \left(u\,\nabla^2 v + \nabla u \cdot \nabla v\right) dV = \iint\limits_S u\,\nabla v \cdot \mathbf{n}\, dS,$$

where $\nabla^2 v = \nabla \cdot \nabla v$. You may apply Gauss' Formula in Exercise 48 to $\mathbf{F} = \nabla v$ or apply the Divergence Theorem to $\mathbf{F} = u\,\nabla v$.

51. Green's Second Identity Prove Green's Second Identity for scalar-valued functions u and v defined on a region D:

$$\iiint\limits_D \left(u\,\nabla^2 v - v\,\nabla^2 u\right) dV = \iint\limits_S (u\,\nabla v - v\,\nabla u) \cdot \mathbf{n}\, dS.$$

(*Hint:* Reverse the roles of u and v in Green's First Identity.)

52-54. Harmonic functions *A scalar-valued function ϕ is* ***harmonic*** *on a region D if* $\nabla^2 \phi = \nabla \cdot \nabla \phi = 0$ *at all points of D.*

52. Show that the potential function $\phi(x, y, z) = |\mathbf{r}|^{-p}$ is harmonic provided $p = 0$ or $p = 1$, where $\mathbf{r} = \langle x, y, z\rangle$. To what vector fields do these potentials correspond?

53. Show that if ϕ is harmonic on a region D enclosed by a surface S, then $\iint_S \nabla \phi \cdot \mathbf{n}\, dS = 0$.

54. Show that if u is harmonic on a region D enclosed by a surface S, then $\iint_S u\,\nabla u \cdot \mathbf{n}\, dS = \iiint\limits_D |\nabla u|^2\, dV$.

55. Miscellaneous integral identities Prove the following identities.

a. $\iiint_D \nabla \times \mathbf{F}\, dV = \iint_S (\mathbf{n} \times \mathbf{F})\, dS$ (*Hint:* Apply the Divergence Theorem to each component of the identity.)

b. $\iint_S (\mathbf{n} \times \nabla \phi)\, dS = \oint_C \phi\, d\mathbf{r}$ (*Hint:* Apply Stokes' Theorem to each component of the identity.)