Discussion problems for November 14

Complete 1 through 4 for the vector fields $\mathbf{F}(x, y) =< 1, y >$ and $\mathbf{F}(x, y) =< 1, x >$.

(1) Sketch $\mathbf{F}$.
(2) Sketch some flow lines of $\mathbf{F}$.
(3) Find the equation of a flow line passing through the point (x_0, y_0).
(4) (a) Is there a flow line passing through every point of the plane?
 (b) How many flow lines pass through each point of the plane?

Partial Solution I'll show how to do part 3 for the first vector field. By the definition I gave in class, a function $\mathbf{r}(t) =< x(t), y(t) >$ describes a flow line at time t if $\mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t))$ in a neighborhood of t. Therefore a flow line corresponds, in the first case, to a solution to the system of differential equations

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = y.$$

To solve this system, separate variables and integrate: $dx = dt$, so $x = t + C$ for some constant C. For the second equation, we have $\frac{dy}{y} = dt$, so $\ln |y| = t + D$ for some constant D. Substituting we have

$$\ln |y| = x - C + D.$$

Setting the constant $-C + D = \ln K$, we have

$$|y| = e^{x+\ln K} = Ke^x.$$

The constant K can be any positive real number, so we have flow lines

$$y = \pm Ke^x.$$

It is easily checked that the function $\mathbf{r}(t) =< x, 0 >$ also satisfies the system of differential equations, so is also a flow line. Therefore $y = Ae^x$ is a flow line for any real number A. The flow line passes through the point (x_0, y_0) when $A = y_0/e^{x_0}$.