October 15

(1) If $w = f(x, y, z)$ is a differentiable function, prove that $\nabla_{(x_0,y_0,z_0)} f$ is orthogonal to the level surface of f passing through the point (x_0, y_0, z_0).

See proof on page 640

(2) Find the points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where the tangent plane is parallel to the plane $dx + ey + fz = 0$.

See lecture notes, quiz 6, etc...

(3) Find three numbers x, y, and z such that $x^a y^b z^c$ is maximum where $x + y + z = 100$.

Solution: We need to maximize the function $f(x, y, z) = x^a y^b z^c$ under the given restraint. Reduce f to a function of two variables by substituting $z = 100 - x - y$ to obtain

$$g(x, y) = x^a y^b (100 - x - y)^c.$$

Calculate the gradient of g:

$$\nabla g = < ax^{a-1} y^b (100 - x - y)^c - cx^a y^b (100 - x - y)^{c-1},$$

$$bx^a y^{b-1} (100 - x - y)^c - cx^a y^b (100 - x - y)^{c-1} > .$$

To find x and y so that $\nabla g = \mathbf{0}$, notice that we aren't interested in the solutions where x, y, or z are zero, so we can freely divide by any of these quantities. First set the x partial equal to zero:

$$ax^{a-1} y^b (100 - x - y)^c - cx^a y^b (100 - x - y)^{c-1} = 0,$$

and divide through by $x^{a-1} y^b (100 - x - y)^{c-1}$:

$$a(100 - x - y) - cx = 0,$$

giving

$$x = \frac{a(100 - y)}{a + c}.$$

Setting the y partial equal to zero and simplifying, we get

$$b(100 - x - y) - cy = 0.$$

Solving for y we have

$$y = \frac{b(100 - x)}{b + c}.$$

Substituting in the value of x that we found above, and simplifying, we get

$$y = \frac{100b(a+c) - a(100-y)}{(a+c)(b+c)},$$

so solving for y, we have

$$y = \frac{100(b(a+c)) - 100a}{(a+c)(b+c) - 1}.$$

To get the values for x and z, plug this value into the previously obtained equations.