Intersection Theory on the Hilbert Scheme of Points in the Projective Plane

by

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Thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2017

Chicago, Illinois

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To my nephews,

Jack and Levi:

may you explore whatever realms you so desire.

ACKNOWLEDGEMENTS

ACKNOWLEDGEMENTS (Continued)

you, UIC would not have been the comfortable, compelling, and thriving environment that it has been for me.

Finally, I am grateful to the National Science Foundation and UIC for financial support.

AJS

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SUMMARY

Let σ be an arbitrary element of the Mallavibarrena and Sols (MS) basis for the Chow ring of the Hilbert scheme of points in the projective plane. Let H be the divisor of schemes whose support meets a fixed general line. Let τ be an element of the MS basis which has complementary codimension with σ. We have two main results:

1. There is an explicit algorithm to compute the intersection number $\sigma \cdot \tau$.

2. There is an explicit algorithm to compute the class of the intersection $H \cdot \sigma$ in the MS basis.

CHAPTER 1

INTRODUCTION

In this thesis, we will describe the intersection product on the Hilbert scheme of points in the projective plane in a basis due to Mallavibarrena and Sols (MS). A description of this basis can be found in Section 2.4.

The Hilbert scheme $X^{[N]}$ of N points on a projective variety X was introduced by Grothendieck as a natural compactification of the space of unordered N-tuples of distinct points on X (1). When X is a curve, $X^{[N]}$ is the N-th symmetric product of X and if $\dim X \geq 3$, then $X^{[N]}$ can have many irreducible components (2). However, when X is a smooth projective surface, Fogarty proved that the Hilbert scheme $X^{[N]}$ is a smooth irreducible projective variety of dimension $2N$, and considered the problem of studying the geometry of $X^{[N]}$ (3; 4).

Many interesting invariants arise as intersection products in the cohomology ring $H^*(X^{[N]})$ for a smooth complex projective surface X. For instance, if $D_1, \ldots, D_k$ are divisors on $X^{[N]}$ whose corresponding linear functionals generate the codimension one cohomology, and if one can compute the degree of the intersection products $D_1^{i_1} D_2^{i_2} \cdots D_k^{i_k}$ for $\sum i_k = 2N$, then one can give explicitly the volume of nef divisors on $X^{[N]}$, a birational invariant for which not many examples are known (5). Additionally, through a method of Gholampour and Seshmani, these top intersection products would allow one to compute Donaldson-Thomas invariants of 2-dimensional sheaves inside threefolds, a class of invariants motivated by mathematical physics and string theory (6; 7). Another classical example of such invariants would be the following: given a linear system $|L|$ of dimension $3N - 2$ on X inducing a map $X \to \mathbb{P}^{3N-2}$, how many

N-secants of dimension $N - 2$ are there to the image of X? This number can be computed as the degree of the top Segre class $s_{2N}(L^{[N]})$ of the tautological bundle $L^{[N]}$ associated to L.

To that end, there have been many attempts to describe the ring structure of the cohomology ring $H^*(X^{[N]})$ (8; 9; 10; 11; 12; 13). Most notably, after an observation of Vafa and Witten, Nakajima and Gojnowski provided a geometric construction of the ring $\oplus_{N \in \mathbb{N}} H^*(X^{[N]})$ as a representation of a Heisenberg algebra thus linking Hilbert schemes of points on surfaces to representation theory and theoretical physics (14; 15; 16; 17). Lehn was able to use this description to compute the top Segre classes for $N \leq 7$ using Nakajima's description and algorithmically for $N > 7$, although he notes that it is computationally intense (18). More recently, Marian, Oprea, and Pandharipande were able to provide an explicit formula for a more general class of intersections using a novel technique in the case where X is a K3 surface (19).

Our results aim to provide an intuitive and concrete description of the intersection product by specifying to the projective plane. Our main theorem describes the action of a natural divisor on any element of the MS basis. The divisor H is the class of the locus of schemes whose support meets a fixed general line. Let σ be any element of the MS basis.

Main Theorem 1. *There is an explicit algorithm to compute the class in the MS basis of the intersection $H \cdot \sigma$.*

We compute the support of the special fiber of two degenerations of loci in the Hilbert scheme along the way, and describe an explicit process by which we can determine the coefficients of the classes of each irreducible component of the support of the special fiber.

We also show that the natural representative loci of any two elements of complementary codimension in the Mallavibarrena and Sols basis intersect transversely, and provide an explicit

algorithm to compute the class of these intersections. Let σ and τ be elements of the MS basis such that they are of complementary codimension.

Main Theorem 2. *There is an explicit algorithm to compute the intersection number $\sigma \cdot \tau$.*

CHAPTER 2

PRELIMINARIES & BACKGROUND

Before we begin, we will fix some conventions and notation for the remainder of this document. We will work over the field of complex numbers $\mathbb{C}$. Unless otherwise indicated, N will be a fixed positive integer. We will use (sub)variety to mean (sub)algebraic set, so that they are not necessarily irreducible.

Let A be a nonnegative integer. A *partition* $\mathbf{a}$ of A is a non-increasing sequence of integers $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots)$ such that their sum $\sum \mathbf{a}_i = A$. The *length* $\ell(\mathbf{a})$ of the partition $\mathbf{a}$ is the number of nonzero entries[1]. It will often be useful to have an *index i* for $\mathbf{a}$ which we will take to be any integer $1 \leq i \leq \ell(\mathbf{a})$. In this way, we can refer to any nonzero entry $\mathbf{a}_i$ of the partition $\mathbf{a}$.

2.1   A Little Bit About the Chow Ring

A more complete discussion of what follows in this section can be found in *3264 & All That* by Eisenbud and Harris or in *Intersection Theory* by Fulton (20; 21).

Let X be an arbitrary smooth irreducible projective variety of dimension n. The *group of cycles* $Z(X)$ is the free abelian group generated by its irreducible subvarieties. The group is graded by both dimension, and since X is irreducible, codimension. That is,

$$Z(X) = \bigoplus_{k=0}^{n} Z_k(X) = \bigoplus_{k=0}^{n} Z^{n-k}(X)$$

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[1]It will be necessary to alter this convention in Section 4.3.

where $Z_k(X) = Z^{n-k}(X)$ is the subgroup generated by subvarieties of dimension k and is called the *group of k-cycles*.

Two cycles α and β are *rationally equivalent* if there is an irreducible subvariety Ω of $\mathbb{P}^1 \times X$ such that the projection from Ω onto $\mathbb{P}^1$ is dominant and such that $\Omega \cap (\{0\} \times X) = \alpha$ and $\Omega \cap (\{\infty\} \times X) = \beta$. We form the *subgroup of rationally equivalent cycles* $R(X) \subseteq Z(X)$ as the subgroup generated by differences of subvarieties

$$(\Omega \cap (\{0\} \times X)) - (\Omega \cap (\{\infty\} \times X)).$$

Definition 2.1.1. The *Chow group* of X is the quotient

$$A(X) = Z(X)/R(X).$$

If Y is a subvariety of X, we will refer to its rational equivalence class in $A(X)$ as $[Y]$.

The subgroup $R(X)$ respects the grading on the Chow group, so that $A(X)$ is also graded by dimension and codimension. Precisely,

$$A(X) = \bigoplus_{k=0}^{n} A_k(X) = \bigoplus_{k=0}^{n} A^{n-k}(X)$$

where $A_k(X) = A^{n-k}(X) = Z_k(X)/R_k(X)$ is the *group of k-cycles modulo rational equivalence*.

If W and Y are two subvarieties of X, then it is natural to try to define the product of their classes as $[W] \cdot [Y] = [W \cap Y]$ in $A(X)$. Unfortunately, things are not always so simple. For instance, fix a smooth conic C in $\mathbb{P}^2$ and two lines L and M such that L is not tangent to C and M is tangent to C. It is not hard to see that the two lines are rationally equivalent so that

$[L] = [M]$ (one simply projects from $L \cap M$), but $M \cap C$ and $L \cap C$ are a single point and two distinct points, respectively, and are hence not rationally equivalent in $A(\mathbb{P}^2)$.

To fix this, let W_i be a finite collection of subvarieties of X. The intersection $\bigcap W_i$ is *transverse at* p if p is a smooth point of each W_i and if

$$\operatorname{codim}\left(\bigcap T_p W_i\right) = \sum_i \operatorname{codim} T_p W_i.$$

It is *generically transverse* if it is transverse at a dense set of points in the intersection. Two cycles $\sum W_i$ and $\sum Y_j$ are generically transverse if each W_i is generically transverse to each Y_j. In our example above, the line L is transverse to C while the line M is not.

Lemma 2.1.2 (Moving Lemma)**.** *For every* $\eta, \nu \in A(X)$ *there are generically transverse cycles* $W, Y \in Z(X)$ *such that* $[W] = \eta$ *and* $[Y] = \nu$*. Furthermore, the class* $[W \cap Y]$ *is independent of the choice* W *and* Y*.*

Theorem 2.1.3. *There is a unique product structure on* $A(X)$ *such that if* W *and* Y *are generically transverse subvarieties of* X*, then* $[W] \cdot [Y] = [W \cap Y]$*. This makes* $A(X)$ *into a commutative ring called the* Chow ring[1]*.*

2.2   The Hilbert Scheme of Points on a Surface

The Hilbert scheme $X^{[N]}$ of N points on a scheme X represents the functor of flat families of zero dimensional subschemes of length N in X. Hilbert schemes were first constructed by Grothendieck (22; 1). It is well known that when X is a smooth curve, the Hilbert scheme

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[1]*Historically, this theorem was proved as a consequence of the Moving Lemma. Logically, however, it seems the reverse is true. See* 3264 & All That *for this discussion and* Intersection Theory *for the proof (20; 21).*

$X^{[N]}$ is identified with the symmetric product $X^{(N)}$ of the curve. When the dimension of X is greater than two or when X is not smooth, the Hilbert scheme can be quite complicated: it can be singular, reducible, have nonreduced components, etc (2; 23; 24; 25).

Now restrict to the case that X is a smooth irreducible projective surface.

Theorem 2.2.1 (Fogarty (3; 4))**.** *The Hilbert scheme $X^{[N]}$ of N points on X is a smooth irreducible projective variety of dimension $2N$.*

It follows from Fogarty's theorem that there is a natural birational morphism $\mathfrak{h} : X^{[N]} \to X^{(N)}$ from the Hilbert scheme to the symmetric product $X^{(n)}$ which is a resolution of singularities and sends a subscheme Z to its Chow cycle $\sum_{p \in X} \ell(\mathcal{O}_Z(p))\, p$. Consequently, a general point of $X^{[N]}$ corresponds to a subscheme of X consisting of N distinct points in X.

2.2.1 Chow Rings and Hilbert Schemes of Points

Ellingsrud and Strømme computed the Betti numbers for $\mathbb{P}^{2[N]}$, showed that the Chow groups $A^k(\mathbb{P}^{2[N]})$ are free, and provided a basis for these groups (26; 12). See Theorem 2.2.2 and Table I. More generally, Göttsche computed the Betti numbers for the Hilbert scheme $X^{[N]}$ of an arbitrary smooth irreducible projective surface X (27).

Theorem 2.2.2 (Ellingsrud and Strømme (26))**.** *The dimension of $A_k(\mathbb{P}^{2[N]})$ is*

$$\sum_{d_0+d_1+d_2=N} \sum_{p+r=k-d_1} P(p, d_0 - p)P(d_1)P(2d_2 - r, r - d_2)$$

where $P(m, n)$ is the number of partitions of m with entries bounded above by n, and zero when n or m is negative.

k \ d	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	2	3								
3	1	2	5	6							
4	1	2	6	10	13						
5	1	2	6	12	21	24					
6	1	2	6	13	26	39	47				
7	1	2	6	13	28	49	74	83			
8	1	2	6	13	29	54	94	131	150		
9	1	2	6	13	29	56	105	167	232	257	
10	1	2	6	13	29	57	110	198	298	395	440

TABLE I: The first several Betti numbers of $\mathbb{P}^{2[N]}$.

Computing the intersection product on $A(X^{[N]})$ is more difficult. Elencwajg and LeBarz completely computed the intersection product in $A(\mathbb{P}^{2[3]})$ using the basis of Ellingsrud and Strømme, but noted that there were often delicate computations due to nonreduced and excess intersections (9; 11; 10). More generally, Fantechi and Göttsche computed the intersection product for the Chow ring of three points on a general smooth irreducible surface (13).

Nakajima and Grojnowski finally succeeded in providing a description of the Chow ring of the Hilbert scheme for arbitrary N (14; 15; 17). Specifically, they provide an explicit geometric

construction of $\mathbb{H} = \bigoplus_N H(X^{[N]})$ as an irreducible representation of a Heisenberg algebra. Lehn was able to give explicit series to calculate the action of certain Chern classes on $\mathbb{H}$ and used this to describe some intersection products in $H(X^{[N]})$ (18).

2.3   Nakajima's Description – Briefly

We take a moment to provide a basic introduction to the ideas of Nakajima and Grojnowski, as no discussion of the Hilbert scheme of points on a surface would be complete without a proper section on this work. Of course, one should refer to *Lectures on Hilbert Schemes of Points on Surfaces* by Nakajima for a more complete discussion (15). We will follow the approach of Ellingsurd and Göttsche in their notes *Hilbert Schemes of Points and Heisenberg Algebras* (28).

The idea comes from an observation of Vafa and Witten that Göttsche's formula for the Betti numbers of the Hilbert scheme of points on X agrees with the character formula for the representation of the product of a Heisenberg algebra and Clifford algebra (16). The idea then is to consider the direct sum $\mathbb{H} = \bigoplus_{N \geq 0} H(X^{[N]})$ and define natural geometric operators on $\mathbb{H}$ which give it this structure.

To do that, let i be a positive integer, and consider the incidence subscheme

$$\left\{ (W, W') : W \subseteq W' \right\} := X^{[N, N+i]} \subseteq X^{[N]} \times X^{[N+i]}.$$

Let p and q be the restricted projections to the first and second factor. There is a difference-cycle map $\rho : X^{[N, N+i]} \to X^{(i)}$ from this incidence scheme to the symmetric i-th power of X which sends a pair (W, W') to its difference $W' - W$ with multiplicities. Here, $W' - W$ is the scheme corresponding to the finite length $\mathcal{O}_X$-module $I_W / I_{W'}$. The smallest diagonal $\Delta \subseteq X^{(i)}$

is the collection of points $i \cdot P$ for $P \in X$ and is isomorphic to X via the map that sends a cycle

to its support.

The pullback $\rho^* \Delta$ is not necessarily reduced or irreducible, so let $Z_{N,i}$ be the underlying

variety of the component of this pullback which consists of pairs (W, W') such that $W' - W$

is supported at a single point $P \in X$. Let μ be the restriction of ρ to $Z_{N,i}$. We get the

commutative diagram in Figure 1.

$$
\begin{array}{ccc}
X \xrightarrow{\simeq} \Delta & \hookrightarrow & X^{(i)} \\
\mu \uparrow & & \rho \uparrow \\
Z_{N,i} \xrightarrow{\ j\ } & X^{[N,N+i]} & \hookrightarrow \quad X^{[N]} \times X^{[N+i]} \\
\swarrow p & \searrow q & \\
X^{[N]} & & X^{[N+i]}
\end{array}
$$

Figure 1: The commutative diagram used to define the Nakajima operators.

Now let α be a cohomology class on X, let Φ be a cohomology class on $X^{[N]}$, and let Ψ be

a cohomology class on $X^{[N+i]}$. Define the following *Nakajima operators*:

$$P_\alpha[i](\Phi) := q_*(j_*(\mu^*\alpha \cap Z_{N,i}) \cap p^*\Phi), \text{ and} \tag{2.1}$$

$$P_\alpha[-i](\Psi) := p_*(j_*(\mu^*\alpha \cap Z_{N,i}) \cap q^*\Psi). \tag{2.2}$$

To understand this, we envision α, Φ, and Ψ as subvarieties representing each (dual) homology

class. The pull back $\mu^*\alpha \cap Z_{N,i}$ is all pairs (W, W') such that the difference scheme $W' - W$ is

supported at a point of α. The pullbacks $p^*\Phi$ and $q^*\Psi$ are all pairs (W, W') such that $W \in \Phi$ and $W' \in \Psi$, respectively. So $P_\alpha[i](\Phi)$ takes a class on $X^{[N]}$ and forms a class on $X^{[N+i]}$ by adjoining to each scheme in Φ a subscheme of length i supported at a single point of α. Conversely, $P_\alpha[-i](\Psi)$ takes a class on $X^{[N+i]}$ and forms a class on $X^{[N]}$ by deleting from each scheme in Ψ any subscheme of length i supported at a point of α.

Theorem 2.3.1 (Nakajima (14), Grojnowski (17)). *There are relations*

$$P_\alpha[i]P_\beta[j] - P_\beta[j]P_\alpha[i] = i \cdot d_{i+j,0} \cdot \int_X \alpha\beta \cdot \mathbf{1}$$

where $\mathbf{1}$ is the identity operator on $\mathbb{H}$, $\int_X \alpha\beta$ is the intersection product on X, and $d_{i+j,0}$ is the delta function[1].

Corollary 2.3.2. *The space $\mathbb{H}$ with operators $P_\alpha[i], P_\alpha[-i]$ as α ranges over all cohomology classes on X and for all integers $i > 0$ is an irreducible representation of the product of a Heisenberg algebra and a Clifford algebra.*

2.4 Mallavibarrena and Sols' basis for $\mathbb{P}^{2[N]}$

The purpose of this section is to define the basis of Mallavibarrena and Sols (30). This basis forms the foundation for the novel results contained within this thesis.

Fix a triple of partitions $\alpha = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ of nonnegative integers A, B, C such that $A + B + C = N$, respectively. Let r, s, t be the lengths of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and let e, f, g be indices for $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. We will associate to this data a locally closed subset U_α of the open set in $\mathbb{P}^{2[N]}$

---

[1] *The coefficient i in the relations was originally computed by Ellingsrud and Strømme (29).*

of reduced subschemes. The class $\sigma_\alpha = \overline{[U_\alpha]}$ of its closure in $\mathbb{P}^{2[N]}$ will be an element of the MS basis.

Fix r general lines L_e with r general points $P_e \in L_e$, s general lines M_f, and a general point Q. The locus U_α is the locus of reduced schemes Z in $\mathbb{P}^{2[N]}$ which can be written as the disjoint union of three subschemes $Z = Z_1 \cup Z_2 \cup Z_3$ such that:

1. Z does not contain Q or any point of intersection of any pair of fixed lines.

2. Z_1 contains each point P_e, and meets each line L_e in $\mathbf{a}_e$ points.

3. Z_2 meets each line M_f in $\mathbf{b}_f$ points.

4. Z_3 contains t disjoint subschemes $Z_{3,g}$ consisting of $\mathbf{c}_g$ points collinear with Q.

The codimension of U_α in $\mathbb{P}^{2[N]}$ is $N+r-t$ so that the class σ_α is an element of $A^{N+r-t}(\mathbb{P}^{2[N]})$.

Theorem 2.4.1 (Mallavibarrena and Sols (30))**.** *The collection of classes $\{\sigma_\alpha\}$ is a basis for the Chow ring $A\left(\mathbb{P}^{2[N]}\right)$ as α ranges over all triples of partitions of all triples of nonnegative integers whose sum is N.*

We will often refer to the lines spanned by the subschemes $Z_{3,g}$ as *moving lines* and think of them as lines through the point Q which vary.

2.4.1   Some Examples of the Basis

Consider the triple of partitions $\alpha = (0, (1), (1, \ldots, 1))$. We fix a general line M_1 and a point Q, and consider the subset of schemes Z such that Z can be written as the disjoint union of two subschemes Z_2 and Z_3 where:

1. Z_2 meets the line M_1, and

2. Z_3 contains $N-1$ distinct subschemes each of which is collinear with Q.

The codimension of the associated class σ_α is $N + 0 - (N - 1) = 1$. It follows that σ_α is the locus of schemes which meet a fixed general line in the plane and is the divisor associated to the pullback $\mathfrak{h}^* \mathcal{O}_{\mathbb{P}^{2(N)}}(1)$. We will refer to this divisor by H from here on out.

Consider now the triple of partitions $\beta = (0, 0, (2, 1, \ldots, 1))$. In this case, we fix only a general point Q. The locally open set U_β is the collection of schemes Z such that it contains a subscheme of length two collinear with Q. As with α, the codimension of the associated class σ_β has codimension $N + 0 - (N - 1) = 1$ in the Hilbert scheme. This class, along with H, generates the Picard group of $\mathbb{P}^{2[N]}$.

It is often useful to draw schematic pictures for these classes. Fixed lines and points will be solid, points which are allowed to vary will be hollow, and lines which are moving will be dashed. Points through which the moving lines vary will be marked as a thick "X". We conclude this section with Figure 2 which contains heuristic pictures for each element of the MS basis for the Chow ring of the Hilbert scheme of 3 points.
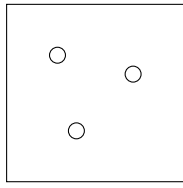
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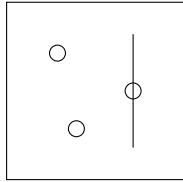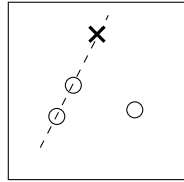
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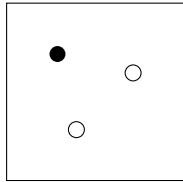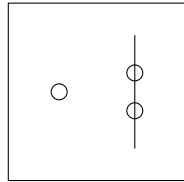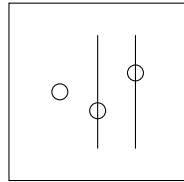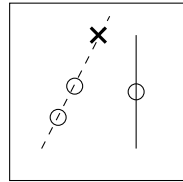
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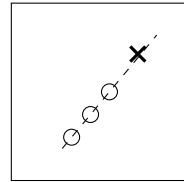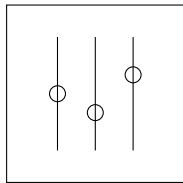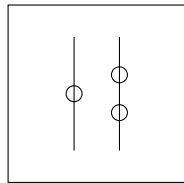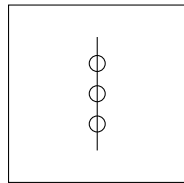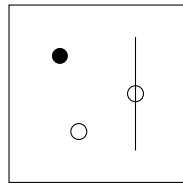
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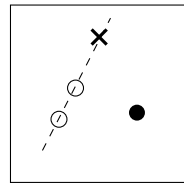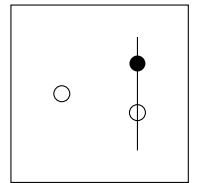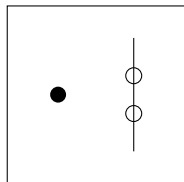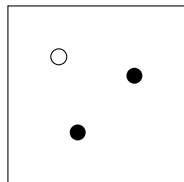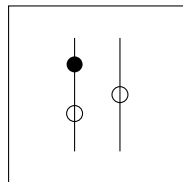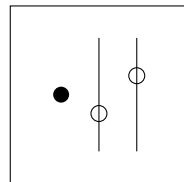
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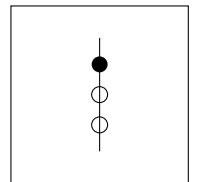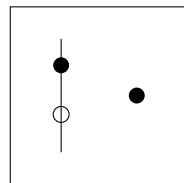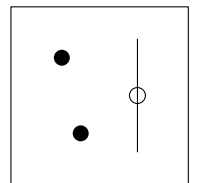
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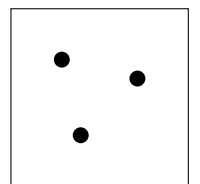
(v) ((1,1,1),0,0)

Figure 2: The full set of basis elements for the Chow ring of $\mathbb{P}^{2[3]}$.

COMPLEMENTARY CODIMENSION INTERSECTIONS

We set out here to prove Main Theorem 2:

Main Theorem 2. *Let σ and τ be two elements of the MS basis of complementary codimension. There is an explicit algorithm to compute the intersection number $\sigma \cdot \tau$.*

3.1   Some Initial Computations

We will consider some illuminating examples before proving the result. The simplest intersections within the MS basis occur when the classes are defined by loci of schemes whose points vary on fixed lines. For instance, in $\mathbb{P}^{2[3]}$ the basis element $\sigma_{(0,(2,1),0)}$ (Figure 2j) has codimension three, and is defined by two points varying on one fixed line and one point varying on another fixed line. Its self intersection number is one, as can be seen in Figure 3.
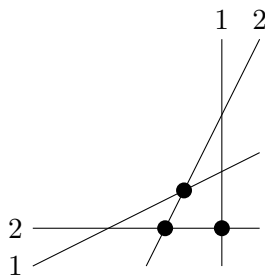
Figure 3: The intersection $\sigma^2_{(0,(2,1),0)}$ in $\mathbb{P}^{2[3]}$.

It is possible to compute this number directly in this case: simply observe that for generally chosen lines in the plane, the unique scheme in the intersection must consist of three distinct points, none of which is the intersection of the pair of lines defining either class, so that locally around this point the two classes are products of lines. It follows that the intersection is transverse.

As another example, we can compute the self intersection of the class $\sigma_{((1),0,(2))}$ (Figure 2m) in $\mathbb{P}^{2[3]}$. The self intersection is again one, as can be seen in Figure 4. In the figure, one class is defined as the class of the locus U of schemes containing A and a subscheme of length two collinear with Q_A, and the other is the class of the locus V of schemes containing B and a subscheme of length two collinear with Q_B.
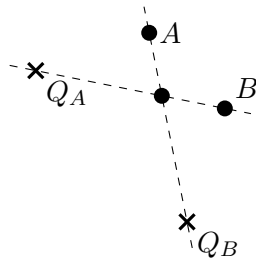
Figure 4: The intersection $\sigma^2_{((1),0,(2))}$ in $\mathbb{P}^{2[3]}$.

In this case, the fixed point A defining one class must be contained in the subscheme of length two collinear with the point Q_B (and similarly the fixed point B defining the other class must be contained in the subscheme of length two collinear with Q_A). This determines the two moving lines as the spans $\overline{AQ_B}$ and $\overline{BQ_A}$, respectively, and consequently the unique point Z in

$\mathbb{P}^{2[3]}$ of the intersection. Without loss of generality we may as well set $A = (1, 0)$, $Q_A = (0, 1)$, $B = (0, -1)$, and $Q_B = (-1, 0)$. The scheme Z is then the set $\{(0, -1), (0, 0), (1, 0)\}$, so that locally around this scheme there are charts isomorphic to $\mathbb{A}^2 \times \mathbb{A}^2 \times \mathbb{A}^2$. If the coordinates on each chart are (x_i, y_i) for $1 \leq i \leq 3$, then the locus U is locally defined by the equations $x_3 = 1, y_3 = 0$ and

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & x_1 & x_2 \\ 1 & y_1 & y_2 \end{bmatrix} = (x_2 - x_1) + x_1 y_2 - x_2 y_1 = 0,$$

and the locus V is similarly described by the equations $x_1 = 0, y_1 = -1$ and

$$\det \begin{bmatrix} 1 & 1 & 1 \\ -1 & x_2 & x_3 \\ 0 & y_2 & y_3 \end{bmatrix} = (y_3 - y_2) + x_2 y_3 - x_3 y_2 = 0.$$

Taking derivatives and considering the kernel of the resulting Jacobian matrix gives that the tangent space to U at Z is defined by the equations

$$x_3 = 1, y_3 = 0, \text{ and } \begin{bmatrix} -1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = -x_1 + 2x_2 = 0.$$

As similar analysis reveals the equations

$$x_1 = 0, y_1 = -1, \text{ and } \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = -2y_2 + y_3 = 0$$

for the tangent space $T_Z V$, and it follows that the intersection $U \cap V$ is transverse at its unique point Z.

We will generalize this argument to show that general representatives of two classes σ and τ of complementary codimension in the MS basis intersect transversally at any point of their intersection.

It is possible to compute these intersection numbers when N is small. We have included the intersection pairing matrices in the MS basis for the Chow ring of $\mathbb{P}^{2[3]}$ and the Chow ring of $\mathbb{P}^{2[5]}$ in Appendix A. These matrices were computed first by hand and later verified by a Python implementation of the algorithm. The Python code can be found in Appendix A Section A.3. It is, however, unreasonable to compute intersections when N is large or when the classes become sufficiently complicated.

3.2 Examples of the Algorithm

We associate to each intersection a diagram consisting of perpendicular lines which we then label. The collections of intersection points which satisfy the different labelings naturally correspond to points in the intersection of general loci representing the classes. To that end,

let us first fix some notation. Let σ_α be an element of the MS basis associated to the triple of partitions $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha)$ and σ_β be an element of the MS basis associated to the triple of partitions $\beta = (\mathbf{a}^\beta, \mathbf{b}^\beta, \mathbf{c}^\beta)$ such that σ_α and σ_β have complementary codimension; that is if $\sigma_\alpha \in A^d(\mathbb{P}^{2[N]})$ then $\sigma_\beta \in A_d(\mathbb{P}^{2[N]}) = A^{2N-d}(\mathbb{P}^{2[N]})$. Let (r, s, t) and (u, v, w) be the lengths of the partitions in α and β, respectively, and let (e, f, g) and (i, j, k) be indices for the partitions in α and β. We will assume that $r = w$ and $t = u$ since otherwise the intersection is necessarily zero and there is nothing to compute.

Definition 3.2.1. (a) The *diagram* $D_{\alpha,\beta}$ associated to the intersection $\sigma_\alpha \cdot \sigma_\beta$ is a set of $r+s+t$ vertical lines and a set of $u+v+w$ horizontal lines along with fixed marked points given as follows. Index the horizontal lines from top to bottom by m and the vertical lines from left to right by n. Add a fixed marked point at the intersection of the m-th horizontal line with the $(r + s + m)$-th vertical line for $1 \leq m \leq t$ and a fixed marked point at the intersection of the $(u + v + n)$-th line with the n-th line for $1 \leq n \leq w$.

(b) An *incidence labeling of* $D_{\alpha,\beta}$ is given as follows. Label the first r vertical lines by the sequence $\mathbf{a}^\alpha$, the next s vertical lines by the sequence $\mathbf{b}^\alpha$, and the last t vertical lines by any permutation of the sequence $\mathbf{c}^\alpha$. Label the horizontal lines similarly: the first u horizontal lines with the sequence $\mathbf{a}^\beta$, the next v horizontal lines with $\mathbf{b}^\beta$, and the last w horizontal lines with any permutation of $\mathbf{c}^\beta$.

Notice that the intersection of classes corresponding to two mixed partitions α and β has exactly one diagram $D_{\alpha,\beta}$ (up to translating the lines – a harmless manipulation) which may admit many incidence labelings. We will often refer to a diagram as D when the context allows no confusion. When $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$ are both constant sequences, there is a unique incidence labeling of D. Figure 5 shows the unique incidence labeling of the diagram for the intersection

$\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$. Notice that there are no fixed marked points in this diagram because $\mathbf{c}^{\alpha}$ and $\mathbf{c}^{\beta}$ are both zero.
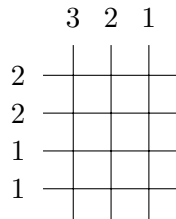
Figure 5: The unique incidence labeling of the diagram for the intersection $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$.

Figure 6 is a more complicated example involving many incidence labelings. It is a list of all possible incidence labelings of the diagram for the intersection $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1))}$ in $\mathbb{P}^{2[9]}$.

Two remarks are in order. First, the incidence labelings are taken over all permutations of both sequences $\mathbf{c}^{\alpha}$ and $\mathbf{c}^{\beta}$. Second, the sequences $\mathbf{c}^{\alpha}$ and $\mathbf{c}^{\beta}$ may contain repeated integers, so that they may not exhibit $t!$ and $w!$ distinct permutations, respectively. For instance, in Figure 6, there are three distinct permutations of $\mathbf{c}^{\alpha} = (2, 1, 1)$ (as opposed to six), and we get a diagram for each distinct choice of permutations of $\mathbf{c}^{\alpha}$ and $\mathbf{c}^{\beta}$ for a total of six labelings.
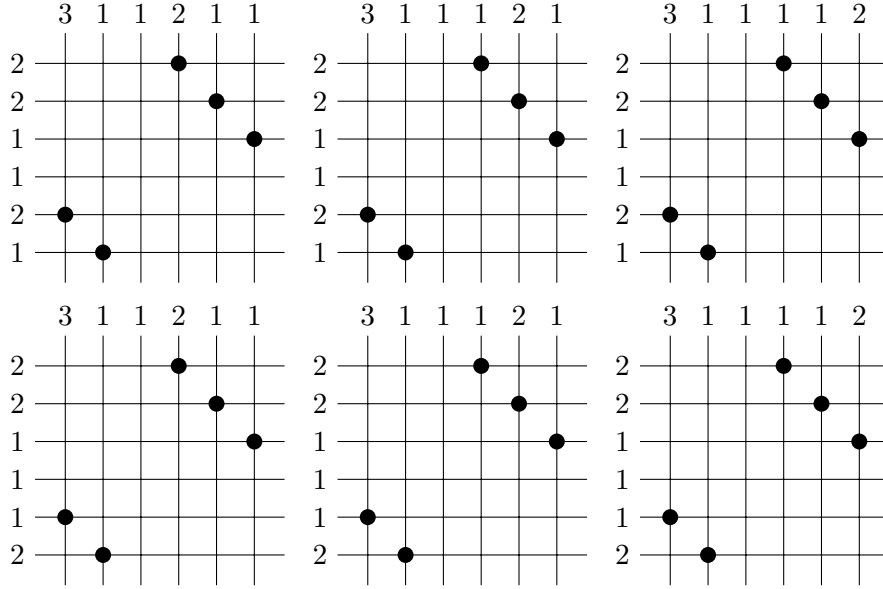
Figure 6: All possible incidence labelings of the diagram for the intersection $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1))}$ in $\mathbb{P}^{2[9]}$.

Definition 3.2.2. The *intersection number* of an incidence labeling of a diagram is the number of ways to choose sets Z consisting of N distinct intersection points of the lines in the diagram satisfying the following conditions:

(i) the number of points on each line is given by its label; and

(ii) Z contains all of the fixed marked points in the diagram.

It is easy to compute the intersection number of an incidence labeling: beginning with the first vertical line, and for each vertical line in the diagram, choose horizontal lines and mark those points of intersection. The label of the vertical line indicates how many horizontal lines

to choose. We keep track of all such choices in a tree whose nodes are the diagram with marked points and whose edges correspond to different choices of points to mark. We will often conflate choices which occur with symmetry and indicate it by labeling the corresponding edge with a multiplicity. It is useful to drop portions of the tree or to ignore choices which obviously will not result in collections of points satisfying all the incidence labels of all the lines.
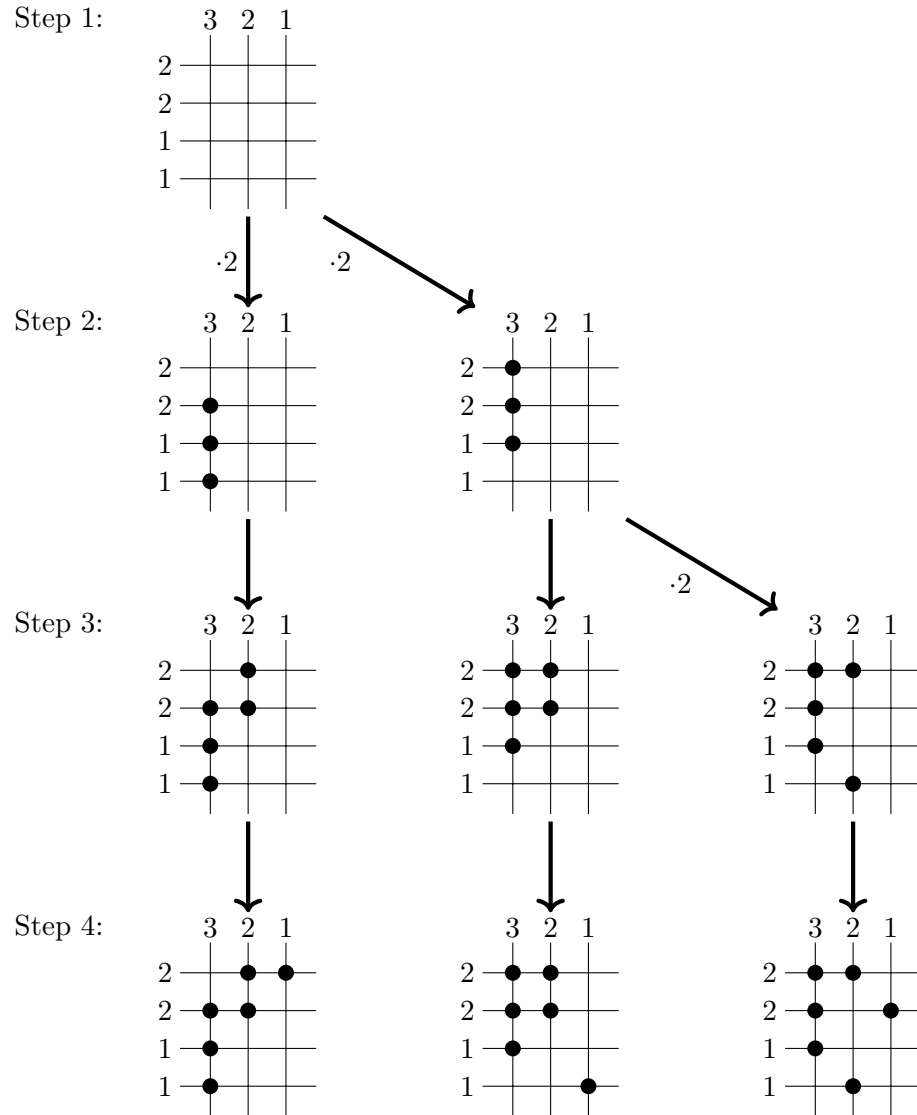
Figure 7: The tree for the unique incidence labeling of the diagram for the product $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$.

For instance, consider Figure 7, where we conduct the process for the unique incidence labeling of the diagram for $\sigma_{(0,(3,2,1),0)} \cdot \sigma_{(0,(2,2,1,1),0)}$ in $\mathbb{P}^{2[6]}$. The first vertical line has a label of three, there are four possible ways to choose three lines from the four horizontal ones, but up to symmetry, only two choices need be expressed and we label the corresponding edges with $\cdot 2$. In the second step, we must choose two lines from the horizontal lines to pair with the second vertical line. We ignore choices which obviously violate the incidence conditions, such as any choice containing either of the bottom two lines in the left node. This leaves just one unique choice for the left node, and three unique choices, two of which are the same up to symmetry, for the right node. Finally, in each of the cases on the third step there is a unique choice of lines satisfying the incidence conditions. We conclude that the intersection number of the incidence labeling is eight.

The intersection number $\sigma_\alpha \cdot \sigma_\beta$ is the sum of the intersection numbers of all incidence labelings of its diagram. We will omit the details of the computation, but the intersection numbers for the incidence labelings in Figure 6 are, from left to right and then top to bottom, 5,5,6,9,9, and 12, so that the intersection number $\sigma_{((3,1),(1),(2,1,1))} \cdot \sigma_{((2,2,1),(1),(2,1))} = 46$.

3.3 Proof of Main Theorem 2

We must first fix some notation. Let σ be an element of the MS basis associated to the triple of partitions $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha)$ and τ be an element of the MS basis associated to the triple of partitions $\beta = (\mathbf{a}^\beta, \mathbf{b}^\beta, \mathbf{c}^\beta)$ such that σ and τ have complementary codimension; that is if $\sigma \in A^d(\mathbb{P}^{2[N]})$ then $\tau \in A_d(\mathbb{P}^{2[N]}) = A^{2N-d}(\mathbb{P}^{2[N]})$. Let (r,s,t) and (u,v,w) be the lengths of the partitions in α and β, respectively, and let (e,f,g) and (i,j,k) be indices for the partitions in α and β.

We will fix general representatives of σ and τ. Fix r general lines L_e^α with r general points $P_e^\alpha \in L_e^\alpha$, fix s general lines M_f^α and fix a general point Q^α. Similarly, fix u general lines L_i^β with u general points $P_i^\beta \in L_i^\beta$, fix s general lines M_j^β and fix a general point Q^β. Define sets U_α and U_β as described in Section 2.4 such that $\sigma = \left[\overline{U_\alpha}\right]$ and $\tau = \left[\overline{U_\beta}\right]$.

Assuming there is no excess intersection, the intersection number $\sigma \cdot \tau$ can be computed as the number of points in $\overline{U_\alpha} \cap \overline{U_\beta}$ counted with multiplicity. The first goal is to establish Proposition 3.3.1.

Proposition 3.3.1. *The intersection number $\sigma \cdot \tau$ is the number of points in the intersection $U_\alpha \cap U_\beta$ counted with multiplicity one.*

To that end, we begin with the following lemma. Recall that in each point $Z \in \overline{U_\alpha}$ (respectively, $\overline{U_\beta}$) there are subschemes $Z_{3,g}^\alpha$ (resp., $Z_{3,k}^\beta$) of length at least $\mathbf{c}_g^\alpha$ (resp., $\mathbf{c}_k^\beta$) collinear with Q^α (resp., Q^β).

Lemma 3.3.2. *Let Z be a scheme in the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$. Each subscheme $Z_{3,g}^\alpha \subset Z$ collinear with Q^α must contain a unique P_i^β. The number t of lines through Q^α spanned by subschemes $Z_{3,g}^\alpha \subseteq Z$ must equal the number u of fixed points defining the other locus U_β. Similarly, the number w of lines through Q^β spanned by subschemes $Z_{3,k}^\beta \subseteq Z$ must equal the number r of fixed points defining the locus U_α.*

Proof. Observe first that the fixed points P_i^β were chosen generally and therefore will not lie in the lines L_e^α or M_f^α defining the locus U_α, and furthermore the span of any two such fixed points does not contain Q^α. Therefore, any scheme Z in the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$ contains, for each i, a subscheme $Z_{3,g}^\alpha$ containing P_i^β. This establishes the inequality $u \leq t$.

By symmetry, we have that $r \leq w$. Now the codimension of σ is $N + r - t$ while the dimension of τ is $N + w - u$, and these are equal, so

$$r + u = w + t$$

and if either $u < t$ or $r < w$, this equality cannot hold. $\qquad \square$

We get an immediate corollary about the support of any such scheme Z in the intersection.

Corollary 3.3.3. *Let Z be scheme in the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$. Z must satisfy all of the following conditions.*

1. *The support of Z cannot contain the intersection point of any two of the $r + s$ fixed lines L_e^α or M_f^α (respectively the $u + v$ fixed lines L_i^β or M_j^β) defining the locus U_α (U_β).*

2. *The support of Z cannot contain Q^α or Q^β.*

3. *The line spanned by any of the subschemes $Z_{3,g}^\alpha \subseteq Z$ (respectively $Z_{3,k}^\beta \subseteq Z$) cannot contain a fixed point P_e^α (P_i^β).*

4. *The support of any of the subschemes $Z_{3,g}^\alpha \subseteq Z$ (respectively $Z_{3,k}^\beta \subseteq Z$) cannot meet any of the lines L_e^α or M_f^α (L_i^β or M_j^β).*

5. *Each subscheme $Z_{3,g}^\alpha \subseteq Z$ (respectively $Z_{3,k}^\beta \subseteq Z$) must span a unique line through Q^α (Q^β).*

Proof. We'll address each individually.

1. The intersection of any two of the $r + s$ fixed lines L_e^α or M_f^α will not be contained in any of the $u + v$ fixed lines L_i^β or M_j^β, nor will they be contained in any of the spans $\langle Q^\beta P_e^\alpha \rangle$.

2. The point Q^α is not contained in any of the $u+v$ fixed lines L_i^β or M_j^β, nor is it contained in any of the spans $\langle Q^\beta P_e^\alpha \rangle$.

3. The points P_e^α are not contained in any spans $\langle Q^\alpha P_i^\beta \rangle$.

4. Fix some subscheme $Z_{3,g}$ and let L be the line it spans (if $Z_{3,g}$ is a single point, then let L be the line it spans with Q^α). The intersection of L with any line L_e^α or M_f^α is not contained in any line L_i^β or M_j^β, nor is it contained in any of the spans $\langle Q^\beta P_e^\alpha \rangle$.

5. The lemma establishes that $t = u$, so every subscheme $Z_{3,g}$ contains a unique P_i^β, and consequently they span distinct lines.

By symmetry, all the conclusions hold for the locus U_β with the appropriate replacements. $\quad\square$

This is turn allows us to conclude the following.

Corollary 3.3.4. *Let Z be a scheme in the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$. Assume that Z contains a nonreduced subscheme W whose support is a single point. W is contained in some fixed line L_e^α or M_f^α, or it is collinear with Q_α.*

Proof. Let $\{Z_t\}$ be a flat family of schemes in U_α whose limit is Z. There is a subfamily $\{W_t\}$ of subschemes $W_t \subseteq Z_t$ such that the flat limit of $\{W_t\}$ is W.

By Corollary 3.3.3, the support of W is not the intersection of any two of the lines L_e^α, M_f^α, or $\langle Q^\alpha P_i^\beta \rangle$, and the support is therefore contained in a unique such line.

Assume for the moment that the support of W is contained in a fixed line $L_{e'}^\alpha$ or $M_{f'}^\alpha$. The general scheme W_t in the family does not meet any line L_e^α or M_f^α where $e \neq e'$ or $f \neq f'$. Furthermore, W_t for a general t cannot meet any of the subschemes $Z_{t,3,g}^\alpha$ collinear with Q^α since the support of the limit of any such scheme would be contained in some of the lines $\langle Q^\alpha P_i^\beta \rangle$. Hence W_t for general t is contained completely in the line $L_{e'}^\alpha$ or $M_{f'}^\alpha$.

Now assume that the support of W is contained in some line $\langle Q^\alpha P_i^\beta \rangle$. Since the support is not contained in any fixed line L_e^α or M_f^β, the subscheme W_t does not meet these fixed lines for any general t, and hence must be the limit of points in subschemes $Z_{t,3,g}^\alpha$. But Corollary 3.3.3 assures us that the support of W is contained in a unique $\langle Q^\alpha P_i^\beta \rangle$ and is not Q^α, and hence W_t for general t must be a subscheme of some $Z_{t,3,g}^\alpha$ for some fixed g. Therefore W is collinear with Q^α. $\qquad\qquad\square$

Corollary 3.3.5. *Let Z be a scheme in the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$. Z cannot contain a nonreduced subscheme. Furthermore, Z is contained in $U_\alpha \cap U_\beta$.*

Proof. Assume that Z contains a nonreduced subscheme W supported at a point P. By Corollary 3.3.4, W is contained in and determines some L_e^α or M_f^α, or is collinear with Q^α. Similarly, W is contained in some L_i^β or M_j^β, or is collinear with Q^β. In any case, these lines are distinct and therefore W cannot be contained in both.

It follows Z is reduced, and when combined with Corollary 3.3.3, that is it contained in the open loci U_α and U_β. $\qquad\qquad\square$

It follows that there are charts for $\mathbb{P}^{2[N]}$ isomorphic to $(\mathbb{A}^2)^N$ around any such Z, and we can locally describe the loci U_α and U_β in these charts explicitly. We will work for the moment solely with U_α, but everything applies analogously to U_β. We first write $(\mathbb{A}^2)^N$ as the product

$$(\mathbb{A}^2)^{\mathbf{a}_1^\alpha} \times \cdots \times (\mathbb{A}^2)^{\mathbf{a}_r^\alpha} \times (\mathbb{A}^2)^{\mathbf{b}_1^\alpha} \times \cdots \times (\mathbb{A}^2)^{\mathbf{b}_s^\alpha} \times (\mathbb{A}^2)^{\mathbf{c}_1^\alpha} \times \cdots \times (\mathbb{A}^2)^{\mathbf{c}_t^\alpha} \qquad (3.1)$$

with coordinates (x_m, y_m) for $1 \leq m \leq N$ on each factor of $\mathbb{A}^2$. For the fixed points $P_e^\alpha = (P_{e,x}^\alpha, P_{e,y}^\alpha)$ in the locus, and consequently in the first factor of $\mathbb{A}^2$ in each grouped factor $(\mathbb{A}^2)^{\mathbf{a}_i^\alpha}$, the locus is defined simply by the equations $x_m = P_{e,x}$ and $y_m = P_{e,y}$. For the remaining points

in each factor $(\mathbb{A}^2)^{\mathbf{a}_e^\alpha}$ or $(\mathbb{A}^2)^{\mathbf{b}_f^\alpha}$ the locus U_α is locally defined in each $\mathbb{A}^2$ factor by the equation of the line L_e^α or M_f^α, respectively.

Three points $Q = (Q_x, Q_y)$, $P = (P_x, P_y)$ and $R = (R_x, R_y)$ in $\mathbb{A}^2$ are collinear if and only if

$$\det \begin{bmatrix} 1 & 1 & 1 \\ Q_x & P_x & R_x \\ Q_y & P_y & R_y \end{bmatrix} = 0.$$

Let $Q^\alpha = (Q_x^\alpha, Q_y^\alpha)$. It follows that the locus U_α in each factor $(\mathbb{A}^2)^{\mathbf{c}_g^\alpha}$ is defined by the set of equations

$$\det \begin{bmatrix} 1 & 1 & 1 \\ Q_x^\alpha & x_n & x_m \\ Q_y^\alpha & y_n & y_m \end{bmatrix} = 0$$

for all $1 \leq n < m \leq \mathbf{c}_g^\alpha$. Indeed, the locus is defined by the $\mathbf{c}_g^\alpha - 1$ independent equations obtained by fixing $n = 1$ and varying m. The Jacobian of this set of equations is then

$$\begin{bmatrix} y_2 - Q_y^\alpha & Q_x^\alpha - x_2 & Q_y^\alpha - y_1 & x_1 - Q_x^\alpha & 0 & 0 & \cdots & 0 \\ y_3 - Q_y^\alpha & Q_x^\alpha - x_3 & 0 & 0 & Q_y^\alpha - y_1 & x_1 - Q_x^\alpha & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ y_{\mathbf{c}_g^\alpha - 1} - Q_y^\alpha & Q_x^\alpha - x_{\mathbf{c}_g^\alpha - 1} & 0 & 0 & \cdots & 0 & Q_y^\alpha - y_1 & x_1 - Q_x^\alpha \end{bmatrix},$$

and evaluating at $Z = \bigcup(z_{x,m}, z_{y,m})$ and taking its kernel gives $\mathbf{c}_g^\alpha - 1$ equations

$$(z_{y,m} - Q_y^\alpha)x_1 + (Q_x^\alpha - z_{x,m})y_1 = (z_{y,1} - Q_y^\alpha)x_m + (Q_x^\alpha - z_{x,1})y_m$$

for the tangent space $T_Z U_\alpha$. This leads us to the following conclusion.

Lemma 3.3.6. *The locus U_α is smooth.*

Proof. The codimension of U_α is $N + r - t$. It suffices to check that the codimension of the tangent space at the point $Z \in U_\alpha$ is the same. But that is clear: there are $\mathbf{c}_g^\alpha - 1$ independent equations defining the tangent space in the factor $(\mathbb{A}^2)^{\mathbf{c}_g^\alpha}$, $\mathbf{b}_f^\alpha$ independent equations defining the tangent space in the factor $(\mathbb{A}^2)^{\mathbf{b}_f^\alpha}$, and $\mathbf{a}_e^\alpha + 1$ independent equations defining the tangent space in the factor $(\mathbb{A}^2)^{\mathbf{a}_e^\alpha}$ for a total of

$$\sum \left(\mathbf{a}_e^\alpha + 1\right) + \sum \left(\mathbf{b}_f^\alpha\right) + \sum \left(\mathbf{c}_g^\alpha - 1\right) = \sum \left(\mathbf{a}_e^\alpha\right) + \sum \left(\mathbf{b}_f^\alpha\right) + \sum \left(\mathbf{c}_g^\alpha\right) + r - t = N + r - t$$

independent equations defining the tangent space to U_α at a point. $\qquad \square$

Lemma 3.3.7. *Let Z be a scheme in the intersection $U_\alpha \cap U_\beta$. The tangent spaces $T_Z U_\alpha$ and $T_Z U_\beta$ intersect in a point, so the intersection $U_\alpha \cap U_\beta$ is transverse.*

Proof. By Corollaries 3.3.3 and 3.3.5, we can work locally in the charts isomorphic to $(\mathbb{A}^2)^N$ around Z. It therefore suffices to check transversality in each factor of $\mathbb{A}^2$ or on smaller sub-products when necessary.

In any factor where the point z of Z is the intersection of a fixed line defining U_α with a fixed line defining U_β, this is clear since the two lines are distinct and the homogeneous equation of the line defines the tangent space to U_α and U_β at the point.

There are three remaining possibilities: in some fixed factor $\mathbb{A}^2$

1. one locus is given by a fixed line and the other is locally given by a "moving" line,

2. both loci are given by "moving" lines, and

3. one loci is locally a fixed point and the other is a "moving" line.

In the final case, the result is clear.

Let $\mathbb{A}^2$ be some fixed factor in the chart, and let $z = (z_x, z_y)$ be the point of Z in this factor. We may as well assume that in U_α the point z is contained in some "moving" line. As such, with respect to the decomposition in Equation 3.1, it lives in some subproduct $(\mathbb{A}^2)^{\mathbf{c}_g^\alpha}$ of which the first factor corresponds to a fixed point P_i^β in U_β. For the moment, we may ignore the remaining $\mathbf{c}_g^\alpha - 2$ factors that do not correspond to our point z and our fixed point P_i^β. Let (x_1, y_1) be coordinates in the this first factor, let (x_2, y_2) be coordinates in the factor containing z, and let Q^α be the point (Q_x^α, Q_y^α). If P_i^β has coordinates $(P_{i,x}^\beta, P_{i,y}^\beta)$ in this chart then in this subproduct $\mathbb{A}^2 \times \mathbb{A}^2$ with local coordinates (x_1, y_1, x_2, y_2), the tangent space to U_α is defined by the single equation

$$(z_y - Q_y^\alpha)x_1 + (Q_x^\alpha - z_x)y_1 = (P_{i,y}^\beta - Q_y^\alpha)x_2 + (Q_x^\alpha - P_{i,x}^\beta)y_2. \qquad (3.2)$$

The equations $x_1 - P_{i,x}^\beta = 0$ and $y_1 - P_{i,y}^\beta = 0$ define the locus U_β and its tangent space is then just the origin in the first factor. The defining equation in the second factor depends on which situation we are in.

In the first case, U_β is locally defined by the equation of the fixed line defining it. The intersection fails to be transverse if the slope of this fixed line coincides with the slope of the line defined by the right hand side of Equation 3.2. This only occurs if all three of the points Q^α, z, and P_i^β reside on the fixed line, which is not the case since neither P_i^β nor Q^α does by assumption.

In the second case, the point z is on some "moving" line in U_β and we must consider a third factor of $\mathbb{A}^2$. There is a decomposition of the charts with respect to U_β as in Equation 3.1

such that there is a subproduct $(\mathbb{A}^2)^{\mathbf{c}_k^\beta}$ and that the first factor contains some fixed point $P_e^\alpha = (P_{e,x}^\alpha, P_{e,y}^\alpha)$ of the locus U_α. Note first that this cannot coincide with either of the two factors of $\mathbb{A}^2$ we're already considering. Let coordinates for this factor be (x_3, y_3). The locus U_α is therefore defined by $x_3 - P_{e,x}^\alpha = 0$ and $y_3 - P_{e,y}^\alpha = 0$ here, and the tangent space to U_β has the additional defining equation

$$(z_y - Q_y^\beta)x_3 + (Q_x^\beta - z_x)y_3 = (P_{e,y}^\alpha - Q_y^\beta)x_2 + (Q_x^\beta - P_{e,x}^\alpha)y_2.$$

The intersection fails to be transverse if the resulting lines defined by

$$(P_{i,y}^\beta - Q_y^\alpha)x_2 + (Q_x^\alpha - P_{i,x}^\beta)y_2 = 0 \text{ and } (P_{e,y}^\alpha - Q_y^\beta)x_2 + (Q_x^\beta - P_{e,x}^\alpha)y_2 = 0$$

coincide. This can only occur if the two lines $\overline{Q^\alpha P_i^\beta}$ and $\overline{Q^\beta P_e^\alpha}$ are equal, since they share the common point z, which is not the case.

In any case, the intersection of the tangent spaces in each of these factors is a single point, and thus we've established the lemma. $\square$

We can now prove Proposition 3.3.1.

Proof of Proposition 3.3.1. By Corollary 3.3.5, the intersection $\overline{U_\alpha} \cap \overline{U_\beta}$ occurs only along the open locus $U_\alpha \cap U_\beta$, and by Lemma 3.3.7 the intersection of the tangent spaces to U_α and U_β at any such intersection point meet in a single point. It follows that there is no excess intersection, and the intersection number $\sigma \cdot \tau$ is the number of points in the intersection $U_\alpha \cap U_\beta$. $\square$

3.4   The Algorithm

We recall Definition 3.2.1 of the diagram and its incidence labelings associated to an inter-section of classes σ and τ of complementary codimension in the MS basis, as well as Definition 3.2.2 of the intersection number of a labeled diagram.

Definition 3.2.1.

(a) The *diagram* $D_{\alpha,\beta}$ associated to the intersection $\sigma_\alpha \cdot \sigma_\beta$ is a set of $r + s + t$ vertical lines and a set of $u + v + w$ horizontal lines along with fixed marked points given as follows. Index the horizontal lines from top to bottom by m and the vertical lines from left to right by n. Add a fixed marked point at the intersection of the m-th horizontal line with the $(r + s + m)$-th vertical line for $1 \leq m \leq t$ and a fixed marked point at the intersection of the $(u + v + n)$-th line with the n-th line for $1 \leq n \leq w$.

(b) An *incidence labeling of* $D_{\alpha,\beta}$ is given as follows. Label the first r vertical lines by the sequence $\mathbf{a}^\alpha$, the next s vertical lines by the sequence $\mathbf{b}^\alpha$, and the last t vertical lines by any permutation of the sequence $\mathbf{c}^\alpha$. Label the horizontal lines similarly: the first u horizontal lines with the sequence $\mathbf{a}^\beta$, the next v horizontal lines with $\mathbf{b}^\beta$, and the last w horizontal lines with any permutation of $\mathbf{c}^\beta$.

Definition 3.2.2. The *intersection number* of an incidence labeling of a diagram is the number of ways to choose sets Z consisting of N distinct intersection points of the lines in the diagram satisfying the following conditions:

 (i) the number of points on each line is given by its label; and

 (ii) Z contains all of the fixed marked points in the diagram.

Proposition 3.4.1. *The intersection number $\sigma \cdot \tau$ of complementary codimension classes σ and τ in the MS basis is the sum of the intersection numbers of all of the incidence labelings of the diagram associated to it.*

Proof. Let $\mathcal{D}$ be the diagram associated to the intersection $\sigma \cdot \tau$. The first $r + s$ vertical lines in $\mathcal{D}$ correspond, in order, to the fixed lines L_e^α and M_f^α defining the locus U_α while the last t vertical lines correspond to the spans $\overline{Q^\alpha P_i^\beta}$. The first $u + v$ horizontal lines in $\mathcal{D}$ correspond, in order, to the fixed lines L_i^β and M_j^β defining U_β, and the last w horizontal lines correspond to the spans $\overline{Q^\beta P_e^\alpha}$. The fixed marked points in $\mathcal{D}$ correspond to the points P_e^α and P_i^β.

Under this correspondence, a collection Σ of N points which are intersection points of the vertical and horizontal lines in $\mathcal{D}$ corresponds to a reduced scheme $Z \in \mathbb{P}^{2[N]}$ given by the intersections of the corresponding lines. If we further require that Σ is an intersection point of an incidence labeling of $\mathcal{D}$, i.e., that Σ contains all the fixed marked points and that the number of points in Σ on each line in the diagram is given by its label, then the corresponding scheme Z is indeed a point of the intersection $U_\alpha \cap U_\beta$.

This correspondence is injective since two such collections Σ and Ω differ by the number of points incident to some line in the diagram and therefore in the image. Furthermore, given a point $Z \in U_\alpha \cap U_\beta$, it is naturally the image under this correspondence of some intersection point Σ of some incidence labeling of $\mathcal{D}$ since Lemma 3.3.2 and Corollary 3.3.5 classify the schemes in the intersection $U_\alpha \cap U_\beta$ as precisely these.

This bijective correspondence combined with Proposition 3.3.1 proves the claim. $\qquad\square$

3.4.1 Step 1 – Diagrams

Fix a set of $r + s + t$ vertical lines indexed from top to bottom and a set of $u + v + w$ horizontal lines indexed from left to right in the plane. Mark the fixed intersection points of the i-th horizontal line with the $(u + v + i)$-th vertical line and the $(r + s + j)$-th horizontal line with the j-th vertical line for $1 \leq i \leq w$ and $1 \leq j \leq t$. This establishes the diagram $\mathcal{D}$ as in Definition 3.2.1. See Figure 8.
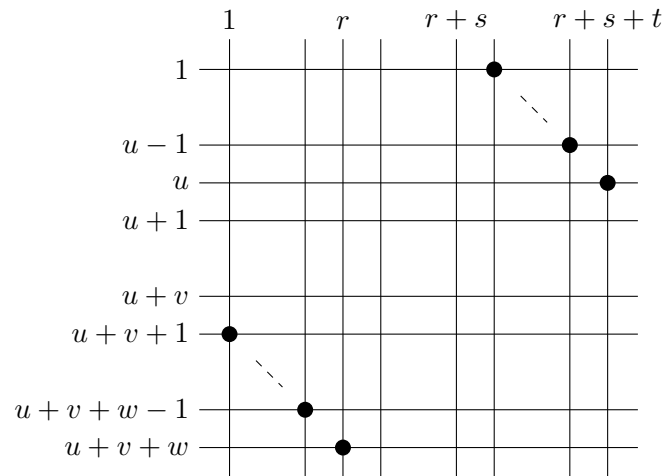
Figure 8: The diagram for an arbitrary σ and τ. Lines are labeled by their indices. Indices are omitted on some of the vertical lines for readability.

3.4.2   Step 2 – Labels

Produce all incidence labelings of $\mathcal{D}$ (as in Definition 3.2.1): label the first r vertical lines with the sequence $\mathbf{a}^\alpha$, the next s vertical lines with $\mathbf{b}^\alpha$, the first u horizontal lines with $\mathbf{a}^\beta$, and the next v horizontal lines with $\mathbf{b}^\beta$. For any pair $(\mathbf{d}^\alpha, \mathbf{d}^\beta)$ of permutations of $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$, respectively, obtain an incidence labeling by labeling the last t vertical lines by $\mathbf{d}^\alpha$ and the last w horizontal lines by $\mathbf{d}^\beta$. Let Ad be the set of all admissibly labeled diagrams. See Figure 9.
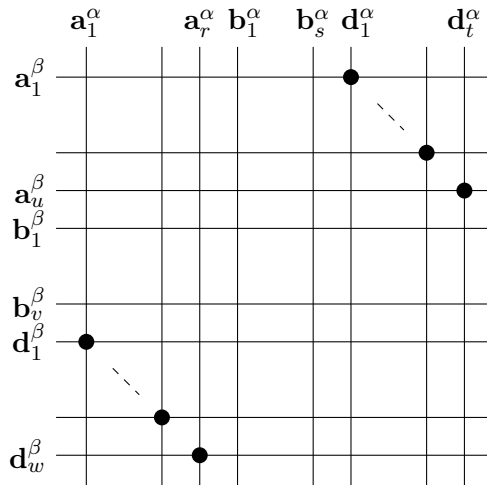
Figure 9: An arbitrary incidence labeling of $\mathcal{D}$. The sequences $\mathbf{d}^\alpha$ and $\mathbf{d}^\beta$ are any permutations of $\mathbf{c}^\alpha$ and $\mathbf{c}^\beta$, respectively.

3.4.3 Step 3 – Counting Intersections in Trees

Fix an incidence labeling ℓ of $\mathcal{D}$. Generate a tree T_ℓ as follows. The root of the tree consists of the labeled diagram ℓ.

Each node consists of ℓ with extra marked points. The number of marked points on each line must not exceed its label. Let $\mathcal{N}$ be a node of depth d. Let z be the integer label of the $(d+1)^{\text{st}}$ vertical line. Each child of $\mathcal{N}$ consists of $\mathcal{N}$ and an additional z marked points on the $(d+1)^{\text{st}}$ vertical line. The extra points are the intersection of the $(d+1)^{\text{st}}$ vertical line with z horizontal lines such that the resulting number of points on each horizontal line does not exceed its label. There is one child for each such choice. If there are no such choices, then $\mathcal{N}$ has no children.

3.4.4 Step 4 – Removing Erroneous Leaves

Beginning with a tree T_ℓ, repeatedly remove any leaves which do not contain the correct number of marked points, given by the label, on each line.

Lemma 3.4.2. *The intersection number i_ℓ of the incidence labeling ℓ (see Definition 3.2.2) is the number of remaining leaves.*

Proof. A leaf in the tree is either

1. an intersection point of the labeled diagram ℓ, i.e., a collection of N points which are the intersection points of the horizontal and vertical lines such that it contains all of the fixed marked points and such that the number of points on each line is given by its label.

2. the labeled diagram ℓ, possibly with extra marked points, but with fewer than N marked points since some choice results in a situation in which there are no choices of horizontal lines that can still satisfy the labeling.

After removing the leaves of type two in 3.4.4, the number of remaining leaves is precisely the intersection number. □

We can now prove our main theorem.

Proof of Main Theorem 2. By Proposition 3.4.1 it suffices to count intersection points of $\sigma \cdot \tau$ as collections of N points satisfying each possible incidence labeling, as constructed in Step 1 (3.4.1) and Step 2 (3.4.2). By Lemma 3.4.2, it suffices to count the leaves of the trees constructed in Step 3 (3.4.3) and modified in Step 4 (3.4.4).

Everything here is explicit, and this obviously terminates since all the choices are finite. □

3.5 Further Remarks

One nice consequence of this algorithm is that it is positive in the strong sense that it counts objects which correspond one to one with points in the intersection. Colloquially, one might say that "there are no negative signs" in the count. It is also purely combinatorial in that it requires no knowledge of the geometry of the intersection to count. One can simply run a process and arrive at a number.

CHAPTER 4

THE ACTION OF H ON THE CHOW RING

We now move on to Main Theorem 1. Fix a general line in $\mathbb{P}^2$, and let H be the class of the locus of schemes in $\mathbb{P}^{2[N]}$ such that their support meets this line. Let σ be any element of the MS basis.

Main Theorem 1. *There is an explicit algorithm to compute the class in the MS basis of the intersection $H \cdot \sigma$.*

As was our approach before, we begin with a barrage of examples, each of which steadily increases in complexity until one which approximately demonstrates the algorithm.

4.1   Initial Computations

Let $\alpha = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a triple of partitions, let σ be the class in the MS basis associated to α, and let U_α be some general representative of σ (See Section 2.4). When U_α contains no moving lines, the intersection with H is relatively straightforward to compute.

For instance, consider the class in $A^4(\mathbb{P}^{2[3]})$ defined by the triple of partitions $\alpha = ((2), (1), 0)$ (see 2q). Fix two general lines L_1 and L_2 and a general point $P \in L_1$, so that the representative U_α is the closure of the locus of schemes in $\mathbb{P}^{2[3]}$ which contain P, meet L_1 in an additional point, and meet L_2. H is obtained by considering the locus of schemes meeting some third general line M. The intersection $H \cap U_\alpha$ consists of the union of two loci. The first loci consists of schemes which contain the intersection point $L_1 \cap M$ and the second loci consists of schemes which contain the intersection point $L_2 \cap M$. See Figure 10.

(a) (0,(1),(1,1))   (b) ((2),(1),0)   (c) ((2,1),0,0)   (d) ((1,1),(1),0)
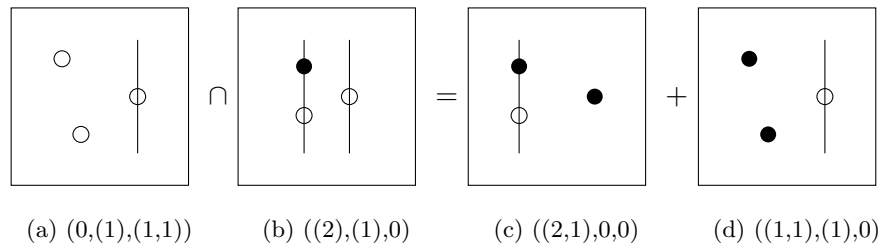
Figure 10: The intersection $H \cap U_\alpha$.

4.1.1   A Simple Example of the MS Degeneration

The situation becomes more complex when there are lines containing fixed points which contain subschemes of length greater than two along this line. For example, consider the triple of partitions $\beta = ((3), 0, 0)$ and the class associated to this triple in the MS basis (see Figure 2s). Fix a point P and a line L through this point and consider the locus $U_\beta \subset \mathbb{P}^{2[3]}$ of schemes contained in L and containing P. If H is the locus of schemes whose support meets a fixed general line M, then the intersection consists of the schemes contained in L, and which contain P and the intersection point $L \cap M$. See Figure 11. We will use a modified version of a degeneration of Mallavibarrena and Sols to compute the class of this locus.

(a) (0,(1),(1,1)) (b) ((3),0,0) (c)
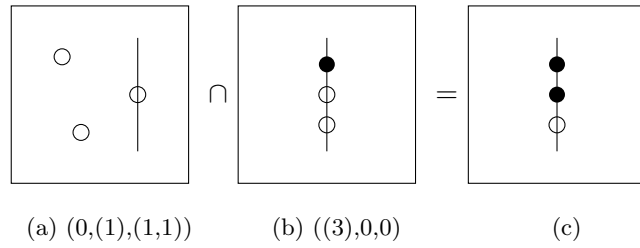
Figure 11: The intersection $H \cap U_\beta$.

Consider a locus defined by the triple of partitions $((2, 1), 0, 0)$ (see Figure 2t) with respect to these fixed objects. We degenerate the fixed point onto the line, and obtain two loci as in Figure 12.

(a) ((2,1),0,0) (b) (c)
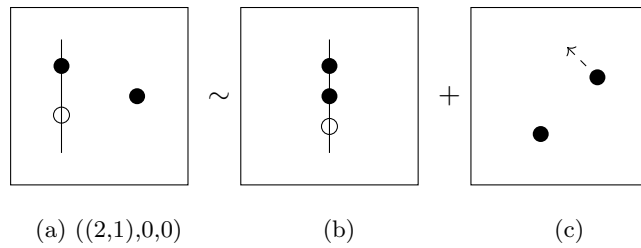
Figure 12: An example of the modified degeneration of Mallavibarrena and Sols.

The first is the locus of interest, and the second is the locus of schemes containing two fixed points, one of which supports a nonreduced structure of length two. We repeat a similar degeneration to resolve the second locus. We begin with a locus defined by the triple of partitions

$((1,1),(1),0)$ (see Figure 2u) and degenerate one of the points onto the line. We again get two

loci as in Figure 13.

(a) ((1,1),(1),0)   (b) ((2,1),0,0)   (c)
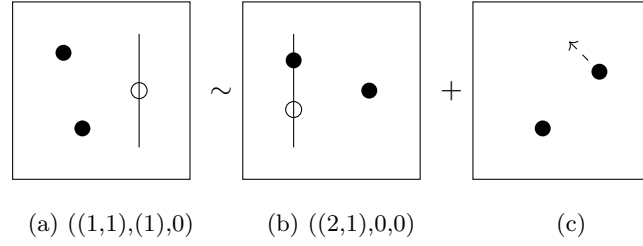
Figure 13: An example of the degeneration of Mallavibarrena and Sols.

Combining all this, we get that the class of the intersection is as seen in Figure 14.

(a) (0,(1),(1,1))   (b) ((3),0,0)   (c) ((2,1),0,0)   (d) ((1,1),(1),0)
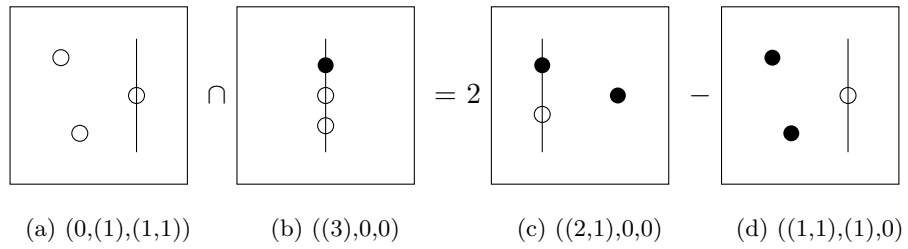
Figure 14: The class of the intersection $H \cdot \sigma_\alpha$ in the MS basis.

The general case of these degenerations was computed by Mallavibarrena and Sols. See Proposition 4.3.7.

More complicated examples involve elements with "moving lines". Consider now the class in $A^1(\mathbb{P}^{2[3]})$ given by the triple of partitions $\gamma = (0, 0, (2, 1))$ (see Figure 2c). To obtain a representative U_γ, we fix a general point Q and let U_γ be the locus of schemes which contain a subscheme of length two collinear with Q. Again let H be the locus of schemes which meet a general fixed line M. The intersection consists of two loci. The first locus is that of schemes which contain a subscheme of length two collinear with Q and whose remaining point meets M. The second locus is the schemes which contain a subcheme of length two collinear with Q which also meets M. See Figure 15.

(a) (0,(1),(1,1))   (b) (0,0,(2,1))   (c) (0,(1),(2))   (d) X
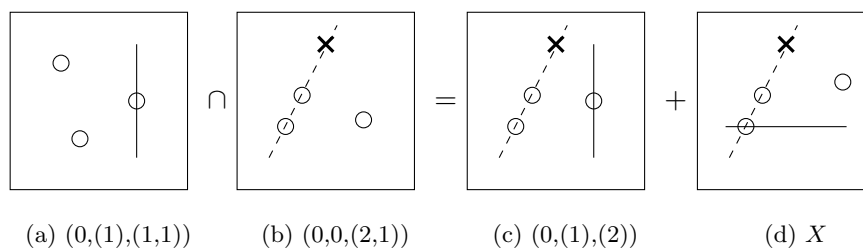
Figure 15: The intersection $H \cap U_\gamma$.

Let the final locus (see Figure 15d) be X. To realize the class $[X]$ in the MS basis, we degenerate the point Q onto the line M. The special fiber of this degeneration consists of two components. The first is the locus of schemes containing a subscheme of length two contained

in the line M, and the second is the locus of schemes containing the point Q. These have classes given by the triples of partitions $(0, (2), (1))$ and $((1), 0, (1, 1))$, respectively. See Figure 16.

(a) X (b) (0,(2),(1)) (c) ((1),0,(1,1))
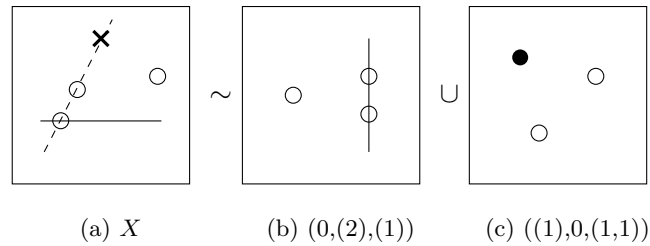
Figure 16: The degeneration of the locus X into two components.

It follows that $[X] = a \cdot \sigma_{(0,(2),(1))} + b \cdot \sigma_{(1),0,(1,1))}$ for undetermined coefficients a and b. To determine them, we pair both sides with the classes corresponding to the triples of partitions $((1), (1, 1), 0)$ and $((1, 1), 0, (1))$. These classes are determined from the triples of partitions representing the loci in the special fiber of the degeneration. The intersection numbers are straightforward to compute and we record them in the Table II.

X	(0,(2),(1))	((1),0,(1,1))
((1),(1,1),0)	2	1	0
((1,1),0,(1))	2	0	1

TABLE II: The intersection numbers which determine the coefficients of the two components of the special fiber of the degeneration of the locus X.

Putting it all together yields the equivalence of classes in Figure 17.

(a) (0,(1),(1,1))   (b) (0,0,(2,1))   (c) (0,(1),(2))   (d) (0,(2),(1))   (e) ((1),0,(1,1))

Figure 17: The class of the product $H \cdot \sigma_{(0,0,(2,1))}$ in the MS basis.

4.2 A More Complicated Example

More generally, for $\alpha = (\mathbf{a}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha)$ any reasonable triple of partitions and let σ be its associated class in the MS basis. We can consider the intersection $H \cdot \sigma$ along each condition for the locus U_α separately and then take the sum of the resulting classes. That is to say, we

can intersect it with each fixed line coming from $\mathbf{a}^\alpha$, each fixed line from $\mathbf{b}^\alpha$, and each moving line from $\mathbf{c}^\alpha$ individually, and then sum to achieve the class of the product. Since the process for intersection along the fixed lines is trivial in the case of those containing no fixed points of the schemes in the locus and a known result of Mallavibarrena and Sols for the others, we'll focus primarily on the intersections with moving lines.

To that end, let $\alpha = (0, 0, (3, 2, 1))$ and σ be the associated class in the MS basis for $A(\mathbb{P}^{2[6]})$. Fix a line L and let H be the locus of schemes whose support meets L, and fix a point P not on L and let U be the locus of schemes which contain distinct subschemes of length one, two, and three collinear with P. See Figure 18.
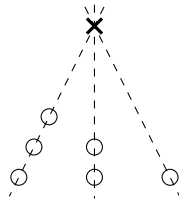
Figure 18: The schematic diagram for σ.

The intersection $H \cap U$ consists of three irreducible components: the first corresponding to when the free point of U lies on L, the second corresponding to when of the points in the subscheme of length two collinear with P lies on L, and finally when one of the points of the subscheme of length three collinear with P lies on L. See Figure 19 for the corresponding pictures. We have labeled each component with names that we will define precisely later.
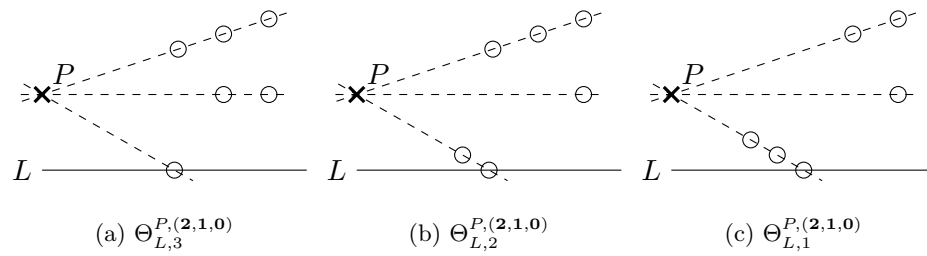
Figure 19: The components of the intersection $H \cdot \sigma$.

The first locus is already the class associated to the triple of partitions $(0, (1), (3, 2))$ in the MS basis. We will resolve the class of the second and third locus via a series of degenerations. We will start with the first component.

To do this, we degenerate the point P onto the line L. There are three components in the limit each consisting of at most one of the points on each moving line colliding with the point P. More precisely, they are the locus of schemes with a subscheme of length three varying on L and a subscheme of length two collinear with P, the locus of subschemes containing P and two disjoint subschemes of length two collinear with P, and the final component consisting of subschemes containing a subscheme of length two supported at P and a subscheme of length two collinear with P. See Figure 20 for the corresponding pictures. The first component is numerically equivalent to the element of the MS basis associated to $(0, (3), (2, 1))$.
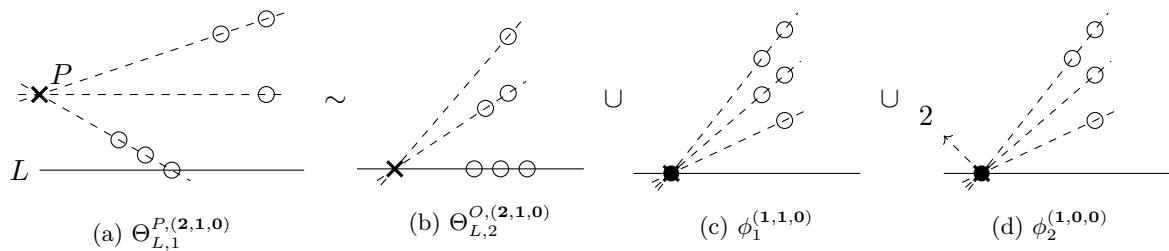
Figure 20: The three loci of the first degeneration.

To determine the multiplicities of each component, we pair both sides with explicit classes in the MS basis. For the first component (Figure 20b), we will use the class associated to the triple $((1, 1), (2, 1, 1), 0)$. To see why we chose this class, we look at the locus and fix two general points to determine the moving lines. We then think of the three lines in the picture (two determined moving lines and one fixed line) as horizontal. Since one of the moving lines only needs to contain a single point, it is now satisfied. The other moving line still needs to contain an additional point, and the fixed line must contain three points. We now fix a set of general lines which we think of as vertical. The first vertical line we require to contain two points to account for one point on each of the remaining unsatisfied two lines, and the remaining two vertical lines we require to contain one point each to account for the two remaining points on fixed line. Alternatively, take the partition $(2, 1)$ and insert the 3 from the fixed line to obtain $(3, 2, 1)$, then subtract one from each entry coming from a moving line (for each fixed point) to get $(3, 1, 0)$, and take the conjugate partition $(2, 1, 1)$. This will be the second entry of the triple of partitions for the class.

We repeat a similar process for the remaining two loci. We fix three general points to determine the moving lines, and then view all of the lines as horizontal. For the second locus (Figure 20c), we also fix a general line which must contain two points and allow the last point of the locus to freely vary. For the third locus (Figure 20d), we instead fix a general line and require it contain just a single point, but fix a final general point and require our schemes contain a subscheme of length two collinear with this point. This forces there to be a single such subscheme which is supported at the fixed point of the third locus, thus making the intersections easy to compute. We get the classes associated to the triples $((1,1,1),(2),(1))$ and $((1,1,1),(1),(2))$. We count each of the intersections by hand, and can even compute the tangent spaces to each locus directly in this case. In general, we show that that these intersections occur transversely at only points which are sufficiently nice. In Table III we record the intersection of this class with the four loci from the degeneration, as well as the multiplicity of each component (obtained by solving the resulting linear equation).

	$\Theta_{L,1}^{P,(\mathbf{2,1,0})}$	$\Theta_{L,2}^{O,(\mathbf{2,1,0})}$	$\phi_1^{(\mathbf{1,1,0})}$	$\phi_2^{(\mathbf{1,0,0})}$
$((1,1),(2,1,1),0)$	6	2	0	0
$((1,1,1),(2),(1))$	6	0	3	0
$((1,1,1),(1),(2))$	6	0	0	3
Mult. of Comp.		3	2	2

TABLE III: The intersection numbers which determine the coefficients of the components of the degeneration in Figure 20.

Notice that none of these classes meets any of the other components due to the length of the subschemes supported at P. In general, there may be multiple different components which have the same length subscheme supported at P, and in such a situation we still get explicitly determined classes for each component, but must solve a linear system resulting from pairing both sides with these classes.

To write the remaining classes in the MS basis, we degenerate again. We start with a locus which is identical except the subscheme of positive length is supported at a fixed point Q distinct from P. We degenerate Q onto P, and obtain the locus we want as one of the irreducible components. We always start with the loci with the lowest length at P (when multiple loci appear with the same length at P, we do them all simultaneously). In our case, we degenerate the second locus (Figure 20c) first. Starting with the class of the locus of subschemes containing a fixed general point Q and two subschemes of length two collinear with the fixed general point P, we degenerate Q onto P. The special fiber of this degeneration is supported along three irreducible components consisting of at most one of the points in each moving subscheme of length two colliding with P. Precisely, the first is the locus of schemes containing P and two subschemes of length two collinear with P, the second is the locus of schemes containing a length two subcheme supported at P and a subscheme of length two collinear with Q, and the third is the locus of schemes containing a length three subscheme supported at P. See Figure 21.
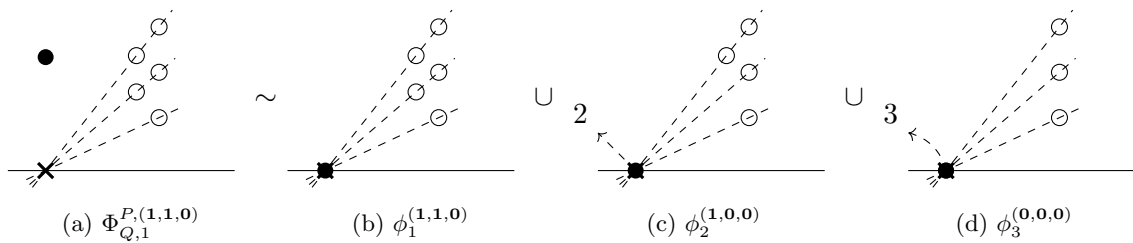
Figure 21: The three loci of the degeneration.

We will again pair both sides with classes to determine the multiplicities of each of the components in the limit of the degeneration. We start with three fixed general points to determine each of the moving lines. For the first component (Figure 21b) the class we want will have an additional general fixed line containing a subscheme of length two. The class for the second component (Figure 21c) will also consist of schemes whose support meets a general fixed line and which contain a subscheme of length two collinear with a fixed general point. The last component (Figure 21d) we want to pair with the locus of schemes containing these three fixed general points and which contain subschemes of length three collinear with a fixed general point. These are the three classes in the MS basis associated to the triples $((1, 1, 1), (2), (1))$, $((1, 1, 1), (1), (2))$, and $((1, 1, 1), 0, (3))$. The results of this pairing we record in Table IV.

	$\Phi_{Q,1}^{P,(\mathbf{1,1,0})}$	$\phi_1^{(\mathbf{1,1,0})}$	$\phi_2^{(\mathbf{1,0,0})}$	$\phi_3^{(\mathbf{0,0,0})}$
((1,1,1),(2),(1))	3	3	0	0
((1,1,1),(1),(2))	6	0	3	0
((1,1,1),0,(3))	3	0	0	1
Mult. of Comp.		1	2	3

TABLE IV: The intersection numbers which determine the coefficients of the components of the degeneration in Figure 20.

Notice again that each class only meets the specific component we want it to because of the length of the scheme contained at the point P. Again, in general, there may be multiple components in the degeneration with the same length nonreduced subscheme supported at P, and in this case we must solve a system of equations to obtain the multiplicities.

Combining the result of these computations gives the equivalence of classes in Figure 22.
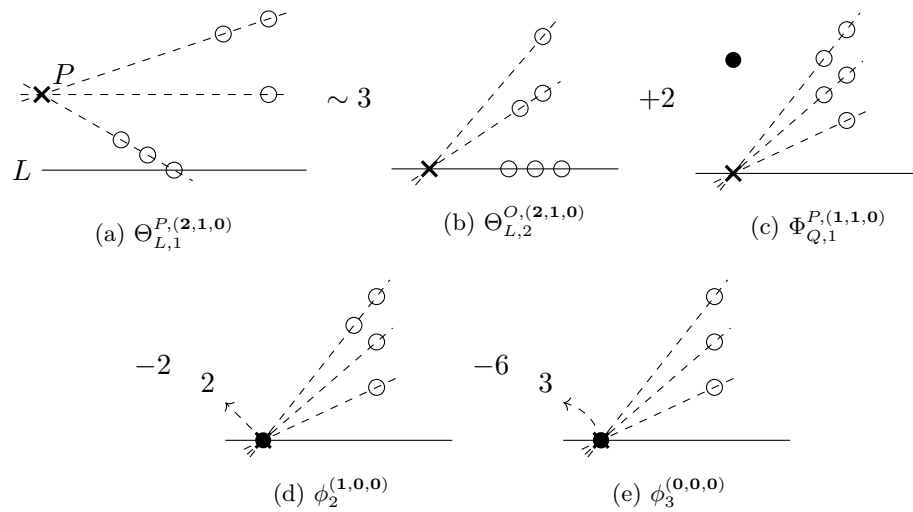
Figure 22: The equivalence of classes roughly halfway through determining the class in the MS basis of the component in Figure 19a.

The point here is that we have replaced a class in the sum with a class in the MS basis at the expense of adding classes comprised of schemes which contain nonreduced subschemes of higher length at the point P. Unfortunately, we must also accept the possibility of negative signs, and as a result our process is not positive. Since this length is bounded, the process eventually terminates. We continue the process with the locus whose subscheme at P has length two (Figure 22d).

We start with a locus which is identical except the nonreduced subscheme of length two is supported a fixed point Q distinct from P. We degenerate the point Q onto P and obtain two components of the special fiber of the degeneration. The first is the locus of schemes containing a nonreduced subscheme of length two at P and a subscheme of length two collinear with P, and

the second is the locus of schemes containing a nonreduced subscheme of length three at P. See Figure 23. The classes we pair with these to determine the multiplicities of each component are the same we used before, that is to say, they are the classes associated with $((1,1,1),(1),(2))$ and $((1,1,1),0,(3))$, so it suffices to compute only the intersection of these classes with the locus on the left hand side. These intersection numbers are both three, so the first component appears with multiplicity one and the second with multiplicity three. Combining this with the equality in Figure 22 we arrive at the equality in Figure 24. Notice that the contribution of the class whose schemes contain subchemes of length three from this last degeneration cancel with those we already have. At the moment, we have no conjecture as to how to detect when this phenomenon occurs.

(a) $\Phi_{Q,2}^{P,(1,0,0)}$ (b) $\phi_2^{(1,0,0)}$ (c) $\phi_3^{(0,0,0)}$
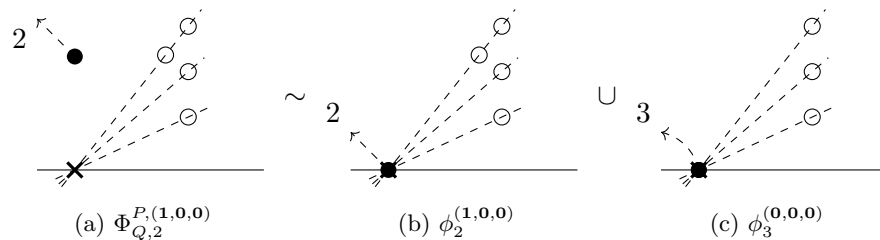
Figure 23: The loci of the second degeneration.
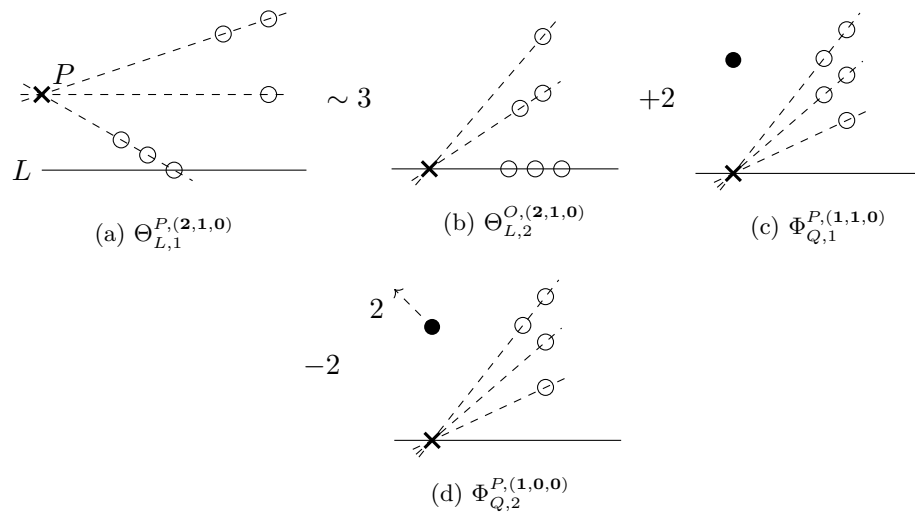
Figure 24: The equivalence in Figure 22 after substituting in the results of the second degeneration.

At this point, we rely on the degeneration of Mallavibarrena and Sols to break up classes of nonreduced subschemes supported at a fixed point into classes whose general member is comprised of distinct points contained in lines. Figure 25 shows the final result of our computation after applying this degeneration.

(a) $\sigma_{(0,(3),(2,1))}$　　(b) $\sigma_{((1),0,(2,2,1))}$　　(c) $\sigma_{((1),(1),(2,1,1))}$　　(d) $\sigma_{((2),0,(2,1,1))}$
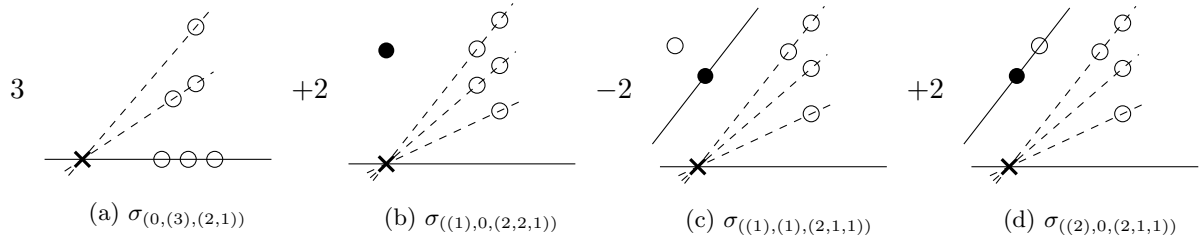
Figure 25: The results of our hard work: the class of the component in Figure 19a in the MS basis.

The computation of the class of the second component (Figure 19b) of the intersection is completely analogous. The result of that computation is contained in Figure 26.

(a) $\sigma_{(0,(3),(2,1))}$　　(b) $\sigma_{(0,(2),(3,1))}$　　(c) $\sigma_{((1),0,(3,1,1))}$
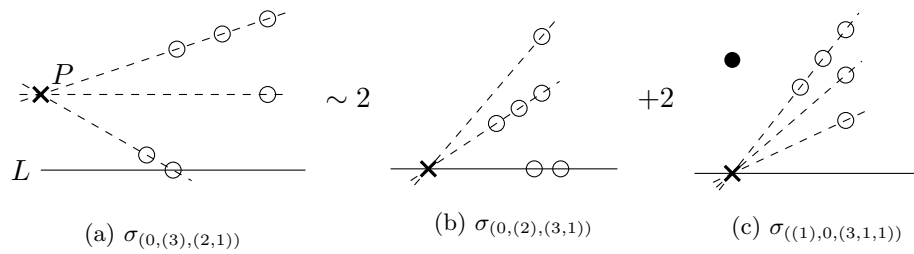
Figure 26: The class of the component in Figure 19b in the MS basis.

4.3　Degenerations

We will now begin to prove the results necessary to establish Main Theorem 1. Brace yourselves. Are you ready?

We need notation for a few loci in the Hilbert scheme before we can describe the degenerations. For that purpose, fix a partition $\mathbf{m}$ of N of length r. For the purpose of this section, we allow partitions to have finitely many zeros and consider them when counting the length. Fix i such that $1 \leq i \leq r$ and such that $\mathbf{m}_i > \mathbf{m}_{i+1}$. Let P and Q be distinct points in $\mathbb{P}^2$, let L be a line in $\mathbb{P}^2$, and let q be a positive integer.

Let $\Theta_{L,i}^{P,\mathbf{m}}$ be the closure of the locus in $\mathbb{P}^{2[N+r]}$ of schemes Z such that:

- for each $1 \leq j \leq r$, Z contains a subscheme Z_j of length $\mathbf{m}_j + 1$ collinear with P and spanning a line L_j such that the support of Z_j does not meet L_k for $k \neq j$, and

- the support of Z_i meets L.

See Figure 27 for an example of this locus when $\mathbf{m} = (3, 2, 1)$ and $i = 2$.
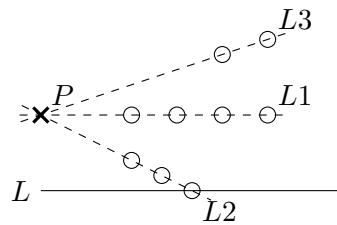
Figure 27: A schematic picture for the scheme $\Theta_{L,2}^{P,(3,2,1)}$.

Notice that the general scheme in $\Theta_{L,i}^{P,\mathbf{m}}$ is supported at $N + r$ distinct points. There are $\mathbf{m}_j + 2$ degrees of freedom for each collinear subscheme except when $j = i$. In this case there are $\mathbf{m}_i + 1$ degrees of freedom, so the dimension of $\Theta_{L,i}^{P,\mathbf{m}}$ is

$$\mathbf{m}_i + 1 + \sum_{j \neq i} (\mathbf{m}_j + 2) = 2r + N - 1.$$

(a) $\Phi_{Q,2}^{P,(3,2,1)}$ (b) $\phi_2^{(3,2,1)}$
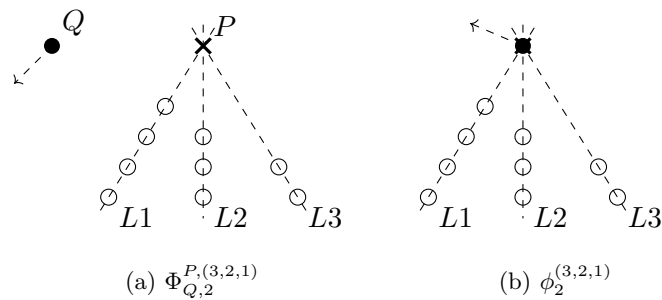
Figure 28: Pictures for the scheme $\Phi_{Q,2}^{P,(3,2,1)}$ and $\phi_2^{(3,2,1)}$.

Let $\Phi_{Q,q}^{P,\mathbf{m}}$ be the closure of the locus in $\mathbb{P}^{2[N+r+q]}$ of schemes Z such that:

- Z contains a nonreduced subscheme Z_Q of length q at Q;

- for each $1 \leq j \leq r$, Z contains a subscheme Z_j of length $\mathbf{m}_j + 1$ collinear with P and spanning a line L_j such that the support of any Z_j does not meet L_j for $k \neq j$;

- and Q does not lie on L_j for any j.

Finally, let $\phi_q^{\mathbf{m}}$ be the closure of the locus in $\mathbb{P}^{2[N+q+r]}$ of schemes Z such that:

- Z contains a nonreduced subscheme Z_O of length q at the origin;

- and for each $1 \leq j \leq r$, Z contains a subscheme Z_j of length $\mathbf{m}_j + 2$ collinear with the origin and spanning a line L_j such that the support of any Z_j does not meet L_k away from the origin for $k \neq j$.

Note that $\phi_q^{\mathbf{m}}$ differs from $\Phi_{Q,q}^{P,\mathbf{m}}$ since P and Q are distinct. See Figure 28 for an example of each locus when $\mathbf{m} = (3, 2, 1)$ and $q = 2$. In either case, there are $q - 1$ degrees of freedom defining the length q subscheme at Q and $\mathbf{m}_i + 2$ degrees of freedom for each collinear subscheme, so the dimension of $\Phi_{Q,q}^{P,\mathbf{m}}$ is

$$(q - 1) + \sum_{i=1}^{r} (\mathbf{m}_i + 2) = q + 2r + N - 1.$$

Notice that the general scheme in either locus is supported at $N + r + 1$ distinct points. It is important to observe the following fact.

Lemma 4.3.1. *The loci $\phi_q^{\mathbf{m}}$ and $\Phi_{Q,q}^{P,\mathbf{m}}$ are irreducible.*

Proof. The loci $\phi_q^{\mathbf{m}}$ and $\Phi_{Q,q}^{P,\mathbf{m}}$ are defined as closures of loci which are isomorphic to the product of a Briançon scheme and open sets defining elements of the MS basis, so they are irreducible of the given dimensions. $\square$

Let Σ_j be the set of partitions obtained from some partition $\mathbf{m}$ by subtracting one j times, at most once from each entry. Expressed as Young diagrams, the partitions obtained from $\mathbf{m} = (2, 1, 1)$ are shown in Figure 29.
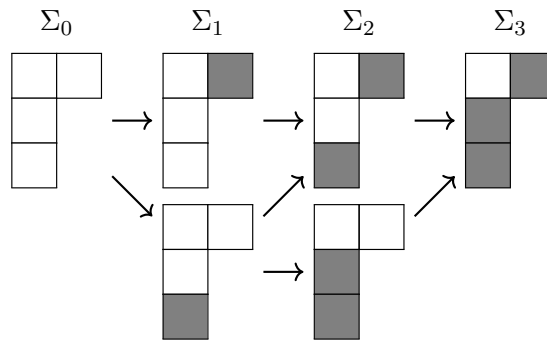
Figure 29: The sets Σ_j for $\mathbf{m} = (2, 1, 1)$.

Recall that for the purpose of this section our partitions may have zeros, so that, for instance, the unique partition of Σ_3 obtained from the partition $\mathbf{m} = (2, 1, 1)$ is $(1, 0, 0)$ and we consider it to be of length three.

Additionally, let Σ_j^i be the set of partitions obtained from some partition $\mathbf{m}$ by subtracting one j times, at most once from each entry, and always beginning with $\mathbf{m}_i$. This causes no issues because our choice of i was such that $\mathbf{m}_i > \mathbf{m}_{i+1}$. For instance, if $\mathbf{m} = (2, 1, 1)$ as above, then $\Sigma_j^2 = \Sigma_j^3 = \Sigma_j$ for $j = 0$ and $j > 1$, but $\Sigma_1^2 = \Sigma_1^3$ is a singleton consisting of only the lower partition shown under Σ_1 in Figure 29.

4.3.1   The First Degeneration

Let O be the origin, let P_t be the point $(0, t)$ in the plane with $P = P_1 = (0, 1)$, and let L be the line $\{y = 0\}$. Let F be the family in $\mathbb{P}^{2[N+r]} \times \mathbb{C}^*$ such that the fiber F_t over $t \in \mathbb{C}^*$ is $\Theta_{L,i}^{P_t, \mathbf{m}}$. The closure $\hat{F}$ of F in $\mathbb{P}^{2[N+r]} \times \mathbb{C}$ is a flat family with special fiber $\hat{F}_0$.

Proposition 4.3.2. *The support of the special fiber $\hat{F}_0$ is contained in the union*

$$\Theta_{L,i}^{O,\mathbf{m}} \cup \bigcup_{j=1}^{r} \bigcup_{\lambda \in \Sigma_j^i} \phi_j^\lambda.$$

Corollary 4.3.3. *There is an equivalence of cycles*

$$\left[\Theta_{L,i}^{P,\mathbf{m}}\right] \sim a \left[\Theta_{L,i}^{O,\mathbf{m}}\right] + \sum_{j=1}^{r} \sum_{\lambda \in \Sigma_j^i} c_j^\lambda \left[\phi_j^\lambda\right]$$

for nonnegative integers a and c_j^λ.

Notice that every scheme in the component $\Theta_{L,i}^{O,\mathbf{m}}$ must have a subscheme of length $\mathbf{m}_i$ contained in L.

For an example, when the general fiber of the family is $\Theta_{L,2}^{P,(2,1,1)}$, Proposition 4.3.2 says that the special fiber is supported on the irreducible components given by the Σ_j^2. See Figure 30 for pictures, but note that, for example, the locus $\phi_4^{(0,0,0)}$ does not appear.
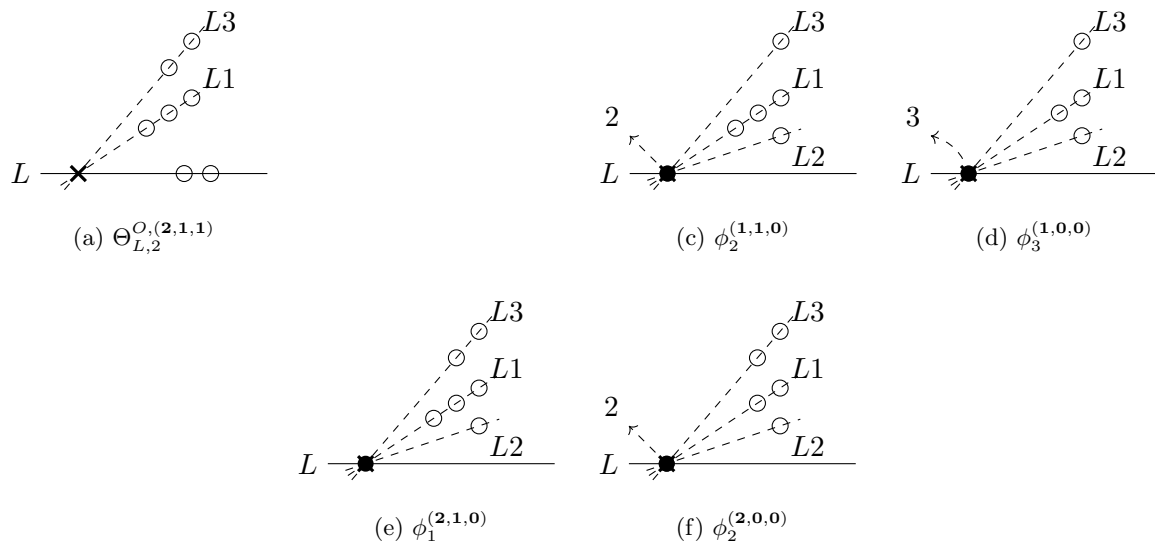
Figure 30: The irreducible components of the support of the special fiber.

We will use the following lemma in the proof of Proposition 4.3.2.

Lemma 4.3.4. *A general nonreduced subscheme Z of length $k > 0$ supported at the origin has length one along any general line L through the origin.*

Proof. The general subscheme Z of length k supported at the origin is contained in a smooth curve ζ of degree $k - 1$. The length $\ell(Z \cap L)$ of the scheme along the line L is the intersection multiplicity $I_O(L, \zeta)$ at the origin. Since the line L is general, this is readily seen to be one. $\square$

Proof of Proposition 4.3.2. Let Z_0 be a general point of the special fiber $\hat{F}_0$ and let γ be a curve in F specializing to Z_0 such that the general point γ_t is contained in F_t. The general γ_t contains subschemes $\gamma_{t,j}$ of length $\mathbf{m}_j + 1$ for $1 \leq j \leq r$ such that $\gamma_{t,j}$ spans a line $L_{t,j}$ through P_t. Furthermore, the support of the subscheme $\gamma_{t,j}$ meets L. Each family of lines

$\{L_{t,j}\}$ specializes to a line $L_{0,j}$ through the origin O, and the support of Z_0 is contained in the union of these lines with the lengths $\ell(Z_0 \cap L_{0,j})$ of Z_0 along each line L_j at least $\mathbf{m}_j + 1$. It follows that if the support of Z_0 does not contain the origin, then $Z_0 \in \Theta_{L,i}^{O,\mathbf{m}}$. Otherwise, Z_0 contains a subscheme of length $l > 0$ supported at the origin and subschemes Z_j of length $\mathbf{m}_j - l_j + 1$ disjoint from and collinear with the origin such that $\sum l_j = l$.

With that in mind, let C be a component of $\hat{F}_0$ described by a partition $\mathbf{m}' = (\mathbf{m}_1 - l_1, \mathbf{m}_2 - l_2, \ldots, \mathbf{m}_r - l_r)$ for nonnegative integers $l_i \leq \mathbf{m}_i$ such that C is contained in the locus $\phi_l^{\mathbf{m}'}$. It follows that

$$\dim C \leq l - 1 + \sum_{j=1}^{r}(\mathbf{m}'_j + 2) = l - 1 + \sum_{j=1}^{r}(\mathbf{m}_j - l_j + 2) = N + 2r - 1$$

with equality holding if and only if $C = \phi_l^{\mathbf{m}'}$. The general scheme F_t in the family has dimension $N + 2r - 1$, so C must too, and therefore $C = \phi_l^{\mathbf{m}'}$.

Now, the general scheme $Z \in \phi_l^{\mathbf{m}'}$ consists of subschemes Z_j of $\mathbf{m}'_j$ distinct points spanning a line L_j through the origin and a general length l subscheme Z_O at the origin. By Lemma 4.3.4, the length of Z_O along any of the lines L_j is one, so that

$$\mathbf{m}_j + 1 \leq \ell(Z \cap L_j) = (\mathbf{m}'_j + 1) + \ell(Z_O \cap L_j) = (\mathbf{m}_j - l_j + 1) + 1,$$

and it follows that $l_j \leq 1$.

Additionally, since Z is the limit of some curve in F whose general member is contained in F_t, the subscheme $Z_i \subseteq Z$ is the limit of subschemes $Z_{t,i}$ all meeting L and therefore must meet L also. But then the support of Z_i must meet the origin for general Z. $\qquad\square$

4.3.2   The Second Degeneration

Now let q be a positive integer, let Q_t be the point $(0, t)$ in the plane, let $Q = Q_1 = (0, 1)$ and let O be the origin. Consider the family F in $\mathbb{P}^{2[N+q+r]} \times \mathbb{C}^*$ such that the fiber F_t over $t \in \mathbb{C}^*$ is $\Phi_{Q_t,q}^{O,\mathbf{m}}$. The closure $\hat{F}$ of F in $\mathbb{P}^{2[N+q+r]} \times \mathbb{C}$ is a flat family with special fiber $\hat{F}_0$.

Proposition 4.3.5. *The support of the special fiber $\hat{F}_0$ is contained in the union*

$$\bigcup_{j=0}^{r} \bigcup_{\lambda \in \Sigma_j} \phi_{q+j}^{\lambda}.$$

Corollary 4.3.6. *There is an equivalence of cycles*

$$\left[\Phi_{Q,q}^{O,\mathbf{m}} \right] \sim \sum_{j=0}^{r} \sum_{\lambda \in \Sigma_j} c_j^{\lambda} \left[\phi_{q+j}^{\lambda} \right]$$

for nonnegative integers c_j^{λ}.

For example, when the general fiber of the family is $\Phi_{Q,2}^{O,(2,1,1)}$, Proposition 4.3.5 says that the special fiber $\hat{F}_0$ is supported on the irreducible components given by the Σ_j in Figure 29. See Figure 31 for the pictures of each locus, but note that, for instance, the locus $\phi_6^{(0,0,0)}$ does not appear.
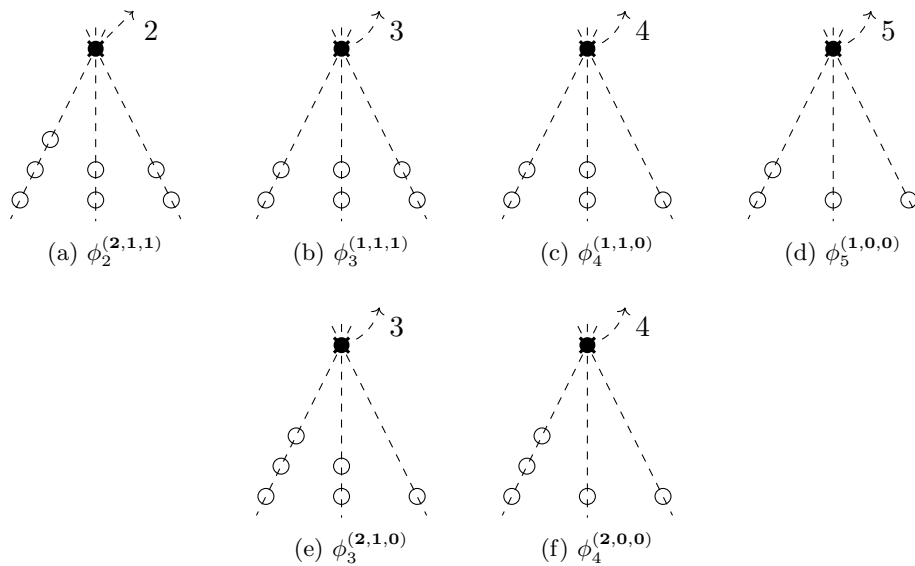
Figure 31: The irreducible components of the support of the special fiber.

Proof of Proposition 4.3.5. Let Z_0 be a general point of the special fiber $\hat{F}_0$ and let γ be a curve in F specializing to Z_0 such that the general point γ_t is contained in F_t. The general γ_t contains subschemes $\gamma_{t,i}$ of length $\mathbf{m}_i + 1$ for $1 \leq i \leq r$ such that $\gamma_{t,i}$ spans a line $L_{t,i}$ through the origin and a subscheme γ_{t,Q_t} of length q supported at Q_t. The family $L_{t,i}$ for each i specializes to a line $L_{0,i}$ and Q_t specializes to the origin O, so that the support of Z_0 is contained in $\{O\} \cup \bigcup_{i=1}^r L_{0,i}$. Furthermore, the length of Z_0 along any of the $L_{0,i}$ is $\ell(Z_0 \cap L_{0,i}) \geq \mathbf{m}_i + 1$ and the length of Z_0 at O is $\ell(Z_0)_O \geq q$.

The components of $\hat{F}_0$ therefore consist of loci comprised of schemes containing subschemes whose support is disjoint from the origin, but which are collinear with the origin, of length less than or equal to $\mathbf{m}_i$ for each $1 \leq i \leq r$ and containing a subscheme of length $q + j$ at the origin for $j \geq 0$. With that in mind, let C be a component of $\hat{F}_0$ described by a partition

$\mathbf{m}' = (\mathbf{m}_1 - l_1, \mathbf{m}_2 - l_2, \ldots, \mathbf{m}_r - l_r)$ for nonnegative integers l_i, and let l be the sum of the l_i such that C is the contained in the locus $\phi_{q+l}^{\mathbf{m}'}$. It follows that

$$\dim C \leq (q + l - 1) + \sum_{i=1}^{r} (\mathbf{m}_i' + 2) = (q + l - 1) + \sum_{i=1}^{r} (\mathbf{m}_i - l_i + 2) = q + N + 2r - 1.$$

Equality holds if and only if $C = \phi_{q+l}^{\mathbf{m}'}$. Since the dimension of each fiber F_t has dimension $q + N + 2r - 1$, the dimension of C must be $q + N + 2r - 1$. Hence, the general point $Z_0 \in C$ contains subschemes $Z_{0,i}$ consisting of $\mathbf{m}_i' + 1$ distinct points collinear with the origin and a general subscheme $Z_{0,O}$ of length $q + l$ supported at the origin.

Assume Z_0 is a general point of C. As before, let $L_{0,i}$ be the lines through the origin spanned by the subschemes $Z_{0,i}$ and $Z_{0,O}$ be the subscheme of length $q + l$ supported at the origin. By Lemma 4.3.4, $\ell(Z_{0,O} \cap L_{0,i}) = 1$ for each i. On the other hand,

$$\mathbf{m}_i + 1 \leq \ell(Z_0 \cap L_{0,i}) = (\mathbf{m}_i' + 1) + \ell(Z_{0,O} \cap L_{0,i}) = (\mathbf{m}_i - l_i + 1) + 1,$$

and it follows that $l_i \leq 1$, as desired. $\qquad\square$

4.3.3 The MS Degeneration

We finish this section with a final degeneration which is due to Mallavibarrena and Sols (30).

Let L be a line in $\mathbb{P}^2$, $Q \in \mathbb{P}^2$ a point, and let l and q be nonnegative integers. Define the subset $H_{Q,q}^{L,l}$ to be the locus of schemes $Z \in \mathbb{P}^{2[l+q]}$ such that

- the length $\ell(Z \cap L) \geq l$; and

• the length $\ell(Z)_Q \geq q$.
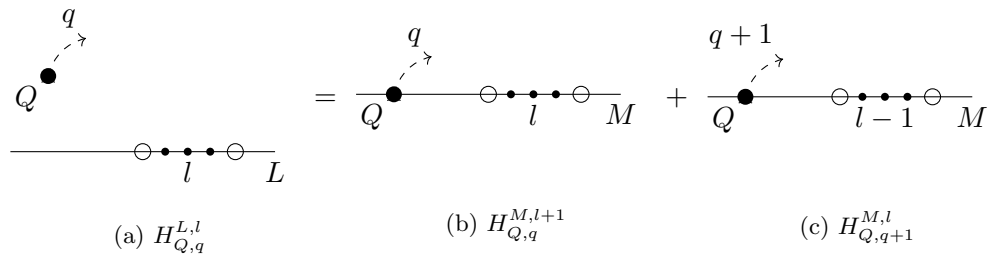
Figure 32: The degeneration of Mallavibarrena and Sols.

Proposition 4.3.7. *Let Q be a point of $\mathbb{P}^2$, L a line not containing Q, and M a line containing Q. Let l and q be nonnegative integers. There is an equivalence of classes on $\mathbb{P}^{2[l+q]}$*

$$\left[H_{Q,q}^{L,l}\right] \sim \left[H_{Q,q+1}^{M,l}\right] + \left[H_{Q,q}^{M,l+1}\right].$$

For a proof, see Mallavibarrena and Sols (30). A schematic picture can be found in Figure 32. We obtain the obvious corollary.

Corollary 4.3.8. *There is an equivalence of classes*

$$\left[H_{Q,q}^{M,1}\right] = \sum_{i=1}^{q} (-1)^{i+1} \left[H_{Q,q-i}^{L,i}\right] + (-1)^{q-1} \left[H_{Q,1}^{M,q}\right].$$

A slight modification of the degeneration in Proposition 4.3.7 will allow us to resolve the class of loci appearing naturally in the intersections $H \cdot \sigma$. To that end, let Q be a point of $\mathbb{P}^2$,

L be a line containing a fixed point P, and l, q be nonnegative integers. Define the locus $G_{Q,q}^{P,L,l}$ to be the schemes $Z \in \mathbb{P}^{2[l+q+1]}$ such that

- $P \in Z$;

- the length $\ell(Z \cap L) \geq l$; and

- the length $\ell(Z)_Q \geq q$.

Notice that $G_{Q,q}^{P,L,l}$ differs from $H_{Q,q}^{L,l}$ since its members contain the additional fixed point $P \in L$. By fixing a point on L in Proposition 4.3.7, we obtain the following corollary.

Corollary 4.3.9. *Let P and Q be distinct fixed points in $\mathbb{P}^2$, let L be the line they span, and let R be a fixed point not contained in L. There is an equivalence of classes*

$$\left[G_{Q,1}^{P,L,l+2} \right] = \sum_{i=0}^{l} (-1)^i \left[G_{R,i}^{P,L,l+1-i} \right].$$

4.4   The Algorithm

In this section, we describe how to compute the class of the intersection of the divisor $\sigma_{(0,1,(N-1))}$ with any other class σ_α for a triple of partitions $\alpha = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ in the Hilbert scheme $\mathbb{P}^{2[N]}$. We begin with some immediate observations and reductions.

First, we fix a general representative U_α for σ_α and H for $\sigma_{(0,1,(N-1))}$. The intersection of these two representatives is generically transverse as long as the lines and points defining U_α and the line defining H are chosen generally with respect to each other, as is the case. The irreducible components of the intersection are thus easy to identify: the point moving on the line defining H must satisfy one of the three different types of conditions defining U_α which we call type A, B, and C, respectively. Type A intersections contain as a fixed point the intersection

point of the line defining H and one of the lines through a fixed point defining U_α. Type B intersections contain as a fixed point the intersection point of the line defining H and one of the fixed lines defining U_α which does not pass through a fixed point defining U_α. Type C intersections occur when one of the points on a moving line through Q defining U_α resides on the line defining H. See Figure 33 for schematic diagrams of these intersections.

(a) Type A Intersections   (b) Type B Intersections   (c) Type C Intersections
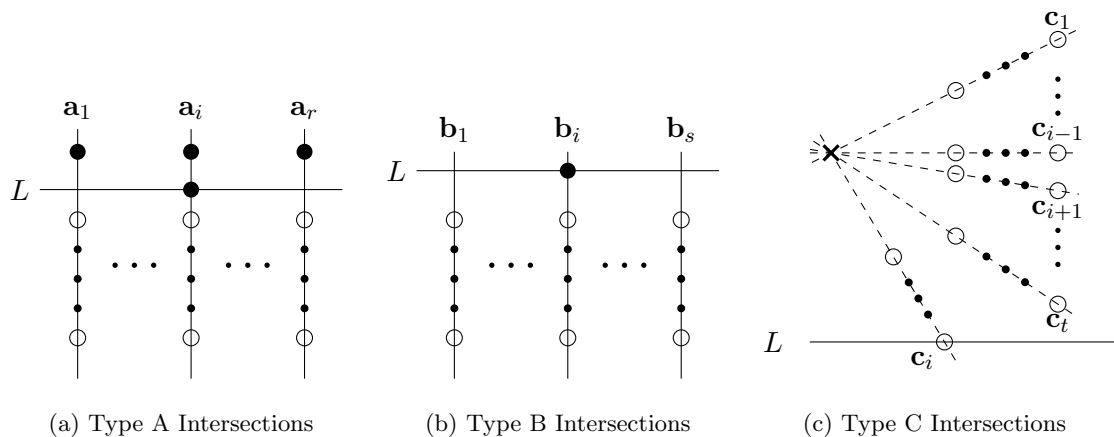
Figure 33: The three different types of intersections.

The class $\sigma_{(\mathbf{a},\mathbf{b},\mathbf{c})} \cdot \sigma_{(0,1,(N-1))}$ is the sum of the classes of each component of the intersection of H with U_α. It remains to compute these classes. We will dispatch the easiest case first.

4.4.1 Type B

The simplest case is when the point moving on the line defining H meets any of the fixed lines defining U_α coming from the partition $\mathbf{b}$ as shown in Figure 33b. The classes of these components are already elements of the MS basis, and there is very little work to be done.

If $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_s)$, then the resulting class describing the intersection along these types is given by the sum

$$\sum_{i=1}^{s} \sigma_{(\mathbf{a}^i, \widehat{\mathbf{b}^i}, \mathbf{c})}$$

where $\widehat{\mathbf{b}^i}$ is $\mathbf{b}$ with $\mathbf{b_i}$ omitted and $\mathbf{a}^i$ is $\mathbf{a}$ with $\mathbf{b}_i$ inserted. Notice that if an entry of $\mathbf{b}$ is repeated then there are repeated summands.

4.4.2 Type A

This occurs when the point moving on the line defining H meets any of the lines in U_α through a fixed point of U_α as defined by $\mathbf{a}$ as shown in Figure 33a. Because the resulting locus contains schemes with two fixed points on the same line, possibly with or without remaining moving points on that line, we must resolve the class of this locus into elements of the MS basis.
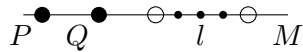
Figure 34: A locus of type A, or $G_{Q,1}^{P,L,l+2}$.

For this we turn to the degeneration of Mallavibarrena and Sols given in Proposition 4.3.7. Specifically, Corollary 4.3.9 allows us to write the class of the locus $G_{Q,1}^{P,L,l+2}$ (Figure 34) of the intersection as an alternating sum of the classes of loci consisting of a fixed point P with some number of points on a line through P and a nonreduced subscheme supported at a point away the line. The class of these loci can then be written in the MS basis by repeatedly applying Corollary 4.3.8. Refer to Section 4.1.1 for a simple example of this process.

4.4.3   Type C

Let $\mathbf{c}'$ be the partition given by $\mathbf{c} - 1 := (\mathbf{c}_1 - 1, \mathbf{c}_2 - 1, \ldots, \mathbf{c}_t - 1)$. This shift is necessary to align with our notation from Section 4.3. Any locus of Type C as shown in Figure 33c is consequently the locus $\Theta_{L,i}^{Q,\mathbf{c}'}$ for general line L defining the locus H and general point Q defining U_α.

The general steps are as follows.

1. We apply the degeneration from Proposition 4.3.2 and Corollary 4.3.3 to write the class $[\Theta_{L,i}^{Q,\mathbf{c}'}]$ as a sum of classes of loci of type ϕ_j^λ. We determine the coefficients c_j^λ in Corollary 4.3.3 by pairing with specific natural classes in the MS basis and solving the resulting systems of equations; this process will be explained more precisely in Section 4.4.3.1.

2. We then resolve the classes $[\phi_j^\lambda]$ in order of increasing j. To each locus we apply the degeneration in Proposition 4.3.5 and Corollary 4.3.6. In doing so, we replace $[\phi_j^\lambda]$ with a class of $[\Phi_{Q,j}^\lambda]$ at the cost of adding additional classes of loci $\phi_k^{\lambda'}$ for $k > j$. We again determine the coefficients $c_k^{\lambda'}$ in Corollary 4.3.6 by pairing with explicit classes as described in Section 4.4.3.1.

3. Repeat step 2 increasing the length of the subscheme common to all schemes in the locus at each step. Since this is bounded above by the total number of points C, eventually this terminates in the class $[\phi_C^{(0,0,\ldots,0)}] = [\Phi_{Q,C}^{(0,0,\ldots,0)}]$.

4. We replace each class $[\Phi_{Q,j}^\lambda]$ with a sum of classes in the MS basis using the degeneration of Mallavibarrena and Sols in Proposition 4.3.7 and Corollary 4.3.8.

4.4.3.1   Determining the Multiplicities

At the moment, we see no way to determine the coefficients c_λ^j of each component in Corollary 4.3.3 or 4.3.6 except by explicit computation by hand, as it seems to depend on delicate combinatorial information. See Section 4.2 for examples of this behavior. We end the section with Conjecture 4.4.8 which is an explicit determination of these coefficients.

Fix an integer $j \geq 0$ and let λ be a partition of a positive integer M possibly containing zeros as our convention allows. Each component ϕ_j^λ naturally determines a class in the MS basis which we use to determine the coefficients of the components with a nonreduced subscheme at the origin of length j. Let r be the length of λ, let $\mathbf{a} = (1, 1, \ldots, 1)$ be the unique partition of length r consisting of only ones, and let λ' be the conjugate partition to λ. Let $\tau_j^{\lambda'}$ be the class in the MS basis defined by the triple of partitions $(\mathbf{a}, \lambda', (j))$. Let e and f be indices for $\mathbf{a}$ and λ' and let s be the length of λ'. Let U be a general representative of $\tau_j^{\lambda'}$ defined by fixing r general points P_e, a general line L_f for each entry λ_f', and a general fixed point Q with which the subschemes of length j must be collinear.

Lemma 4.4.1. *The locus $\phi_q^{\mathbf{m}}$ does not meet a general representative of $\tau_j^{\lambda'}$ for $j \neq q$.*

Proof. First assume that $j < q$. The subschemes of any general member $Z \in U$ which are not collinear with Q are supported at the fixed points or on the fixed lines defining Σ which do not pass through the origin. As such, the maximum length of any scheme in U at the origin is j.

Now assume that $j > q$. Any scheme Z in the locus $\phi_q^{\mathbf{m}}$ has subschemes Z_k of length at least $\mathbf{m}_k + 1$ collinear with the origin. If Z is also a member of U, then each such subscheme must contain one of the r general points defining U, so that each line defined by Z_k is distinct from the others and is the span of the fixed point and the origin. Additionally, these lines do not pass through Q.

If Z contains a subscheme Y of length j at the origin, then this subscheme is the limit of smaller subschemes $W_{k,t}$ contained in the subschemes $Z_{k,t}$ of schemes Z_t in $\phi_q^{\mathbf{m}}$, so that the length of Y along any of the lines defined by the $Z_{k,0} = Z_k$ is at least the length of the subschemes $W_{k,t}$ for a general t. It follows that the length of any scheme Y is at least two along some line through the origin and one of the fixed points defining U. But any such scheme is contained in the line spanned by Q and the origin, and therefore had length one along any other line. Hence, there is no such scheme. $\qquad\square$

The above lemma allows us to pair each side of the equivalences in Corollaries 4.3.3 and 4.3.6 with well chosen classes in the MS basis to select out only those coefficients of the components on the right hand side whose general member contains a subscheme of the correct length at the origin. This reduces the problem of computing the classes to that of solving relatively small systems of linear equations. Unfortunately, we see no way of pairing both sides of the equivalences to obtain the coefficient of just a single component on the right.

Now let ρ be a partition of M also of length r and possibly with zeros as our convention allows. Our immediate goal is to prove the following lemma.

Lemma 4.4.2. *The intersection $\phi_j^\rho \cap U$ is transverse.*

The outline of the proof is as follows. We first characterize the points of the intersection using the fact that the classes have complementary codimension in the Hilbert scheme. This allows us to put charts on the Hilbert scheme around each intersection point which give it the local structure of a product. We then prove that that the intersection is transverse by computing the tangent spaces directly in each of these factors. Note the similarity to the approach and arguments in Section 3.3.

Lemma 4.4.3. *Any point Z of the intersection $\phi_j^\rho \cap U$ consists of $M + r$ distinct points and a length j linear subscheme supported at the origin.*

Proof. Let Z be a point of the intersection $\phi_j^\rho \cap U$. No two of the fixed points P_e defining U span a line through the origin, so Z contains subschemes Z_e of length at least $\rho_e + 1$ each of which contains P_e and is contained in the line N_e spanned by P_e and the origin for each $1 \le e \le r$. Additionally, the origin is not contained in any fixed line L_f defining U and Z must therefore contain a subscheme Z' of length j supported at the origin and contained in the line spanned by the origin and Q. Furthermore, Z must meet the union of the lines L_f in M points, and the common intersection points of the L_f do not lie on any of the lines N_e, so each Z_e is supported at the intersection points of the line N_e and some of the L_f's.

Let Y be a subscheme of Z supported at the intersection point R of some N_e with some L_f. R is not the intersection point of two distinct L_f's, and R is not contained in the subscheme of Z collinear with Q, so Y must be contained in the line L_f since it is the limit of distinct points contained in L_f. Y must also be contained in N_e since it is the limit of distinct points spanning a line through the origin and the limit in $\mathbb{P}^{2*}$ of these spans is N_e. If the length of Y is greater than one, then Y determines the line, but the lines L_f and N_e are distinct, so Y must be reduced. $\qquad\square$

We will need the following lemma.

Lemma 4.4.4. *The intersection in $\mathbb{P}^{2[j]}$ of the class of the locus of schemes of length j supported at a fixed point and the class of the locus of schemes of length j collinear with a fixed point is the class of a single reduced point.*
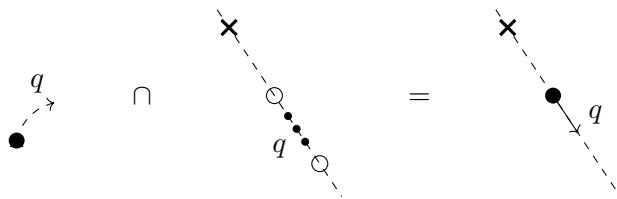
Figure 35: The intersection in Lemma 4.4.4.

Proof. Let the first class be that of locus of subschemes of length q supported at the origin. Let the second class be that of the locus of subschemes collinear with $(-1, 0)$. The unique subscheme in the intersection is given by the ideal (x^j, y). There are charts for $\mathbb{P}^{2[j]}$ isomorphic to $\mathbb{A}^{2[j]}$ around this point with coordinates given by coefficients of generators for the ideals

$$(x^j - \gamma_{j-1}x^{j-1} - \cdots - \gamma_1 x - \gamma_0, y - c_{j-1}x^{j-1} - \cdots - c_1 x - c_0).$$

The first locus is described by the equations $\gamma_i = c_0 = 0$ for all $0 \leq i \leq q-1$. The second locus can be described as those ideals containing the equation of a line through $(-1, 0)$, and in particular has equations $c_i = 0$ for $2 \leq i \leq j-1$ and $c_1 = c_0$. It follows that the loci are locally distinct linear spaces of complementary codimension, so they intersect transversally. $\qquad\square$

Proof of 4.4.2. Let Z be a scheme in the intersection $\phi_j^\rho \cap U$. Lemma 4.4.3 allows us to describe the tangent spaces to ϕ_j^ρ and U around Z. Since Z consists of a length j subscheme at the origin and distinct points otherwise, $\mathbb{P}^{N+r}$ is locally isomorphic to $\mathbb{P}^{2[j]} \times \mathbb{A}^{M+r}$. We can check transversality in each of these factors individually. Lemma 4.4.4 shows that the intersection in $\mathbb{P}^{2[j]}$ is transverse.

For the $\mathbb{A}^{M+r}$ factor, the argument is identical to the one in the proof of Lemma 3.3.7. $\square$

It remains to show the following.

Lemma 4.4.5. *The classes $\tau_j^{\lambda'}$ are independent on the span of the classes $[\phi_j^\lambda]$ as λ ranges over* Σ_j.

Proof. First, let the cardinality of Σ_j be m. Order the partitions $\lambda_i \in \Sigma_j$ lexicographically so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. It suffices to show that the pairing matrix with columns $\tau_j^{\lambda_i'}$ and rows $\phi_j^{\lambda_i}$ is upper triangular.

In particular, it is enough to check that:

1. $\tau_j^{\lambda'} \cdot [\phi_j^\lambda] \neq 0$, and

2. $[\phi_j^\rho] \cdot \tau_j^{\lambda'} = 0$ if $\rho' < \lambda'$ for two partitions ρ and λ in Σ_j.

The first is straightforward: the fixed points of τ determine the moving lines of ϕ and each subsequent fixed line of τ meets precisely the number of moving lines in ϕ necessary to form an intersection – indeed, this intersection number is one once the moving lines of ϕ are ordered, so that the intersection number is precisely the number of ways to order the moving lines in ϕ.

The second is similar: if $\rho' < \lambda'$ then there is an i such that $\rho_i' < \lambda_i'$ and $\rho_k' = \lambda_k'$ for all $k < i$. Now, the moving lines in ϕ are determined by the fixed points of τ, and ρ_1' is the number of moving lines in ϕ which contain more than just a fixed point of τ. If $i = 1$ and $\rho_1' < \lambda_1'$, then no intersection can occur since the condition that the first fixed line of τ must contain λ_i' points cannot be satisfied. Otherwise, the first $i - 1$ fixed lines of τ are forced to meet all the moving lines of ϕ possible, and then since $\rho_i' < \lambda_i'$ the i-th fixed line in τ cannot contain a subscheme of length λ_i'.

$\square$

We have the following two immediate results.

Lemma 4.4.6. *The coefficient a in Corollary 4.3.3 is* $\mathbf{m}_i + 1$.

Proof. We prove this by intersecting both sides of the equivalence with a class τ in the MS basis which does not meet any of the other classes on the right hand side but the class $\left[\Theta_{L,i}^{\mathbf{m}}\right]$.

Let $\mathbf{m}^2$ be the partition obtained from $\mathbf{m}$ by removing $\mathbf{m}_i$ and inserting $\mathbf{m}_i + 1$, and let $\mathbf{n}$ be the conjugate partition to $\mathbf{m}^2$ obtained by taking the columns in $\mathbf{m}'$ to be the rows of $\mathbf{n}$. Let r be the length of $\mathbf{m}^2$ and let s be the length of $\mathbf{n}$.

Let τ be the element of the MS basis defined by the mixed partition $((1, \ldots, 1), \mathbf{n}, 0)$ where the first partition consists of $r-1$ entries. The codimension of τ is $2(r-1) + \sum_{j=1}^{s} \mathbf{n}_j = 2r + N - 1$. Let T be a general representative (defined by fixing s general lines and $r - 1$ general points) of τ. T does not meet any of the components ϕ_j^λ since the origin is not contained in any of the lines defining T, so

$$\tau \cdot \left[\Theta_{L,i}^{P,\mathbf{m}}\right] = \tau \cdot a \left[\Theta_{L,i}^{\mathbf{m}}\right].$$

Let $\{P_e\}$ be the fixed points and $\{M_f\}$ be the fixed lines defining T. Let Z be an intersection point of either $\Theta_{L,i}^{\mathbf{m}} \cap T$ or $\Theta_{L,i}^{P,\mathbf{m}} \cap T$. The subscheme Z_i of length $\mathbf{m}_i + 1$ which meets L does not contain any of the P_e or it would also need to contain the intersection of the line L and the line through the origin and P_e (or P and P_e, respectively) which is not contained in any of the lines M_f. Hence the lines defined by the subschemes Z_j of length $\mathbf{m}_j + 1$ in Z are the spans of the origin and some P_e (or the spans of P and P_e, respectively). For each choice of lines $\overline{OP_e}$ to contain each Z_j there is a unique choice of lines $\overline{PP_e}$ and Z_j, so fix once and for all the ordering so that Z_j is contained in the line $\overline{OP_j}$ or $\overline{PP_j}$, respectively. We can assume from here on out that $\mathbf{m}$ has no zero entries, since $P_j \in Z_j$ and if $\mathbf{m}_j = 0$ then $\ell(Z_j) \geq \mathbf{m}_j + 1$ is satisfied.

Now we restrict ourselves to the study of $Z \in \Theta_{L,i}^{\mathbf{m}} \cap T$ for the moment. Notice that the subscheme Z_i of Z must be contained in L. Each entry $\mathbf{n}_k$ is the number of entries in $\mathbf{m}^2$ which are less than or equal to k. For $i \neq j \leq k$, the subschemes Z_j must contain the intersection points $M_k \cap \overline{OP_j}$ and for $k \leq \mathbf{m}_i + 1$, the subscheme Z_i must contain $M_k \cap L$. This determines Z uniquely, and the intersection is one.

For $Z \in \Theta_{L,i}^{P,\mathbf{m}} \cap T$, Z_i need not be contained in L. There are $\mathbf{m}_i + 1$ choices to consider corresponding to Z_i containing the intersection of L with M_f for $f \leq \mathbf{m}_i + 1$. Once this choice is made, the line defined by Z_i is determined, and similar reasoning to the above shows that there is a unique scheme Z in the intersection.

This determines the coefficient a to be $\mathbf{m}_i + 1$. $\qquad\qquad\square$

Lemma 4.4.7. *The coefficient $c_{\mathbf{m}}^0$ in Corollary 4.3.6 is 1.*

Proof. In this case, $\lambda = \mathbf{m}$, the class we will pair both sides with is $\sigma_{(\mathbf{a},\mathbf{m}',q)}$ where $\mathbf{m}'$ here is the conjugate partition to $\mathbf{m}$. Let U be a general representative for this class, and let R be the fixed point defining U such that each scheme in U must have a subscheme of length q collinear with R. We will show that the intersection of both $\Phi_{Q,q}^{P,\mathbf{m}}$ and $\phi_q^{\mathbf{m}}$ with U consist of the same number of points.

Let Z be an intersection point of U with either $\Phi_{Q,q}^{P,\mathbf{m}}$ or $\phi_q^{\mathbf{m}}$. The r fixed points of U are general, so in particular the lines any pair of them span do not contain the origin or P. As such, each must reside in a single subscheme Z_j of Z of length $\mathbf{m}_j$. That is to say, for all intersection points Z in either intersection, the lines L_j spanned by each Z_j must be the spans of the fixed points defining U with P or the origin, respectively. The subscheme Z_q of Z of length q collinear with R must be the unique length q subscheme of the plane supported at P or the origin, respectively, and Z_q must be contained in the line spanned by this point and

R. It follows that the remaining points of Z must be the intersections of the lines L_j with the fixed lines defining U. Since the lines L_j for either case have the same number of incidence conditions (the labels $\mathbf{m}_j$), and since U was chosen generally, the intersection number is the same for either $\Phi_{Q,q}^{P,\mathbf{m}}$ or $\phi_q^{\mathbf{m}}$. $\hfill\square$

4.4.3.2   Determining the Coefficients in a Degeneration

The process to determine the coefficients c_j^λ is then as follows.

1. Form the collection of classes $\tau_j^{\lambda'}$.

2. Choose general representatives for each (as defined in section 2.4).

3. Compute the intersection with both sides of the equivalences in Corollary 4.3.3 or Corollary 4.3.6. This is simply a combinatorial count of collections of distinct points which satisfy the incidence conditions on all the lines, as well as multipliers for ordering the lines. For each λ, we obtain a linear equation in the c_j^λ's in this way.

4. Solve the resulting system of linear equations. Lemma 4.4.5 establishes that the linear equations obtained in the above step are independent.

Finally, we have a conjecture about the coefficients in the degenerations.

Conjecture 4.4.8. *The multiplicity of each component ϕ_{q+i}^λ in the second degeneration is the number of ways to choose i entries of λ such that adding one to each results in the original partition $\mathbf{m}$.*

APPENDIX

SOME COMPUTATIONS AND INTERSECTION NUMBERS

A.1 The Matrices for $\mathbb{P}^{2[3]}$

We refer to the elements of the MS basis for $A(\mathbb{P}^{2[3]})$ by their subfigure labels in Figure 2.

	2t	2u
2b	1	1
2c	1	2

TABLE V: The pairing matrix for $A^1(\mathbb{P}^{2[3]}) \times A_1(\mathbb{P}^{2[3]})$.

	2p	2s	2o	2q	2r
2d	1	0	0	0	0
2f	0	1	1	2	2
2g	0	0	1	1	2
2e	0	0	0	1	1
2h	0	0	0	0	1

TABLE VI: The pairing matrix for $A^2(\mathbb{P}^{2[3]}) \times A_2(\mathbb{P}^{2[3]})$.

APPENDIX (Continued)

	2i	2j	2k	2n	2l	2m
2i	6	3	1	0	0	0
2j	3	1	0	0	0	0
2k	1	0	0	0	0	0
2n	0	0	0	1	1	0
2l	0	0	0	1	1	1
2m	0	0	0	0	1	1

TABLE VII: The pairing matrix for $A^3(\mathbb{P}^{2[3]}) \times A_3(\mathbb{P}^{2[3]})$.

A.2 The Matrices for $\mathbb{P}^{2[5]}$

We list below the intersection matrices for $\mathbb{P}^{2[5]}$. We order the basis elements in ascending lexicographic order given as follows:

If $\varphi = (\varphi_1, \ldots, \varphi_r)$ and $\psi = (\psi_1, \ldots, \psi_s)$ are partitions of integers R_1 and R_2, respectively, then $\varphi < \psi$ if $R_1 < R_2$, or if $R_1 = R_2$, then $\varphi < \psi$ if for some $i \leq r$ we have $\varphi_j = \psi_j$ for $j < i$ and $\varphi_i < \psi_i$.

If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are mixed partitions of an integer N, then $\alpha < \beta$ if for some $i \leq 3$ we have $\alpha_j = \beta_j$ and $\alpha_i < \beta_i$.

In each matrix, the dimension d basis elements index the columns, while the codimension d elements index the rows.

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

TABLE VIII: The pairing matrix for $A^1(\mathbb{P}^{2[5]}) \times A_1(\mathbb{P}^{2[5]})$.

APPENDIX (Continued)

$$\begin{bmatrix} 6 & 3 & 0 & 4 & 3 & 1 \\ 3 & 0 & 0 & 2 & 1 & 0 \\ 6 & 3 & 0 & 5 & 4 & 2 \\ 2 & 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

TABLE IX: The pairing matrix for $A^2(\mathbb{P}^{2[5]}) \times A_2(\mathbb{P}^{2[5]})$.

$$\begin{bmatrix} 6 & 2 & 0 & 0 & 0 & 3 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 & 4 & 2 & 0 & 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 0 & 0 & 4 & 1 & 0 & 2 & 1 & 0 & 0 \\ 12 & 6 & 2 & 0 & 0 & 10 & 5 & 0 & 8 & 4 & 3 & 1 \\ 6 & 2 & 0 & 0 & 0 & 5 & 2 & 0 & 4 & 1 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 & 6 & 3 & 0 & 6 & 3 & 3 & 1 \\ 3 & 1 & 0 & 0 & 0 & 3 & 1 & 0 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

TABLE X: The pairing matrix for $A^3(\mathbb{P}^{2[5]}) \times A_3(\mathbb{P}^{2[5]})$.

APPENDIX (Continued)

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 12 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 18 & 9 & 3 & 0 & 0 & 0 & 6 & 3 & 0 & 0 & 1 & 0 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 9 & 4 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 12 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 24 & 12 & 4 & 0 & 0 & 0 & 12 & 6 & 0 & 0 & 4 & 1 \\
12 & 5 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 12 & 5 & 1 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 1 & 0 \\
6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
$$

TABLE XI: The pairing matrix for $A^4(\mathbb{P}^{2[5]}) \times A_4(\mathbb{P}^{2[5]})$.

APPENDIX (Continued)

120	60	30	20	10	5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60	27	12	7	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	12	5	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	7	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	3	1	3	1	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	1	3	6	3	6	3	1	0	0	2	4	4	2	0	1	1	0
0	0	0	0	0	0	0	0	1	3	1	3	1	0	0	0	1	2	2	1	0	0	0	0
0	0	0	0	0	0	0	1	3	6	3	6	3	1	0	0	3	6	6	3	0	3	3	1
0	0	0	0	0	0	0	0	1	3	1	3	1	0	0	0	1	3	3	1	0	1	1	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	2	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	2	1	3	1	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	4	2	6	3	1	0	0	1	3	4	2	0	1	1	0
0	0	0	0	0	0	0	0	1	4	2	6	3	1	0	0	1	4	6	3	0	2	3	1
0	0	0	0	0	0	0	0	0	2	1	3	1	0	0	0	0	2	3	1	0	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1	0	3	1	0	0	0	0	1	2	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	3	1	0	0	0	0	1	3	1	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

TABLE XII: The pairing matrix for $A^5(\mathbb{P}^{2[5]}) \times A_5(\mathbb{P}^{2[5]})$.

A.3   Some Python Code

The following code is an implementation of the algorithm found in Section 3.4. The function `compute_intersection_number` takes as input two triples of partitions (given as two lists of three lists each) which define classes σ and τ of complementary codimension and computes the intersection number between them. The function `compute_intersection_matrix` takes as input a number N and a codimension d and computes the entire intersection matrix between classes of dimension d and codimension d.

```python
#!/usr/bin/python

from itertools import *

####
## A Simple Tree Data Structure
##

```

APPENDIX (Continued)

```python
 9  class Tree():
10      def __init__(self, parent, horiz, vert):
11          self.parent = parent
12          self.children = []
13          self.horiz = horiz
14          self.vert = vert
15          self.numLeaf = 0
16
17  # generates mixed partitions for n
18  def mixed_partition(n):
19      partition = [0 for i in range(n+1)]
20      k = 1
21      y = n - 1
22      while k != 0:
23          x = partition[k - 1] + 1
24          k -= 1
25          while 2*x <= y:
26              partition[k] = x
27              y -= x
28              k += 1
29          l = k + 1
30          while x <= y:
31              partition[k] = x
32              partition[l] = y
33              if k + 2 <= 3:
34                  output = list(reversed(partition[:k+2]))
35                  output = output + [0] * (1-k)
36                  yield output
37              x += 1
38              y -= 1
39          partition[k] = x + y
40          y = x + y - 1
41          if k+1 <= 3:
42              output = list(reversed(partition[:k+1]))
43              output = output + [0] * (2-k)
44              yield output
45
46  # Code from http://jeromekelleher.net/partitions.php cite later
47  # modified above to only emit partitions of length m if specified
48  def partition(n, m = None):
49      partition = [0 for i in range(n+1)]
50      k = 1
51      y = n - 1
52      while k != 0:
53          x = partition[k - 1] + 1
```

```
54        k -= 1
55        while 2*x <= y:
56            partition[k] = x
57            y -= x
58            k += 1
59        l = k + 1
60        while x <= y:
61            partition[k] = x
62            partition[l] = y
63            if not m or (k + 2) == m:
64                yield list(reversed(partition[:k + 2]))
65            x += 1
66            y -= 1
67        partition[k] = x + y
68        y = x + y - 1
69        if not m or k+1 == m:
70            yield list(reversed(partition[:k + 1]))
71
72  # A diagram is a list containing exactly three members:
73  # 0: A nonincreasing sequence of positive integers representing incidence
        conditions on horizontal lines
74  # 1: A nonincreasing sequence of positive integers representing incidence
        conditions on vertical lines
75  # 2: a set of (i,j) pairs corresponding to when the ith horizontal line must meet
        the jth vertical line
76  # it's assumed that the sum of the entries in the vertical list is equal the sum of
         the entries in the horizontal list
77  def generate_tree( diagram ):
78      deg_tuple1 = diagram[1]
79      deg_tuple2 = diagram[0]
80      point_set = diagram[2]
81
82      t = Tree( None, deg_tuple1, deg_tuple2 )
83      generate_tree_worker( t, point_set, 0 )
84      return t
85
86  def generate_tree_worker( t, point_set, height ):
87
88      if all( i == 0 for i in t.horiz):
89          if all( degree == 0 for degree in t.vert ):
90              return 1
91          else:
92              return 0
93
94      index_list = [i for i, e in enumerate(t.vert) if e != 0]
```

APPENDIX (Continued)

```
 95    line = next((i for i,e in enumerate(t.horiz) if e != 0), None)
 96    required_points = {(x,y) for (x,y) in point_set if x == line}
 97    choices = t.horiz[line]
 98    if choices > len( index_list ):
 99        return 0
100
101    number_viable = 0
102    child_deg_tuple_zero = list(t.horiz)
103    child_deg_tuple_zero[line] = 0
104    subsequences = combinations(index_list, choices)
105
106    for subsequence in subsequences:
107        if any(y not in subsequence for (x,y) in required_points):
108            continue
109
110        new_list = list( t.vert )
111        for i in subsequence:
112            new_list[i] -= 1
113
114        child = Tree(t, child_deg_tuple_zero, new_list)
115        t.children.append(child)
116        number_viable += generate_tree_worker(child, point_set, height+1)
117
118    t.numLeaf = number_viable;
119    return number_viable
120 # this function takes as input two mixed partitions and returns all the diagrams to
        compute the interseciton number for those corresponding basis elements
121 def generate_diagrams(sigma, tau):
122    sigma_fixed = sigma[0] + sigma[1]
123    tau_fixed = tau[0] + tau[1]
124
125    list_of_diagrams = []
126
127    point_set_sigma = set( (len(sigma_fixed)+i,i) for i in range(0,len(tau[0])) )
128    point_set_tau = set( (i,len(tau_fixed)+i) for i in range(0, len(sigma[0])) )
129
130    point_set = point_set_sigma | point_set_tau
131
132    all_permutations_sigma = {x for x in permutations(sigma[2])}
133    all_permutations_tau = {x for x in permutations(tau[2])}
134
135    for c_tau in all_permutations_tau:
136        for c_sigma in all_permutations_sigma:
137            final_list_sigma = sigma_fixed + list(c_sigma)
138            final_list_tau = tau_fixed + list(c_tau)
```

APPENDIX (Continued)

```
139              list_of_diagrams.append([final_list_tau,final_list_sigma,point_set])
140
141        return list_of_diagrams
142  # this takes as input two mixed partitions and computes the intersection number. it
          is assumed that the length of the first element of the first mixed partition
          is the length of the third element of the second, and vice versa
143  def compute_intersection_number( mixed_partition_1, mixed_partition_2 ):
144        list_of_diagrams = generate_diagrams( mixed_partition_1, mixed_partition_2 )
145        number = 0
146        for diagram in list_of_diagrams:
147            t = generate_tree( diagram )
148            number += t.numLeaf
149        return number
150
151  def print_diagram(diagram):
152        horiz_labels = [str(x) for x in diagram[0]]
153        vert_labels = [str(x) for x in diagram[1]]
154        point_set = diagram[2]
155        col_size = max([len(x) for x in horiz_labels] + [len(x) for x in vert_labels])
156        print(" ".ljust(col_size), " ".join([x.ljust(col_size) for x in vert_labels]))
157        for i in range(0,len(horiz_labels)):
158            string = ""
159            hold = [x for (x,y) in point_set if y == i]
160            if hold:
161                if hold[0] == 0:
162                    string = "*"
163                else:
164                    string = "*".rjust(hold[0]*(col_size+1)  + 1)
165            print (horiz_labels[i].ljust(col_size), string)
166
167  def print_diagrams( list_of_diagrams ):
168        for diagram in list_of_diagrams:
169            print_diagram( diagram )
170
171  def print_dfs(tree):
172        discovered = []
173        stack = []
174        stack.append((tree,0))
175        while stack:
176            nodepair = stack.pop()
177            node = nodepair[0]
178            depth = nodepair[1]
179            if node not in discovered:
180                print (str(depth) + ":: " + str(node.horiz) + "::" + str(node.vert) + "::"
                        + str(node.numLeaf))
```

APPENDIX (Continued)

```
181              discovered.append(node)
182              for child in node.children:
183                  stack.append((child,depth + 1))
184
185  def generate_basis(N, codim):
186      list_of_basis = []
187      fixed_range = codim // 2 + 1
188      for mixed in mixed_partition(N):
189          perms = {x for x in permutations(mixed)}
190          for perm in perms:
191              for fixed in range(0, fixed_range):
192                  moving = fixed + N - codim
193                  on_lines = N - moving - fixed
194                  if perm[0] < fixed or perm[2] < moving or (not fixed and perm[0]) or (
195                      not moving and perm[2]):
196                      continue
196                  for first in partition(perm[0], fixed):
197                      if len(first) == 1 and first[0] == 0:
198                          first = []
199                      for second in partition(perm[1]):
200                          if len(second) == 1 and second[0] == 0:
201                              second = []
202                          for third in partition(perm[2], moving):
203                              if len(third) == 1 and third[0] == 0:
204                                  third = []
205                              list_of_basis.append([first, second, third])
206      return list_of_basis
207
208  def order_basis(list_of_basis):
209      print(list_of_basis.sort())
210  def key_fcn(basis):
211      new_base = []
212      for entry in basis:
213          a = sum(entry)
214          new_base.append([a] + entry)
215      return new_base
216
217
218  def compute_intersection_matrix(N, d):
219      basis = generate_basis(N,d)
220      opposite_basis = [list(reversed(x)) for x in basis]
221
222      basis = sorted(basis, key=key_fcn)
223      opposite_basis = sorted(opposite_basis, key=key_fcn)
224
```

APPENDIX (Continued)

```
225    print(" ——— Basis Codim", d ,"(rows) ——— ")
226    for (i,b) in enumerate(basis):
227        print(i, ":: ", b, sep = "")
228    print(" ——— Basis Codim", 2*N—d ,"(cols) ——— ")
229    for (i,b) in enumerate(opposite_basis):
230        print(i, ":: ", b, sep="")
231
232    matrix = []
233
234    for b in basis:
235        row = []
236        for o in opposite_basis:
237            if len(b[0]) == len(o[2]) and len(b[2]) == len(o[0]):
238                row.append(compute_intersection_number(b,o))
239            else:
240                row.append(0)
241        matrix.append(row)
242
243    return matrix
244
245 matrix = compute_intersection_matrix(5,4)
246
247 for row in matrix:
248    print(row)
```

A.4   Some triple intersection products in $\mathbb{P}^{2[6]}$

We include here some of early computations and intersection numbers computed in the investigation of the Main Theorem 2.

APPENDIX (Continued)

	$H \cdot \sigma_{(0,0,(3,2,1))}$	$\Theta_{L,1}^{(2,1,0),P}$	$\Theta_{L,2}^{(2,1,0),P}$	$\Theta_{L,3}^{(2,1,0),P}$	$\phi_1^{(1,1,0)}$	$\phi_1^{(2,0,0)}$	$\phi_2^{(1,0,0)}$	$\phi_3^{(0,0,0)}$
$((1,1),(1,1,1,1),0)$	72	24	24	24	0	0	0	0
$((1,1),(2,1,1),0)$	24	6	8	10	0	0	0	0
$((1,1),(2,2),0)$	8	0	4	4	0	0	0	0
$((1,1),(3,1),0)$	2	0	0	2	0	0	0	0
$((1,1),(4),0)$	0	0	0	0	0	0	0	0
$((1,1,1),0,(3))$	0	0	0	0	0	0	0	1
$((1,1,1),(1),(2))$	12	6	6	0	0	0	3	0
$((1,1,1),(1,1),(1))$	18	12	6	0	6	3	0	0
$((1,1,1),(2),(1))$	6	6	0	0	3	0	0	0
$((2,1),(1,1,1),0)$	54	21	18	15	0	0	0	0
$((2,1),(2,1),0)$	18	6	6	6	0	0	0	0
$((2,1),(3),0)$	1	0	0	1	0	0	0	0
$((1,1,1,1),0,(1,1))$	0	0	0	0	0	0	0	0
$((2,1,1),0,(2))$	8	4	4	0	0	0	2	0
$((2,1,1),(1),(1))$	12	8	4	0	4	2	0	0
$((2,2),(1,1),0)$	40	18	12	10	0	0	0	0
$((2,2),(2),0)$	14	6	4	4	0	0	0	0
$((3,1),(1,1),0)$	10	3	4	3	0	0	0	0
$((3,1),(2),0)$	3	0	2	1	0	0	0	0
$((2,2,1),0,(1))$	8	6	2	0	3	1	0	0
$((3,1,1),0,(1))$	2	2	0	0	1	0	0	0
$((3,2),(1),0)$	7	3	2	2	0	0	0	0
$((4,1),(1),0)$	0	0	0	0	0	0	0	0
$((3,3),0,0)$	0	0	0	0	0	0	0	0
$((4,2),0,0)$	0	0	0	0	0	0	0	0
$((5,1),0,0)$	0	0	0	0	0	0	0	0

TABLE XIII: The complete table of intersections of the loci in the degenerations to resolve the class in the MS basis of the intersection $H \cdot \sigma_{(0,0,(3,2,1))}$.

APPENDIX (Continued)

Figure 36: A scanned in set of notes depicting calculating the intersection product in $\mathbb{P}^{2[3]}$.

APPENDIX (Continued)

Figure 37: A scanned in set of notes depicting calculations of the classes of intersections $H \cdot \sigma$ for many different MS basis elements σ. The triple intersection numbers with the basis elements of the correct codimension are listed in each box with the elements ordered lexicographically by their triples of partitions.

CITED LITERATURE

1. Grothendieck, A.: Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In Séminaire Bourbaki, Vol. 6, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

2. Iarrobino, A.: Reducibility of the families of 0-dimensional schemes on a variety. Invent. Math., 15:72–77, 1972.

3. Fogarty, J.: Algebraic families on an algebraic surface. Amer. J. Math, 90:511–521, 1968.

4. Fogarty, J.: Algebraic families on an algebraic surface. II. The Picard scheme of the punctual Hilbert scheme. Amer. J. Math., 95:660–687, 1973.

5. Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., and Popa, M.: Asymptotic invariants of line bundles. Pure Appl. Math. Q., 1(2, Special Issue: In memory of Armand Borel. Part 1):379–403, 2005.

6. Gholampour, A. and Sheshmani, A.: Donaldson-Thomas Invariants of 2-Dimensional sheaves inside threefolds and modular forms. ArXiv e-prints, April 2015.

7. Gholampour, A. and Sheshmani, A.: Intersection numbers on the relative Hilbert schemes of points on surfaces. ArXiv e-prints, April 2015.

8. Costello, K. and Grojnowski, I.: Hilbert schemes, Hecke algebras and the Calogero-Sutherland system. ArXiv Mathematics e-prints, October 2003.

9. Elencwajg, G. and Le Barz, P.: Une base de $\text{Pic}(\text{Hilb}^k \mathbb{P}^2)$. C. R. Acad. Sci. Paris Sér. I Math., 297(3):175–178, 1983.

10. Elencwajg, G. and Le Barz, P.: Explicit computations in $\text{Hilb}^3\mathbb{P}^2$. In Algebraic geometry (Sundance, UT, 1986), volume 1311 of Lecture Notes in Math., pages 76–100. Springer, Berlin, 1988.

11. Elencwajg, G. and Le Barz, P.: Détermination de l'anneau de Chow de $\text{Hilb}^3\mathbb{P}^2$. C. R. Acad. Sci. Paris Sér. I Math., 301(12):635–638, 1985.

12. Ellingsrud, G. and Strømme, S. A.: On a cell decomposition of the Hilbert scheme of points in the plane. Invent. Math., 91(2):365–370, 1988.

13. Fantechi, B. and Göttsche, L.: The cohomology ring of the Hilbert scheme of 3 points on a smooth projective variety. J. Reine Angew. Math., 439:147–158, 1993.

14. Nakajima, H.: Heisenberg algebra and Hilbert schemes of points on projective surfaces. Ann. of Math. (2), 145(2):379–388, 1997.

15. Nakajima, H.: Lectures on Hilbert schemes of points on surfaces, volume 18 of University Lecture Series. American Mathematical Society, Providence, RI, 1999.

16. Vafa, C. and Witten, E.: A strong coupling test of S-duality. Nuclear Phys. B, 431(1-2):3–77, 1994.

17. Grojnowski, I.: Instantons and affine algebras. I. The Hilbert scheme and vertex operators. Math. Res. Lett., 3(2):275–291, 1996.

18. Lehn, M.: Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. Invent. Math., 136(1):157–207, 1999.

19. Marian, A., Oprea, D., and Pandharipande, R.: Segre classes, and Hilbert schemes of points. ArXiv e-prints, April 2016.

20. Eisenbud, D. and Harris, J.: 3264 & All That: A Second Course in Algebraic Geometry. Cambridge University Press, Cambridge, 2016.

21. Fulton, W.: Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

22. Dieudonné, J. and Grothendieck, A.: Éléments de géométrie algébrique. Inst. Hautes Études Sci. Publ. Math., 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

23. Vakil, R.: Murphy's law in algebraic geometry: badly-behaved deformation spaces. Invent. Math., 164(3):569–590, 2006.

24. Cartwright, D. A., Erman, D., Velasco, M., and Viray, B.: Hilbert schemes of 8 points. Algebra Number Theory, 3(7):763–795, 2009.

25. Erman, D.: Murphy's law for Hilbert function strata in the Hilbert scheme of points. Math. Res. Lett., 19(6):1277–1281, 2012.

26. Ellingsrud, G. and Strømme, S. A.: On the homology of the Hilbert scheme of points in the plane. Invent. Math., 87(2):343–352, 1987.

27. Göttsche, L.: The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann., 286(1-3):193–207, 1990.

28. Ellingsrud, G. and Göttsche, L.: Hilbert schemes of points and Heisenberg algebras. In School on Algebraic Geometry (Trieste, 1999), volume 1 of ICTP Lect. Notes, pages 59–100. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

29. Ellingsrud, G. and Strømme, S. A.: An intersection number for the punctual Hilbert scheme of a surface. Trans. Amer. Math. Soc., 350(6):2547–2552, 1998.

30. Mallavibarrena, R. and Sols, I.: Bases for the homology groups of the Hilbert scheme of points in the plane. Compositio Math., 74(2):169–201, 1990.

31. Briançon, J.: Description de $Hilb^n C\{x, y\}$. Invent. Math., 41(1):45–89, 1977.

32. Grothendieck, A.: Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]. Secrétariat mathématique, Paris, 1962.

33. Haiman, M.: Hilbert schemes, polygraphs and the Macdonald positivity conjecture. J. Amer. Math. Soc., 14(4):941–1006, 2001.

34. Hartshorne, R.: Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

35. Hermoso, C. and Sols, I.: Bases of the homology spaces of the Hilbert scheme of points in an algebraic surface. Rev. Mat. Univ. Complut. Madrid, 9(1):53–66, 1996.

36. Stathis, A. J.: An algorithm for intersections in $\mathbb{P}^{2[N]}$. Communications in Algebra, 45(9):3778–3791, 2017.

37. Stathis, A.: The action of $\mathcal{O}(1)$ on $\mathbb{P}^{2[N]}$. In Preparation, 2017-18.

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