AN ALGORITHM FOR INTERSECTIONS IN $\mathbb{P}^{2[N]}$

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Abstract. We provide an explicit algorithm for computing intersection numbers between basis elements of complementary codimension in the Hilbert scheme of N points in the projective plane.

1. Introduction

The Hilbert scheme $\mathbb{P}^{2[N]}$ of N points in the projective plane is an irreducible, smooth, projective variety of dimension $2N$ [7,8]. Intersection theory on $\mathbb{P}^{2[N]}$ encodes many interesting enumerative questions. In his seminal work, Nakajima showed that for a projective surface X the direct sum of Chow rings $\oplus_{\mathbb{N}} A(X^{[N]})$ is an irreducible representation of the Heisenberg algebra [13,14]. However, using this description to determine the ring structure of $\mathbb{P}^{2[N]}$ is complicated even when N is small. Furthermore, computing explicit geometric or enumerative values, such as the top intersection products of divisors spanning the Picard group, can be difficult.

The first major progress towards computing the intersection products on $\mathbb{P}^{2[N]}$ was made by Ellingsrud and Strømme: they showed that the groups $A^d(\mathbb{P}^{2[N]})$ were free, established that rational, homological, and numerical equivalence coincide, and computed the Betti numbers [5]. They subsequently provided a cell decomposition whose closures form a basis for the groups $A^d(\mathbb{P}^{2[N]})$ [6]. Elencwajg and LeBarz completely computed the intersection product in $A(\mathbb{P}^{2[3]})$ using this basis [2–4]. For higher N, determining the intersection product using the basis of Ellingsrud and Strømme involves computing excess intersections and difficult multiplicity counting. Mallavibarrena and Sols provide a basis much more suited to such computations [12]. We prove the following theorem.

Main Theorem. *Let σ and τ be elements of complementary codimension in the basis of Mallavibarrena and Sols. There is an explicit algorithm to compute the intersection number $\sigma \cdot \tau$.*

Additionally, we show that the intersections between basis elements of complementary codimension can be computed along loci such that the points of intersection are all reduced points corresponding to reduced subschemes of points in the plane. This reduces the problem of computing each entry of the intersection matrices on $\mathbb{P}^{2[N]}$ to counting the number of ways to choose intersection points from two orthogonal sets of parallel lines in the plane such that they satisfy certain incidence conditions. In many cases, this is quick and easy to compute.

One case in which the Chow ring of a projective variety is well understood is the Grassmannian $G(k,n)$ of k-planes in an n-dimensional vector space. A geometric basis for the Chow ring of $G(k,n)$ is specified by partitions whose Young diagram has no row longer than $n-k$ and has at most k rows. The intersection product between these basis elements can be completely described using only this combinatorial description. For instance, if two basis elements have complementary codimension, then their intersection is one if and only if the partitions describing them are dual; otherwise it is zero. There is also Pieri's rule which gives a

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combinatorial description for the intersection of each Schubert class with special classes indexed by single integers. [1, 9].

It would be useful to describe the intersection theory on $\mathbb{P}^{2[N]}$, or more generally any moduli space, analogously to Schubert calculus. In particular, an identity akin to Pieri's rule would provide an easy method to compute the top intersection powers of divisors on $\mathbb{P}^{2[N]}$. The top intersection products of divisors which generatre the Picard group are exactly the coefficients of the polynomial function defining the volume of nef divisors, a birational invariant which is not currently known. More generally, Gholampour and Sheshmani were able to compute some top intersection numbers on the relative Hilbert schemes $(S/C)^{[N]}$ for projective surfaces S and divisors C on S in terms of intersection numbers on the Hilbert scheme of points $S^{[N]}$. The intersection numbers that they compute on $(S/C)^{[N]}$ are closely related to Donaldson-Thomas invariants of two dimensional sheaves on threefolds. Their results link Donaldson-Thomas invariants of $\mathbb{P}^3$ to that of the intersection theory on $\mathbb{P}^{2[N]}$ [10].

The paper is organized as follows. Section 2 describes the basis of Mallavibarrena and Sols for $A^*(\mathbb{P}^{2[N]})$ that we work with throughout the remainder of the paper. Section 3 gives a brief description of the algorithm through some examples. Section 4 proves the necessary results: intersections of complementary basis elements are transverse along nonsingular loci and consist of points corresponding to reduced subschemes of $\mathbb{P}^2$. Section 5 describes the algorithm to compute the intersections. Section 6 contains a link to the algorithm implemented in Python3 and contains the complete list of intersection matrices for $\mathbb{P}^{2[5]}$.

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2. The Basis of Mallavibarrena and Sols

We start with some notation. A partition $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r)$ of a nonnegative integer A is a nonincreasing sequence of nonnegative integers such that their sum is A. The length $\ell(\mathbf{a})$ of $\mathbf{a}$ is the number of nonzero entries. A mixed partition $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of N is a triple of partitions $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ of nonnegative integers A, B, and C such that $A + B + C = N$. The set I_N of all mixed partitions of N has a decomposition into disjoint subsets

$$I_{N,d} = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}) : N + \ell(\mathbf{a}) - \ell(\mathbf{c}) = d\}$$

each of which indexes a basis element for the group of codimension d algebraic cycles $A^d(\mathbb{P}^{2[N]})$[1].

Fix a mixed partition $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in I_{N,d}$. In $\mathbb{P}^2$, fix a set $\mathbb{L}$ of $\ell(\mathbf{a})$ general lines L_i, $\ell(\mathbf{a})$ general points $p_i \in L_i$, a set $\mathbb{M}$ of $\ell(\mathbf{b})$ general lines M_j, and a general point q. Let U be the locally closed subset of all subschemes Z which can be written as a union $Z_a \cup Z_b \cup Z_c$ such that:

- Z_a is a set of A distinct points containing each of the points p_i, and such that for each line L_i, $\mathbf{a}_i$ many of the points of Z_a lie on L_i with none of them the intersection point of any two lines from $\mathbb{L}$ or $\mathbb{M}$.
- Z_b is a set of B distinct points such that $\mathbf{b}_j$ of them lie on each M_j and such that none of them are the intersection of any two lines from $\mathbb{M}$ or $\mathbb{L}$.
- Z_c is a set of C distinct points, not containing q, none of which reside on any line in $\mathbb{L}$ or $\mathbb{M}$, and such that there are $\ell(\mathbf{c})$ distinct lines N_k through q with the property that for each k, $\mathbf{c}_k$ many reside on N_k.

Theorem 2.1 (Mallavibarrena, Sols [12])**.** *The class $\sigma_{(\mathbf{a},\mathbf{b},\mathbf{c})} := [\bar{U}]$ is independent of the choice of lines and points, and the set of $\sigma_{(\mathbf{a},\mathbf{b},\mathbf{c})}$ where $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ ranges over $I_{N,d}$ is a basis of $A^d(\mathbb{P}^{2[N]})$.*

[1]Our convention is different from that of [12]; our partitions are the conjugate partitions to theirs.

Counting dimensions, Z_a, Z_b, and Z_c have $A - \ell(\mathbf{a})$, B, and $C + \ell(\mathbf{c})$ degrees of freedom, so that Z has

$$A - \ell(\mathbf{a}) + B + C + \ell(\mathbf{c}) = N - \ell(\mathbf{a}) + \ell(\mathbf{c})$$

degrees of freedom. As such, the closure $\bar{U}$ is a subscheme of $\mathbb{P}^{2[N]}$ of codimension $N + \ell(\mathbf{a}) - \ell(\mathbf{c})$.

We suggest the following heuristic for thinking about the set U defining $\sigma_{(\mathbf{a},\mathbf{b},\mathbf{c})}$: each $\mathbf{a}_i$ describes that many points on each fixed line $L_i \in \mathbb{L}$ with one fixed and the others varying; each $\mathbf{b}_j$ describes that many points freely varying on each fixed line $M_j \in \mathbb{M}$; and each $\mathbf{c}_k$ describes that many points moving freely on a moving line N_k through q. All of the points are distinct, none of the points is the intersection of any of the lines from $\mathbb{L}$ or $\mathbb{M}$ with any other, and the moving points described by $\mathbf{c}$ do not coincide with q or lie on the lines of $\mathbb{L}$ or $\mathbb{M}$. Of course, in the closure $\bar{U}$, these conditions can be violated.

As an example, consider the mixed partition $(\mathbf{a},\mathbf{b},\mathbf{c}) = ((2,2,1),(3,2),(2,1,1))$ of $N = 14$. We fix general points and lines $p_1 \in L_1, p_2 \in L_2, p_3 \in L_3$, general lines M_1, M_2, and a general point q. We get the class $\sigma_{((2,2,1),(3,2),(2,1,1))} = [\bar{U}] \in A^{14}(\mathbb{P}^{2[14]})$ where U consists of all subschemes Z of $\mathbb{P}^2$ with $Z = Z_a \cup Z_b \cup Z_c$ such that:

- Z_a consists of five distinct points, three of which are p_1, p_2, and p_3, and two varying points, one on L_1 and one on L_2.
- Z_b consists of five distinct points, three of which are varying on M_1 and two of which are varying on M_2.
- Z_c consists of four distinct points, two of which are collinear with q and two of which are freely varying in the plane.

We may find it useful to draw schematic pictures for the classes. We will use solid lines and points to denote ones which are fixed, dashed lines and hollow points for those which are moving, and in the cases where it is necessary, a cross for the point q. To illustrate, Figure 1 is the schematic picture for the example class above.

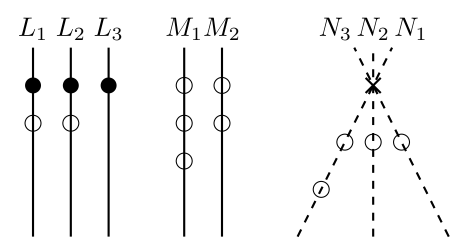

FIGURE 1. Schematic for $\sigma_{((2,2,1),(3,2),(2,1,1))}$.

3. Some Examples of the Algorithm

The simplest elements of the MS basis correspond to loci described by points varying on fixed lines. For instance, in $\mathbb{P}^{2[3]}$ the basis element indexed by the mixed partition $(0,(2,1),0)$ has codimension three, and is defined by two points varying on one fixed line and one point varying on another fixed line. Its self intersection number is one, as can be seen in Figure 2. In the figure, the two lines with labels to the left define one cycle, and the two lines with labels above define the other.

It is possible to compute the intersection in this situation directly. Computing more complicated intersections, however, especially those involving moving lines, will require the following combinatorial objects associated to an intersection.

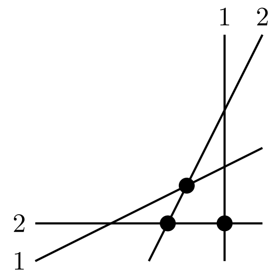

FIGURE 2. The intersection $\sigma^2_{(0,(2,1),0)}$ in $\mathbb{P}^{2[3]}$.

Definition 3.1. Let $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha)$ and $\beta = (\mathbf{a}^\beta, \mathbf{b}^\beta, \mathbf{c}^\beta)$ be mixed partitions of a positive integer N such that the classes they define have complementary codimension. Additionally, assume that $\ell(\mathbf{a}^\alpha) = \ell(\mathbf{c}^\beta)$ and $\ell(\mathbf{a}^\beta) = \ell(\mathbf{c}^\alpha)$.

(a) The *diagram* $D_{\alpha,\beta}$ associated to the intersection $\sigma_\alpha \cdot \sigma_\beta$ is a set of $\ell(\mathbf{a}^\alpha) + \ell(\mathbf{b}^\alpha) + \ell(\mathbf{c}^\alpha)$ vertical lines and a set of $\ell(\mathbf{a}^\beta) + \ell(\mathbf{b}^\beta) + \ell(\mathbf{c}^\beta)$ horizontal lines along with marked points given as follows. Index the horizontal lines from top to bottom and the vertical lines from left to right. The diagram also consists of fixed marked points: add a fixed marked point at the intersection of the i-th horizontal line with the $(\ell(\mathbf{a}^\alpha) + \ell(\mathbf{b}^\alpha) + i)$-th vertical line for $1 \le i \le \ell(\mathbf{c}^\alpha)$ and a fixed marked point at the intersection of the $(\ell(\mathbf{a}^\beta) + \ell(\mathbf{b}^\beta) + j)$-th line with the j-th line for $1 \le j \le \ell(\mathbf{c}^\beta)$.

(b) An *incidence labeling of* $D_{\alpha,\beta}$ is given as follows. Label the first $\ell(\mathbf{a}^\alpha)$ vertical lines by the sequence $\mathbf{a}^\alpha$, the next $\ell(\mathbf{b}^\alpha)$ vertical lines by the sequence $\mathbf{b}^\alpha$, and the last $\ell(\mathbf{c}^\alpha)$ vertical lines by any permutation of the sequence $\mathbf{c}^\alpha$. Label the horizontal lines similarly, except with the sequences $\mathbf{a}^\beta$, $\mathbf{b}^\beta$, and any permutation of $\mathbf{c}^\beta$.

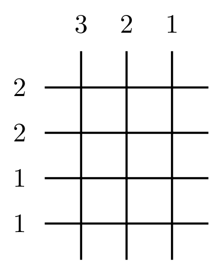

FIGURE 3. The unique incidence labeling of the diagram for the intersection $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$.

Notice that the intersection of classes corresponding to two mixed partitions α and β has exactly one diagram $D_{\alpha,\beta}$ (up to translating the lines – a harmless manipulation) which may admit many incidence labelings. We will often refer to a diagram as D when the context allows no confusion. When $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$ are both constant sequences, there is a unique incidence labeling of D. Figure 3 shows the unique incidence labeling of the diagram for the intersection $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$. Notice that there are no fixed marked points in this diagram because $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$ are both zero.

Figure 4 is a more complicated example involving many incidence labelings. It is a list of all possible incidence labelings of the diagram for the intersection $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1)}$ in $\mathbb{P}^{2[9]}$.

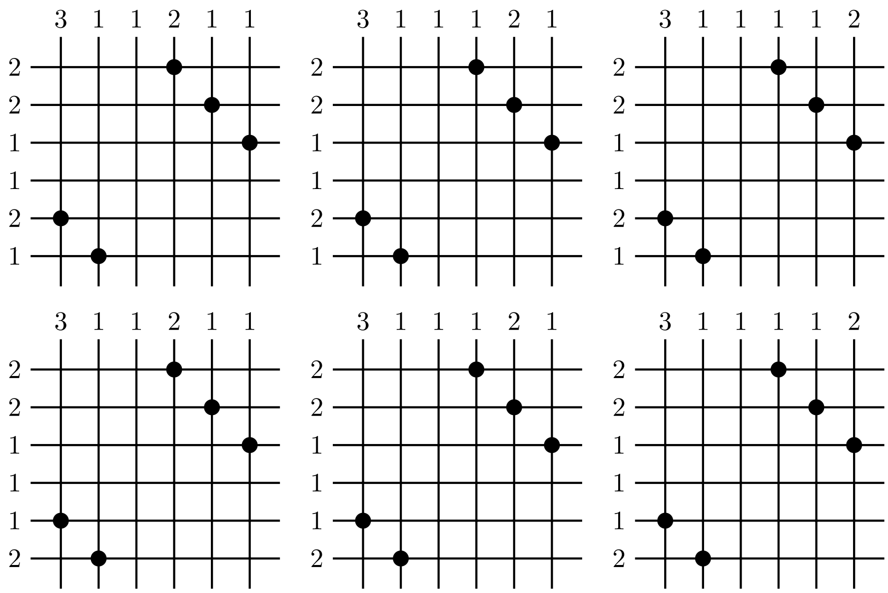

FIGURE 4. All possible incidence labelings of the diagram for the intersection $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1)}$ in $\mathbb{P}^{2[9]}$.

Two remarks are in order. First, the incidence labelings are taken over all permutations of both sequences $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$. Second, the sequences $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$ may contain repeated integers, so that they may not exhibit $\ell(\mathbf{c}^\alpha)!$ and $\ell(\mathbf{c}^\beta)!$ distinct permutations, respectively. Both of these are shown in Figure 4.

Definition 3.2. The *intersection number* of an incidence labeling of a diagram is the number of ways to choose sets Z consisting of N distinct intersection points of the lines in the diagram satisfying the following conditions:

(i) the number of points on each line is given by its label; and
(ii) Z contains all of the fixed marked points in the diagram.

It is easy to compute the intersection number of an incidence labeling: beginning with the first vertical line, and for each vertical line in the diagram, choose horizontal lines and mark those points of intersection. The label of the vertical line indicates how many horizontal lines to choose. We keep track of all such choices in a tree whose nodes are the diagram with marked points and whose edges correspond to different choices of points to mark. We will often conflate choices which occur with symmetry and indicate it by labeling the corresponding edge with a multiplicity. It is useful to drop portions of the tree or to ignore choices which obviously will not result in collections of points satisfying all the incidence labels of all the lines.

For instance, consider Figure 5, where we conduct the process for the unique incidence labeling of the diagram for $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$. The first vertical line has a label of three, there are four possible ways to choose three lines from the four horizontal ones, but up to symmetry, only two choices need be expressed and we label the corresponding edges with ·2. In the second step, we must choose two lines from the horizontal lines to pair with the second vertical line. We ignore choices which obviously violate the incidence conditions, such as any choice containing either of the bottom two lines in the left node. This leaves just one unique choice for the left node, and three unique choices,two of which are the same up to symmetry, for the right node. Finally, in each of the cases on the third step there is a unique choice of lines satisfying the incidence conditions. We conclude that the intersection number of the incidence labeling is eight.

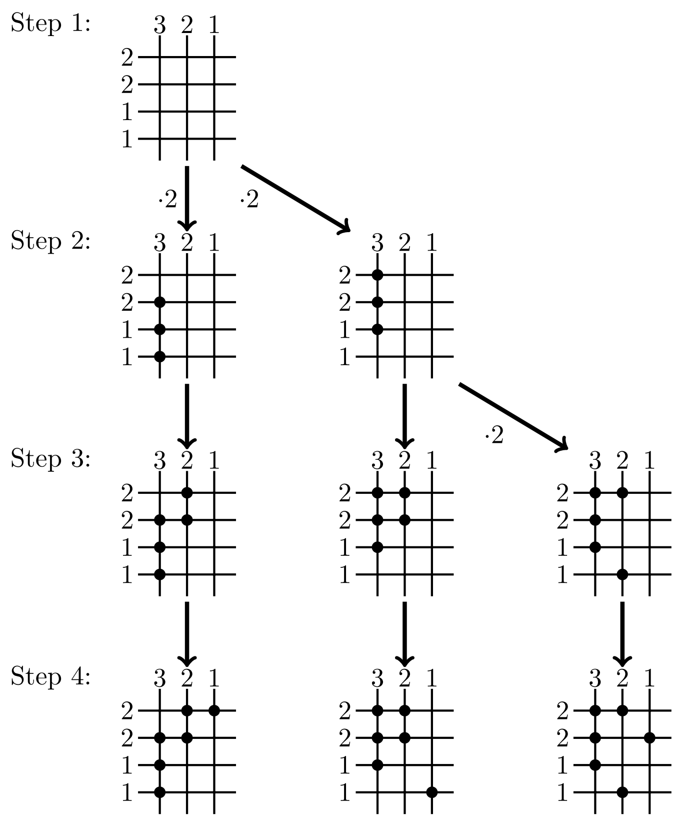

FIGURE 5. The tree for the unique incidence labeling of the diagram for the product $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$.

The intersection number $\sigma_\alpha \cdot \sigma_\beta$ is the sum of the intersection numbers of all incidence labelings of its diagram. We will omit the details of the computation, but the intersection numbers for the incidence labelings in Figure 4 are, from left to right and then top to bottom, 5,5,6,9,9, and 12, so that the intersection number $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1)} = 46$.

4. Transversality of Cycles

Fix a positive integer N, a nonnegative integer $d \leq N$, let $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha) \in I_{N,d}$, and let $\beta = (\mathbf{a}^\beta, \mathbf{b}^\beta, \mathbf{c}^\beta) \in I_{N,N-d}$ be two mixed partitions. Let e, f, and g be indices for $\mathbf{a}^\alpha$, $\mathbf{b}^\alpha$, and $\mathbf{c}^\alpha$, respectively, beginning at one and ending at the length of the partition. Similarly, let i, j, and k be indices for $\mathbf{a}^\beta$, $\mathbf{b}^\beta$, and $\mathbf{c}^\beta$. As was described in Section 2, let $p_e^\alpha \in L_e^\alpha, M_f^\alpha$, and q^α be the general fixed points and lines defining U_α, and let $p_i^\beta \in L_i^\beta, M_j^\beta$, and q^β be the general fixed points and lines defining U_β. Let $\sigma_\alpha = [\bar{U_\alpha}]$ and $\sigma_\beta = [\bar{U_\beta}]$. When necessary, we will refer to the lines defined by a subscheme $Z_c \subset Z$ in U_α or U_β with the appropriate superscript N_g^α or N_k^β as well. The lines N_g^α or N_k^β depend on the scheme Z in the intersection, but we hope any confusion induced by this is worth the reduction in notation. The goal of this section is to prove the following proposition.

Proposition 4.1. *The intersection number $\sigma_\alpha \cdot \sigma_\beta$ is equal to the number of points in the intersection $U_\alpha \cap U_\beta$ counted with multiplicity one.*

It is clear that the intersection number $\sigma_\alpha \cdot \sigma_\beta$ can be computed as the number of points in the intersection $\bar{U_\alpha} \cap \bar{U_\beta}$ with the proper multiplicities, assuming there is no excess intersection.

The idea is as follows. We will first eliminate schemes Z from the intersection which correspond to the support of Z satisfying any of several "support boundary conditions" (Corollary 4.3). This will allow us to then do away with schemes Z which contain nonreduced subschemes (Corollary 4.5). Since we can then conclude any such scheme Z will consist of distinct points, we will be able to give a useful description of the tangent spaces to U_α and U_β at such a scheme Z. We will then show that the intersection is transverse (Lemma 4.7).

4.1. **Support Boundary Conditions.** For the moment, let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a mixed partition of N, and p_i, L_i, and M_j points and lines defining U, the corresponding locally closed subscheme.

Let $Z \in \bar{U} \setminus U$ be a length N zero dimensional subscheme of $\mathbb{P}^2$, and let N_k be the lines defined by the subscheme $Z_c \subseteq Z$. We wish to first eliminate the possibility of such a scheme Z satisfying a small number of boundary conditions. These are the 5 possible cases we wish to consider:

1. The support of Z can contain an intersection point of two lines L_i, M_j or an intersection point of some L_i with some M_j;
2. The support of Z can contain q;
3. A line N_k can contain some point p_i;
4. Points of Z_c can lie on some line L_i or M_j;
5. Lines N_k can collide, so that there are $\ell(\mathbf{c}) - 1$ distinct lines N_k with $\mathbf{c}_{k_1} + \mathbf{c}_{k_2}$ many points residing on a single line.

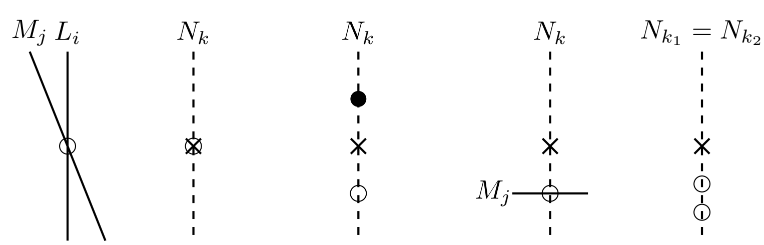

Figure 6. From left to right: the schematic diagrams for the boundary conditions 1-5.

Figure 6 shows the schematic diagrams for each case.

4.2. **Points of the Intersection.** With notation as in the beginning of the section, we prove the following lemma and corollaries.

Lemma 4.2. *Let Z be a scheme in the intersection $\bar{U_\alpha} \cap \bar{U_\beta}$. The number of moving points $\ell(\mathbf{c}^\alpha)$ defining U^α must equal the number of fixed points $\ell(\mathbf{a}^\beta)$ defining the other cell U^β. Furthermore, the lines N_g^α spanned by the points of $Z_c \subset Z$ must be the spans $\langle q^\alpha p_i^\beta \rangle$.*

Proof. The proof begins with the observation that the fixed points and lines used to define U_β can be chosen generally with respect to the fixed lines and points defining U_α, so it follows that the fixed points p_i^β will not lie on any of the fixed lines L_e^α or M_f^α. Furthermore, the lines $\langle p_i^\beta p_j^\beta \rangle$ for any suitable i, j do not contain the fixed point q^α. Therefore, each p_i^β must be contained in a unique moving line N_g^α.

We can also observe that the number $\ell(\mathbf{a}^\beta)$ of fixed points defining U_β must equal the number $\ell(\mathbf{c}^\alpha)$ of moving lines defining U_α: by the previous paragraph, we know that $\ell(\mathbf{a}^\beta) \leq \ell(\mathbf{c}^\alpha)$, and $\ell(\mathbf{a}^\alpha) \leq \ell(\mathbf{c}^\beta)$ by the same reasoning. The dimension of $\bar{U_\beta}$ is $d = N - \ell(\mathbf{a}^\beta) + \ell(\mathbf{c}^\beta)$ while the codimension of $\bar{U_\alpha}$ is $d = N + \ell(\mathbf{a}^\alpha) - \ell(\mathbf{c}^\alpha)$, so that $\ell(\mathbf{a}^\alpha) - \ell(\mathbf{c}^\alpha) = -\ell(\mathbf{a}^\beta) + \ell(\mathbf{c}^\beta)$. Combining this with the inequalities gives that $\ell(\mathbf{a}^\beta) = \ell(\mathbf{c}^\alpha)$. □

Corollary 4.3. *Let Z be a scheme in the intersection $\bar{U_\alpha} \cap \bar{U_\beta}$. Z cannot satisfy any of the five boundary conditions shown above.*

Proof. Lemma 4.2 immediately eliminates such schemes from the intersection:

1. The intersection of two fixed lines L_i^β or M_j^β will not be contained in any of the possible lines $\langle q^\alpha p_i^\beta \rangle$, nor will they be contained in the lines L_e^α or M_f^α.
2. q^β will not be contained in any of the lines $\langle q^\alpha p_i^\beta \rangle$, nor will it be contained in the L_e^α or M_f^α.
3. The lines N_g^α must contain q^α and some p_i^β and will therefore not contain any p_e^α.
4. Fix some N_g^α as the span $\langle q^\alpha p_i^\beta \rangle$; the intersection of this line with any of the L_e^α's or M_g^α's will not occur on the lines L_i^β, M_j^β, or N_k^β (each of which must be the span $\langle q^\beta p_e^\alpha \rangle$ for some p_e^α).
5. The lines N_g^α must each be the span $\langle q^\alpha p_i^\beta \rangle$ for a unique p_i^β, so they cannot coincide.

□

Corollary 4.4. *Let Z be a scheme in the intersection $\bar{U_\alpha} \cap \bar{U_\beta}$. Assume that Z contains a nonreduced subscheme W whose support is a single point. W is contained in the some fixed line L_e^α or M_f^α, or it is collinear with q_α.*

Proof. Let $\{Z_t\}$ be a family of schemes in U_α whose limit is Z. There is a subfamily $\{W_t\}$ whose limit is W. The support p of W is contained in some fixed line L_e^α, M_f^α, or $\langle q^\alpha p_i^\beta \rangle$. Let W be the fiber over $t = t_0$. By Corollary 4.3, p is not the intersection of any two of the lines L_e^α, M_f^α, or $\langle q^\alpha p_i^\beta \rangle$. As a consequence, after possibly shrinking to a small neighborhood of t_0, the support of the schemes W_t must satisfy exactly one of the following:

(i) be contained in L_e^α;
(ii) be contained in M_f^α; or
(iii) be collinear with q^α.

Hence, W must satisfy exactly one of i), ii), and iii) as well. □

Of course, there is nothing special about the choice of α here, and the lemma can be applied to β. We obtain the following corollary.

Corollary 4.5. *Let Z be a scheme in the intersection $\bar{U_\alpha} \cap \bar{U_\beta}$. Z cannot contain a nonreduced subscheme. Furthermore, Z is contained in $U_\alpha \cap U_\beta$.*

Proof. Assume that Z contains a nonreduced subscheme W supported at a point p. By Corollary 4.4, W is contained in and determines some L_e^α or M_f^α, or is collinear with q^α. Similarly, W is contained in some L_i^β or M_j^β, or is collinear with q^β. In any case, these lines are distinct and W cannot be contained in both, unless it is reduced. □

By Corollary 4.5, it remains only to show that U_α and U_β are transverse at each point of their intersection. To that end, we show that the tangent spaces of the locally closed subsets U_α and U_β are in fact closely related to the tangent spaces of their defining lines and points in the plane. This relationship allows us to observe that the subsets U_α and U_β are in fact smooth at any point of their intersection, as well as to observe transversality.

4.3. **Description of the Tangent Spaces of the Locally Closed Subsets U at a General Point.** Let Z be a general point of a locally closed subset U defining the class $\sigma_{(\mathbf{a},\mathbf{b},\mathbf{c})}$ corresponding to a reduced subscheme of $\mathbb{P}^2$. Since Z consists of N distinct points, there is a chart on $\mathbb{P}^{2[N]}$ centered at Z and isomorphic to an open set of $(\mathbb{A}^2)^N$. We write the product $(\mathbb{A}^2)^N$ as the product $(\mathbb{A}^2)^{\ell(\mathbf{a})} \times (\mathbb{A}^2)^{\ell(\mathbf{b})} \times (\mathbb{A}^2)^{\ell(\mathbf{c})}$ and describe the tangent space to U as a product of subspaces of each piece, respectively.

For each fixed point described by $\mathbf{a}$, the tangent space in the corresponding copy of $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^0$. For each moving point described by $\mathbf{a}$ or $\mathbf{b}$, the tangent space is the fixed line on which it lies.

We can further decompose the product $(\mathbb{A}^2)^{\ell(\mathbf{c})}$ into $(\mathbb{A}^2)^{\mathbf{c}_0} \times \cdots \times (\mathbb{A}^2)^{\mathbf{c}_{\ell(\mathbf{c})-1}}$ corresponding to each line N_k. Working in just one such piece $(\mathbb{A}^2)^{\mathbf{c}_k}$, let (x_r, y_r) be points of Z, (u_r, v_r) coordinates around each point, and (q_x, q_y) the coordinates of q. The condition that three distinct points (q_x, q_y), (u_r, v_r), and (u_s, v_s) in the plane are collinear is given locally by

$$\det \begin{bmatrix} 1 & 1 & 1 \\ q_y & v_r & v_s \\ q_x & u_r & u_s \end{bmatrix} = 0,$$

so that the equations for U in $(\mathbb{A}^2)^{\mathbf{c}_k}$ are precisely these determinants for all $1 \le r < s \le \mathbf{c}_k$. Notice that we can view these equations from the perspective of allowing one point $p_1 = (x_1, y_1)$ to vary freely, and the rest in the line defined by q and p_1. In this way, it becomes clear that there are $\mathbf{c}_k - 1$ independent linear relations defining the tangent space at Z to U in $(\mathbb{A}^2)^{\mathbf{c}_k}$.

Lemma 4.6. *U is smooth at Z.*

Proof. Recall that $\mathbf{a}$ is taken to be a partition of A, $\mathbf{b}$ a partition of B, and $\mathbf{c}$ a partition of C for nonnegative integers A, B, and C such that $A+B+C = N$. The culmination of these observations is that the dimension of the tangent space to U at Z is

$$A - \ell(\mathbf{a}) + B + \sum_{1}^{\ell(\mathbf{c})} 2\mathbf{c}_k - \mathbf{c}_k + 1 = A + B + C - \ell(\mathbf{a}) + \ell(\mathbf{c}) = \dim U$$

and that U is smooth at Z. □

4.4. **Tangent Spaces at Points of the Intersection.**

Lemma 4.7. *Let Z be a point of the intersection $U_\alpha \cap U_\beta$. The tangent spaces $T_Z U_\alpha$ and $T_Z U_\beta$ are transverse.*

Proof. Corollaries 4.3 and 4.5 allow us to work in a local chart of $\mathbb{P}^{2[n]}$ around Z isomorphic to $(\mathbb{A}^2)^n$. As such, we can check transversality in each factor individually, or as will be more natural for us, on subproducts.

In these charts, and by our description above, it is clear that the tangent space in any factor corresponding to a point $z \in Z$ which is the intersection of a fixed line defining U_α with a fixed line defining U_β is transverse. It suffices then to check that the intersection of U_α and U_β is transverse in any subproduct $(\mathbb{A}^2)^{\mathbf{c}_k}$ corresponding to some moving line N_g^α or N_k^β. For the sake of the argument, we will assume that the moving is in U_β, and it follows from Lemma 4.2 that one of the relevant points z_1 in Z is fixed in the description of U_α.

In fact, let $z_1 = (x_1, y_1)$ be the point whose chart is contained in first factor of $\mathbb{A}^2$. The equations defining U_β in $(\mathbb{A}^2)^{\mathbf{c}_k}$ can be taken such that they are the determinants

$$\det \begin{bmatrix} 1 & 1 & 1 \\ q_y & v_1 & v_s \\ q_x & u_1 & u_s \end{bmatrix} = 0$$

for $2 \leq s \leq \mathbf{c}_k$. As such, it suffices to check the tangent spaces to U_α and U_β are transverse at any pair of points z_1 and (x_i, y_i) in Z for $2 \leq i \leq \mathbf{c}_k$.

Let $q_\beta = (q_x, q_y)$ be the fixed point describing U_β, $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in Z$ be the points we will be considering, and (u_1, v_1), (u_2, v_2) charts of $\mathbb{A}^2 \times \mathbb{A}^2$ around z_1 and z_2, respectively. Assume that z_1 is a fixed point of U_α, meaning that every scheme in U_α contains z_1, so that the tangent space to U_α in the corresponding copy of $\mathbb{A}^2$ is given by $u_1 = v_1 = 0$. From our analysis above, the tangent space to U_β around z_1 and z_2 is given by

$$q_y u_1 - q_x v_1 = q_y u_2 - q_x v_2$$

which, after substituting $u_1 = v_1 = 0$ reduces to the linear space defined by the right hand side. It follows that the tangent space to U_β in the second copy of $\mathbb{A}^2$ is the line through the origin defined by the slope of the line through q_β. In any case, this intersects the tangent space to U_α at z_1 in just a point, assuming the fixed lines defining U_α and the moving lines N_g^α defined by Z are distinct from the line through q_β and z_0. This is true by Lemma 4.2, and since the tangent spaces intersect in just the origin, they are transverse. □

This establishes Proposition 4.1.

5. The Algorithm

Fix an N and two mixed partitions $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha)$ and $\beta = (\mathbf{a}^\beta, \mathbf{b}^\beta, \mathbf{c}^\beta)$ of complementary codimension. We may assume that $\ell(\mathbf{a}^\alpha) = \ell(\mathbf{c}^\beta)$ and $\ell(\mathbf{a}^\beta) = \ell(\mathbf{c}^\alpha)$ otherwise the intersection is zero. Let $\ell_\alpha = \ell(\mathbf{a}^\alpha) + \ell(\mathbf{b}^\alpha) + \ell(\mathbf{c}^\alpha)$ and $\ell_\beta = \ell(\mathbf{a}^\beta) + \ell(\mathbf{b}^\beta) + \ell(\mathbf{c}^\beta)$. Refer to Section 3 for examples.

5.1. **Step 1 – Diagrams.** Fix a set of ℓ_α vertical lines indexed from top to bottom and a set of ℓ_β horizontal lines indexed from left to right in the plane. Mark the fixed intersection points of the i-th horizontal line with the $(\ell(\mathbf{a}^\beta)+\ell(\mathbf{b}^\beta)+i)$-th vertical line and the $(\ell(\mathbf{a}^\alpha)+\ell(\mathbf{b}^\alpha)+j)$-th horizontal line with the j-th vertical line for $1 \leq i \leq \mathbf{c}^\beta$ and $1 \leq j \leq \mathbf{c}^\alpha$. This establishes the diagram D as in Definition 3.1. See Figure 7.

5.2. **Step 2 – Labels.** Produce all incidence labelings of D (as in Definition 3.1): label the first $\ell(\mathbf{a}^\alpha)$ vertical lines with the sequence $\mathbf{a}^\alpha$, the next $\ell(\mathbf{b}^\alpha)$ vertical lines with $\mathbf{b}^\alpha$, the first $\ell(\mathbf{a}^\beta)$ horizontal lines with $\mathbf{a}^\beta$, and the next $\ell(\mathbf{b}^\beta)$ horizontal lines with $\mathbf{b}^\beta$. For any pair $(\mathbf{d}^\alpha, \mathbf{d}^\beta)$ of permutations of $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$, respectively, obtain an incidence labeling by labeling the last $\ell(\mathbf{c}^\alpha)$ vertical lines by $\mathbf{d}^\alpha$ and the last $\ell(\mathbf{c}^\beta)$ horizontal lines by $\mathbf{d}^\beta$. Let Ad be the set of all admissibly labeled diagrams. See Figure 8.

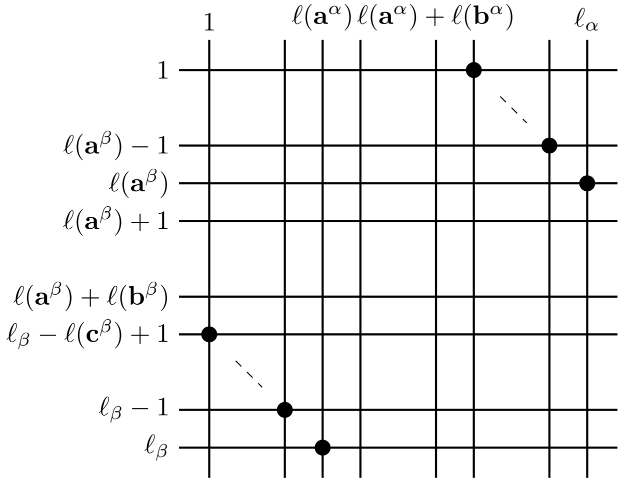

FIGURE 7. The diagram for an arbitrary α and β. Lines are labeled by their indices. Indices are omitted on some of the vertical lines for readability.

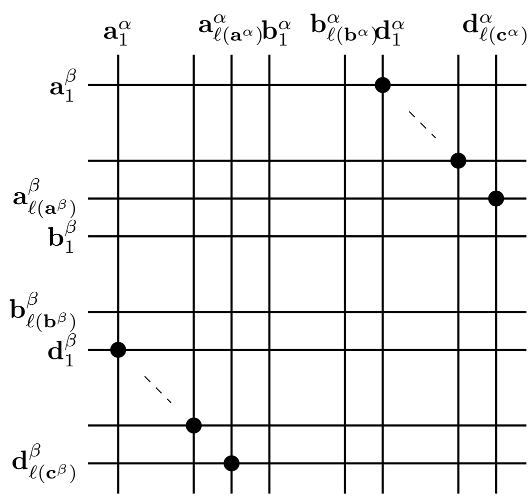

FIGURE 8. An arbitrary incidence labeling of D. The sequences $\mathbf{d}^\alpha$ and $\mathbf{d}^\beta$ are any permutations of $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$, respectively.

5.3. **Step 3 – Trees.** For each incidence labeling a of D in Ad, generate a tree T_a as follows. The root of the tree consists of a.

Each node consists of a with extra marked points. The number of marked points on each line must not exceed its label. Let N be a node of depth d. Let z be the integer label of the $(d+1)^{\text{st}}$ vertical line. Each child of N consists of a with the extra marked points from N and an additional z marked points on the

$(d+1)^{\text{st}}$ vertical line. The extra points are the intersection of the $(d+1)^{\text{st}}$ vertical line with z horizontal lines such that the resulting number of points on each horizontal line does not exceed its label. There is one child for each such choice. If there are no such choices, then N has no children.

5.4. **Step 4 – Removing Erroneous Leaves.** Beginning with a tree T_{a}, repeatedly remove any leaves which do not contain the correct number of marked points, given by the label, on each line.

The intersection number i_{a} of the incidence labeling a (See Definition 3.2) is the number of remaining leaves.

Theorem 5.1. *The intersection number $\sigma_\alpha \cdot \sigma_\beta$ is equal to the sum*

$$\sum_{a \in Ad} i_a.$$

Proof. We will use the notation of Section 4 to refer to the lines, points, and partitions defining U_α and U_β. We define a correspondence between the lines in the diagram $D = D_{\alpha,\beta}$ underlying any $\mathsf{a} \in \mathsf{Ad}$ and the lines L_e^α, L_i^β, M_f^α, M_j^β, as well as the lines $\langle q^\alpha p_i^\beta \rangle$ and $\langle q^\beta p_e^\alpha \rangle$ as follows.

The first $\ell(\mathbf{a}^\alpha)$ vertical lines correspond, in order, to the lines L_e^α, the next $\ell(\mathbf{b}^\alpha)$ vertical lines correspond, in order, to the lines M_f^α, and the last $\ell(\mathbf{c}^\alpha)$ vertical lines correspond to the spans $\langle q^\alpha p_i^\beta \rangle$, also in order. Similarly, the horizontal lines correspond to the lines defining U_β.

Define a map of sets

$$\varphi : \{\text{leaves of the trees } T_{\mathsf{a}}\} \to U_\alpha \cap U_\beta$$

as follows. Let λ be any leaf of any tree T_{a}. Each marked point in λ is the intersection of two lines in D and defines a point in $\mathbb{P}^2$ given as the intersection of the corresponding lines defining U_α and U_β. The image $\varphi(\lambda)$ is the subscheme of N distinct points given as the union over all the marked points in λ of the corresponding points in $\mathbb{P}^2$. This map is well defined since in each λ there are exactly N marked points, and the N points satisfy the incidence conditions on the lines defining U_α and U_β.

The intersection of $\bar{U_\alpha} \cap \bar{U_\beta}$ occurs only along $U_\alpha \cap U_\beta$ by Corollary 4.5, and the intersection $U_\alpha \cap U_\beta$ is zero dimensional and consists of reduced points by Lemma 4.7. Therefore, the intersection number $\sigma_\alpha \cdot \sigma_\beta$ can be computed as the number of points in $U_\alpha \cap U_\beta$, and it suffices to show that φ is a bijection.

Two leaves of different trees T_{a} and T_{b} corresponding to different incidence labelings a and b cannot map to the same point in $U_\alpha \cap U_\beta$ since some line must be labeled differently in a than it is in b and therefore contains a different number of marked points. Two leaves from the same tree result from different choices of intersection points of some line in the diagram D. Hence φ is injective.

Let Z be a scheme in the intersection $U_\alpha \cap U_\beta$. We must show that there is some incidence labeling $\mathsf{a} \in \mathsf{Ad}$ whose corresponding tree T_{a} contains a leaf mapping to Z. By Proposition 4.1, Z is reduced and consists of N distinct points, and by Lemma 4.2 and its Corollary 4.3, each of these points resides on a single line L_e^α, M_f^α, or $\langle q^\alpha p_i^\beta \rangle$. Similarly, each of these points resides on a single line L_i^β, M_j^β, or $\langle q^\beta p_e^\alpha \rangle$. Furthermore, $\mathbf{a}_e^\alpha$ of them must reside on L_e^α, $\mathbf{b}_f^\alpha$ on M_f^α, and the remaining points must reside on the lines $\langle q^\alpha p_i^\beta \rangle$ with a unique nonzero entry of $\mathbf{c}^\alpha$ describing the number of points on each. Completely analogously, $\mathbf{a}_i^\beta$ points lie on the lines L_i^β, $\mathbf{b}_j^\beta$ on the lines M_j^β, and the remaining points residing on the lines $\langle q^\beta p_e^\alpha \rangle$ with unique nonzero entries of $\mathbf{c}^\beta$ describing the number on each line.

Define an incidence labeling of D by labeling each line in D by the number of points in Z lying on its corresponding line describing U_α or U_β. For each point $z \in Z$, mark the intersection of the lines in D corresponding to the unique line containing z describing U_α and the unique line containing z describing U_β. Marking the points on each vertical line in order from left to right allows us to follow the tree to the leaf

λ, which is not removed in Step 4 since it consists of exactly N points with the correct number residing on each line. □

One can ask if it possible to develop a theory of Schubert calculus for moduli spaces such as $\mathbb{P}^{2[N]}$. Our results suggest that it would necessarily be more complex than for the Grassmannians since intersections of complementary codimension Schubert classes in the Grassmannian are always zero or one, while ours can be much larger and are seemingly more difficult to compute. We'll conclude this section with the following question.

Question 5.2. *Can one describe the Chow ring of $\mathbb{P}^{2[N]}$ similarly to Schubert calculus for the Grassmannians? In particular, is there a simple to compute Pieri's rule and/or Giambelli's rule for $\mathbb{P}^{2[N]}$ in any suitable basis?*

6. Computations and Implementation

An implementation of the algorithm in Python3 can be found at http://homepages.math.uic.edu/~astathis/research/IntAlg/count.py.

To demonstrate the ease at which the algorithm performs computations which would otherwise be laborious, we list below the intersection matrices for $\mathbb{P}^{2[5]}$. We order the basis elements in ascending lexicographic order given as follows:

If $\varphi = (\varphi_1, \dots, \varphi_r)$ and $\psi = (\psi_1, \dots, \psi_s)$ are partitions of integers R_1 and R_2, respectively, then $\varphi < \psi$ if $R_1 < R_2$, or if $R_1 = R_2$, then $\varphi < \psi$ if for some $i \leq r$ we have $\varphi_j = \psi_j$ for $j < i$ and $\varphi_i < \psi_i$.

If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are mixed partitions of an integer N, then $\alpha < \beta$ if for some $i \leq 3$ we have $\alpha_j = \beta_j$ and $\alpha_i < \beta_i$.

In each matrix, the dimension d basis elements index the columns, while the codimension d elements index the rows.

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

Figure 9. The pairing matrix for $A^1(\mathbb{P}^{2[5]}) \times A_1(\mathbb{P}^{2[5]})$.

$$\begin{bmatrix} 6 & 3 & 0 & 4 & 3 & 1 \\ 3 & 0 & 0 & 2 & 1 & 0 \\ 6 & 3 & 0 & 5 & 4 & 2 \\ 2 & 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 10. The pairing matrix for $A^2(\mathbb{P}^{2[5]}) \times A_2(\mathbb{P}^{2[5]})$.

$$\begin{bmatrix}
6 & 2 & 0 & 0 & 0 & 3 & 1 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 3 & 1 & 0 & 0 & 4 & 2 & 0 & 3 & 1 & 1 & 0 \\
6 & 2 & 0 & 0 & 0 & 4 & 1 & 0 & 2 & 1 & 0 & 0 \\
12 & 6 & 2 & 0 & 0 & 10 & 5 & 0 & 8 & 4 & 3 & 1 \\
6 & 2 & 0 & 0 & 0 & 5 & 2 & 0 & 4 & 1 & 1 & 0 \\
6 & 3 & 1 & 0 & 0 & 6 & 3 & 0 & 6 & 3 & 3 & 1 \\
3 & 1 & 0 & 0 & 0 & 3 & 1 & 0 & 3 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

FIGURE 11. The pairing matrix for $A^3(\mathbb{P}^{2[5]}) \times A_3(\mathbb{P}^{2[5]})$.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 12 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 18 & 9 & 3 & 0 & 0 & 0 & 6 & 3 & 0 & 0 & 1 & 0 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 12 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 24 & 12 & 4 & 0 & 0 & 0 & 12 & 6 & 0 & 0 & 4 & 1 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 12 & 5 & 1 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 1 & 0 \\
6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

FIGURE 12. The pairing matrix for $A^4(\mathbb{P}^{2[5]}) \times A_4(\mathbb{P}^{2[5]})$.

$$
\begin{bmatrix}
120 & 60 & 30 & 20 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 27 & 12 & 7 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 7 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 3 & 6 & 3 & 1 & 0 & 0 & 2 & 4 & 4 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 3 & 6 & 3 & 1 & 0 & 0 & 3 & 6 & 6 & 3 & 0 & 3 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 2 & 6 & 3 & 1 & 0 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 2 & 6 & 3 & 1 & 0 & 0 & 1 & 4 & 6 & 3 & 0 & 2 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

FIGURE 13. The pairing matrix for $A^5(\mathbb{P}^{2[5]}) \times A_5(\mathbb{P}^{2[5]})$.

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