MIDTERM SOLUTIONS

ALEXANDER J STATHIS

1. Compute

$$\begin{vmatrix} 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 4 & 0 & -1 \end{vmatrix}.$$

Solution. We compute the determinant by expanding along the top row. If we call the matrix in the problem statement A, then

$$\det A = 0 - \begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & -5 \\ 0 & 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 & 0 \\ 1 & 0 & -5 \\ 0 & 4 & -1 \end{vmatrix} + 0.$$

Now expand both of the 3×3-matrices in the previous line along the top row to obtain

$$\det A = -\left(3 \begin{vmatrix} 0 & -5 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -5 \\ 0 & -1 \end{vmatrix}\right) + 2\left(3 \begin{vmatrix} 0 & -5 \\ 4 & -1 \end{vmatrix}\right) = \begin{vmatrix} 1 & -5 \\ 0 & -1 \end{vmatrix} + 6 \begin{vmatrix} 0 & -5 \\ 4 & -1 \end{vmatrix} = -1 + 6(20) = 119.$$

2. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 5 & 3 \\ 0 & -2 & 2 \end{pmatrix}$$

and $B = A^T$.

a Find B.
b Find the LU-factorization of A.
c Find $\det A$, $\det B$, and $\det AB$.
d Find A^{-1}.

Solution.

a B is the matrix

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & -2 \\ 1 & 3 & 2 \end{pmatrix}.$$

b Row reduce A to U by subtracting thrice the first row from the second and then subtracting twice the second row from the third. This gives the matrix

$$E_2 E_1 A = U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and the corresponding elementary operations are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Now

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

so

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

c We start by computing $\det A$ by expanding along the top row:

$$\det A = 1 \begin{vmatrix} 5 & 3 \\ -2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 3 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} = 16 - 12 - 6 = -2.$$

Now, $\det B = \det A = -2$, so that $\det AB = (\det A)(\det B) = (\det A)^2 = 4$.

d Augment A with the identity to get the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right).$$

Now perform these operations in this order: subtract thrice the first row from the second, subtract twice the second row from the third, scale the second row by -1, scale the third row by one half, subtract twice the second row from the first, and subtract the third row from the first. This leaves the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 3 & -\frac{1}{2} \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 & \frac{1}{2} \end{array}\right),$$

where the right hand side is the inverse of A.

3. Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 - x_5 &= 3; \\ -3x_1 + 6x_2 - 3x_3 - 3x_5 &= 6; \\ x_2 - x_4 + x_5 &= 2. \end{aligned}$$

Solution. Begin by creating the augment matrix

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & -1 & 3 \\ -3 & 6 & -3 & 0 & -3 & 6 \\ 0 & 1 & 0 & -1 & 1 & 2 \end{array}\right).$$

Now perform these operations in this order: scale the second row by one third, subtract the first row from the second, swap the second and third rows, add the third row to the second, and add twice the second row to the first. The resulting matrix should be

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right),$$

which corresponds to the system

$$\begin{aligned} x_1 + x_3 - x_5 &= 10; \\ x_2 - x_5 &= 5; \\ x_4 - 2x_5 &= 3. \end{aligned}$$

The third and fifth variables are free, and the solution set is

$$S = \left\{ v \in \mathbb{R}^5 : v = (10 + b - a, b + 5, a, 3 + 2b, b) \right\}.$$

4. Let

$$C = \begin{pmatrix} (5-d) & 1 \\ 3 & (7-d) \end{pmatrix}.$$

a For what values of d (if any) does the matrix C have an inverse?
b For what values of d (if any) will the system $Cx = 0$ be inconsistent?
c How many solutions will the system $Cx = 0$ have if $d = 4$?
d How many solutions will the system $Cx = 0$ have if $d = -4$?

Solution.

a The matrix C is invertible if and only if its determinant is nonzero.

$$\det C = (5-d)(7-d) - 3 = d^2 - 12d + 32 = (d-8)(d-4),$$

so $\det C \neq 0$ when $d \neq 4, 8$.
b This is easier to see geometrically. Augmenting C with zero is equivalent to taking the rows of C to be lines through the origin. Since two lines through the origin always intersect, $Cx = 0$ is never inconsistent.
c If $d = 4$, then the lines through the origin given by each row will be the same line, and there will be an infinite number of solutions.
d If $d = -4$, then the lines will differ and they will share only a single solution, the origin.

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