Hybridization and postprocessing in finite element exterior calculus

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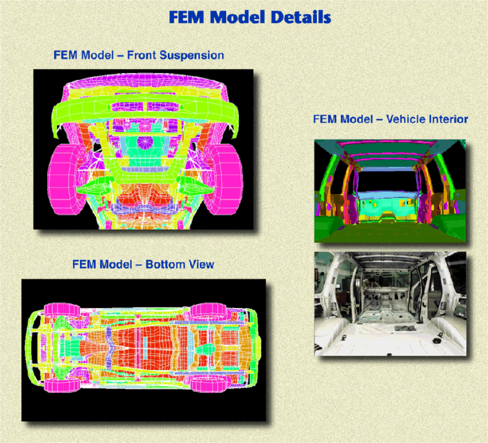
Applications of finite element



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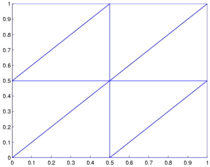
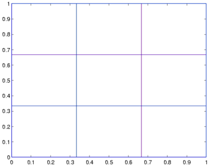
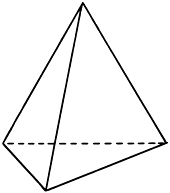
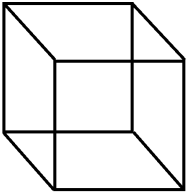
Examples









Model problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{cases}$$

where $\partial\Omega$ will denote the boundary of the bounded domain Ω and Δ denotes the Laplace operator, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Green's identity

$$\int_\Omega (-\operatorname{div} \nabla u) v \; dx = \int_\Omega \nabla u \cdot \nabla v \; dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$$

Take $g = 0$. Find u in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega)$$

Find $u \in V$ such that

$$a(u, v) = F(v), \forall v \in V$$

$V_h \subset V$ conforming finite element (e.g. $V_h \subset H^1(\Omega)$)
Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h), \forall v_h \in V_h$$

A piecewise polynomial is in $H^1(\Omega)$ if it is globally continuous.
On simplices : Lagrange elements, $\mathcal{P}_r$ element
Space of piecewise continuous polynomials of degree r

Finite element spaces

- K is a closed subset of $\mathbb{R}^n$ with a nonempty interior and a Lipshitz continuous boundary
- P_K is a finite dimensional space of vector valued or matrix valued functions defined over the set K
- Θ_K is a finite set of linearly independent linear functionals, $\theta_i, i = 1, \ldots, N$ referred to as degrees of freedom of the finite element, defined over the set P_K.

It is assumed that the set Θ_K is P_K-unisolvent in the sense that

$$\theta_i(p) = 0, i = 1, \ldots, N \Longrightarrow p \equiv 0$$

Mixed finite elements

Main motivation of mixed methods for the Poisson equation $-\Delta u = f \,\mathrm{in}\, \Omega,\ u = g \,\mathrm{on}\, \partial\Omega$: The quantity $\sigma = -\nabla u$ is the one of primary importance.

$$\begin{aligned} \sigma &= -\nabla u \\ \operatorname{div} \sigma &= f \\ u &= g \,\mathrm{on}\, \partial\Omega \end{aligned}$$

Need for $H(\mathrm{div})$ elements : normal component is continuous

Abstract Weak Formulation

$$a(\sigma,\tau) = \int_\Omega \sigma \cdot \tau \qquad b(\sigma,u) = -\int_\Omega u \operatorname{div} \sigma$$

Find $\sigma \in \Sigma = H(\text{div}, \Omega)$, and $u \in V = L^2(\Omega, \mathbb{R})$ such that

$$\begin{cases} a(\sigma,\tau) + b(\tau,u) & = \langle g, \tau \cdot n \rangle \quad \forall \tau \in \Sigma \\ b(\sigma, v) & = (-f, v) \quad \forall v \in V \end{cases}$$

for all $\tau \in \Sigma$ and $v \in V$.

Can we use Lagrange elements to approximate both the scalar variable u and the vector σ?

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Brezzi's stability conditions

$\Sigma_h \subset \Sigma$ and $V_h \subset V$

Sufficient conditions for optimal error bounds

First Brezzi condition $\exists \alpha > 0$ independent of h such that

$$a(\tau, \tau) \geq \alpha \|\tau\|_\Sigma^2$$

for all τ in K_h where

$$K_h = \{\tau \in \Sigma_h : b(\tau, v) = 0, \forall v \in V_h\}$$

Second Brezzi condition $\exists \beta > 0$ independent of h such that

$$\sup_{\tau \in \Sigma_h} \frac{b(\tau, v)}{\|\tau\|_\Sigma} \geq \beta \|v\|_V \quad \forall v \in V_h$$

$$\|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_V \leq \gamma \{\inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_\Sigma + \inf_{v_h \in V_h} \|u - v_h\|_V\}$$

with γ independent of h.

Commutative Diagram

Sufficient conditions for stability

- div $\Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega, \mathbb{R}^n) \to \Sigma_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that with $P_h : L^2(\Omega, \mathbb{R}) \to V_h$ denoting the L^2-projection

$$\begin{array}{ccc} H(\mathrm{div}, \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \downarrow \pi_h & & \downarrow P_h \\ \Sigma_h & \xrightarrow{\mathrm{div}} & V_h \end{array}$$

Model electromagnetic problem Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Given $f \in L^2(\Omega, \mathbb{R}^3)$, with $\operatorname{div} f = 0$, and a positive number ω, the time harmonic Maxwell equation consists in finding a vector field u such that

$$\begin{aligned} \operatorname{curl} \operatorname{curl} u - \omega^2 u &= f \text{ in } \Omega \\ \operatorname{div} u &= 0 \text{ in } \Omega \\ u \times \widehat{n} &= 0 \text{ on } \Omega, \end{aligned}$$

where $\widehat{n}$ is the unit outward normal to $\partial\Omega$.

For $\omega = 0$, we have a prototype of Maxwell's equation. Moreover, enforcing the divergence free condition with a Lagrange multiplier, we obtain the weak formulation : find $u \in H_0(\operatorname{curl}; \Omega)$ and $\sigma \in H_0^1(\Omega)$ such that

$$\begin{aligned} (\operatorname{curl} u, \operatorname{curl} v) + (\operatorname{grad} \sigma, v) &= (f, v) \ \forall v \in H_0(\operatorname{curl}; \Omega) \\ (\operatorname{grad} \sigma, \tau) &= 0, \ \forall \tau \in H_0^1(\Omega). \end{aligned}$$

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$H(\mathrm{curl}, \Omega) = \{E \in L^2(\Omega)^3, \mathrm{curl}\, E \in L^2(\Omega)^3\}$

$H(\mathrm{curl})$ elements : tangential component is continuous

The L^2 de Rham complex

$$\mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0,$$

has a discrete version on simplices

$$\mathbb{R} \xrightarrow{\subset} R_h \xrightarrow{\mathrm{grad}} N_h \xrightarrow{\mathrm{curl}} \Sigma_h \xrightarrow{\mathrm{div}} V_h \to 0.$$

R_h Lagrange elements $\mathcal{P}_r(K, \mathbb{R})$
$N_h : p + \mathbf{x} \times \mathbf{v}, p \in \mathcal{P}_{r-1}(K, \mathbb{R}^3), \mathbf{v} \in \mathcal{H}_{r-1}(K, \mathbb{R}^3)$
$\Sigma_h : p + w\mathbf{x}, p \in \mathcal{P}_{r-2}(K, \mathbb{R}^3), w \in \mathcal{H}_{r-2}(K, \mathbb{R})$

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Hodge Laplacian

Recall the vector Laplacian : for $u = (u_1, u_2, u_3)$,
$(-\Delta u_1, -\Delta u_2, -\Delta u_3) = \operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u$.

There is a closed relation between the Maxwell equations and the vector Laplacian :

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \text{ in } \Omega$$

with boundary conditions

$$\operatorname{div} u = 0, u \times \widehat{n} = 0, \text{ on } \partial\Omega.$$

With $\sigma = \operatorname{div} u$, the weak formulation of the vector Laplacian with electric boundary conditions is : find $u \in H_0(\operatorname{curl}; \Omega)$ and $\sigma \in H_0^1(\Omega)$ such that

$$(\operatorname{curl} u, \operatorname{curl} v) + (\operatorname{grad}\sigma, v) = (f, v)\ \forall v \in H_0(\operatorname{curl}; \Omega)$$

$$(\sigma, \tau) - (\operatorname{grad}\sigma, \tau) = 0,\ \forall \tau \in H_0^1(\Omega).$$

Another set of boundary conditions for the vector Laplacian for which one obtains optimal convergence rates for mixed approximations is :

$$u \cdot \widehat{n} = 0, (\operatorname{curl} u) \times \widehat{n} = 0, \text{ on } \partial\Omega. \quad \textit{magnetic b.c.}$$

As for the (Laplace) equation $-\Delta u = f$, both Dirichlet and Neumann boundary conditions can be imposed with optimal convergence rates for mixed approximations.

It is now known that all these problems have a common structure, referred to as finite element exterior calculus. The point of view is to consider unknowns as differential forms with vector proxies.

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Possible disadvantages of mixed finite element methods.
They give rise to indefinite systems

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} F_h \\ G_h \end{bmatrix}.$$

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For a conforming finite element approximation, one may want to use hybridization. That is, discretize the equation on each element and then enforce the continuity and boundary conditions using Lagrange multipliers on the element boundaries. This approach has several advantages :

- The solution of the hybridized system corresponds to the restriction of the non hybridized problem on each element
- The Lagrange multipliers correspond to weak traces of the solution and its derivatives on element boundaries
- The hybridized system is reduced to one single equation for the Lagrange multipliers. This process is known as *static condensation* and the size of the system to be solved is significantly smaller than the one for the non hybridized system

- Using the Lagrange multipliers on element boundaries, one can recover not only the solution of the non hybridized discrete problem, but also solve other local problems whose solution is an improved approximation. This process is known as postprocessing.

Hybrid methods were known for the Maxwell equations when one uses discontinuous elements and hence nonconforming approximations. i.e, in the context of hybridizable Discontinuous Galerkin methods (HDG).

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Correspondances.

- Continuous Galerkin=Mixed formulation for the Poisson equation with Neumann boundary conditions - Hodge Laplace problem for 0-forms
- Mixed formulation of the vector Laplacian with magnetic boundary conditions - Hodge Laplace problem for 1-forms
- Mixed formulation of the vector Laplacian with electric boundary conditions - Hodge Laplace problem for 2-forms
- Mixed formulation of the Poisson equation with Dirichlet boundary conditions- Hodge Laplace problem for 3-forms

Hybrid methods for the continuous Galerkin method have been discovered by Cockburn, Gopalakrishnan, and Wang while hybrid methods for the mixed formulation of the Poisson equation with Dirichlet boundary conditions were known much earlier by Arnold and Brezzi.

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The L^2 de Rham complex

$$\mathbb{R} \to L^2(\Omega;\mathbb{R}) \xrightarrow{\mathrm{grad}} L^2(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{curl}} L^2(\Omega;\mathbb{R}^3) \xrightarrow{\mathrm{div}} L^2(\Omega;\mathbb{R}) \to 0,$$

will be referred to as the base complex. For the domain complex, we may take

$$\mathbb{R} \to H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0,$$

Recall that the main tool for the derivation of finite element methods is an integration by parts formula, e.g. for $v, \phi \in H(\mathrm{curl}; \Omega)$

$$\int_\Omega (\mathrm{curl}\, v) \cdot \phi = \int_\Omega v \cdot (\mathrm{curl}\, \phi) + \int_{\partial\Omega} v \cdot (\phi \times \widehat{n}).$$

In terms of dual operators, this says that the operator $d^1 = \mathrm{curl}$ with domain $H(\mathrm{curl}; \Omega)$ has a dual δ_2 which is also equal to curl with domain $H_0(\mathrm{curl}; \Omega)$.
The dual of the operator $d^2 = \mathrm{div}$ with domain $H(\mathrm{div}; \Omega)$ is the operator $-\,\mathrm{grad}$ with domain $H_0^1(\Omega)$.

The complex

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$$0 \leftarrow L^2(\Omega) \xleftarrow{-\text{div}} H_0(\text{div};\Omega) \xleftarrow{\text{curl}} H_0(\text{curl};\Omega) \xleftarrow{-\text{grad}} H_0^1(\Omega) \leftarrow 0.$$

The vector Laplacian with magnetic boundary conditions can then be written : find $u \in H(\text{curl};\Omega) \cap H_0(\text{div};\Omega)$ such that $d^1 u \in H_0(\text{curl};\Omega)$, i.e. $u \cdot \widehat{n} = 0$ and $(\text{curl}\, u) \times \widehat{n} = 0$ on $\partial\Omega$ with

$$d^0 \delta_1 u + \delta_2 d^1 u = f \text{ in } \Omega.$$

Unless the domain Ω is simply connected, the above problem is in general not well posed. Denote by $L = d\delta + \delta d$, solutions of the Laplace equation $Lu = 0$ are called harmonic forms.

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Set of objects which I will refer to as space of differential forms $\Lambda^k(\Omega)$ with a basis dx_σ for $\sigma \in \Sigma_k$ such that for $\omega \in \Lambda^k(\Omega)$ and $x \in \Omega$, we have

$$\omega_x = \sum_{\sigma \in \Sigma_k} \omega_\sigma(x) dx_\sigma.$$

The set Σ_k is the collection of subsets of k elements of $\{1, \ldots, n\}$. The space $\Lambda^0(\Omega)$ is defined to be the space of smooth functions on Ω.

There is an operator $\wedge$ such that for $\omega \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^l(\Omega)$, $\omega \wedge \mu \in \Lambda^{k+l}(\Omega)$.

Let $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ be a derivative operator with the property that $d^{k+1} \circ d^k = 0$. One may also require that d^k satisfies a Leibnitz rule and that for $f \in \Lambda^0(\Omega)$, $d^0 f$ is the differential of f. Recall that

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Differential forms

A k-form is a quantity that can be integrated over a k-dimensional region of $\mathbb{R}^n$. $\Lambda^k(\Omega)$ space of smooth k-forms

Functions are 0-forms, integrated by evaluation.

$f(x)\,dx$ 1-form on $[a,b]$, $dG = \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy + \frac{\partial G}{\partial z}\,dz$

associative exterior product : $\omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega), \omega \wedge \eta = (-1)^{jk}\eta \wedge \omega$

$$\omega = \sum_{\sigma(1)<\ldots<\sigma(n)} \omega_\sigma(x)\,dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(k)}, \omega \in \Lambda^k(\Omega)$$

$\omega \in X\Lambda^k(\Omega)$, for $\omega_\sigma(x) \in X$

exterior derivative $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, $d\omega = \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^n \frac{\partial \omega_\sigma}{\partial x_i}\,dx_i \wedge dx_\sigma$

Koszul : $\kappa : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$,

$\kappa\omega = \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^k (-1)^{i+1} f_\sigma x_{\sigma(i)}\,dx_{\sigma(1)} \wedge \ldots \wedge \widehat{dx}_{\sigma(i)} \wedge \ldots \wedge dx_{\sigma(k)}$

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Differential forms

A k-form is a quantity that can be integrated over a k-dimensional region of $\mathbb{R}^n$. $\Lambda^k(\Omega)$ space of smooth k-forms
Functions are 0-forms, integrated by evaluation.
$f(x)\,dx$ 1-form on $[a,b]$, $dG = \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy + \frac{\partial G}{\partial z}\,dz$
associative exterior product : $\omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega), \omega \wedge \eta = (-1)^{jk} \eta \wedge \omega$

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Proxy fields

Identify 0-forms with scalar valued functions

Identify 1-form $\sum_{i=1}^{n} f_i \, dx_i$ with the vector with components $f_i, i = 1, \ldots, n$

For $n = 3$, identify a 2-form $f_1 \, dx_2 \wedge \, dx_3 - f_2 \, dx_1 \wedge \, dx_3 + f_3 \, dx_1 \wedge \, dx_2$ with the vector (f_1, f_2, f_3) and 3-forms with scalar valued functions.

d on 0-forms is grad, d on 1-forms is curl, d on 2-forms is div

trace of a 2-form on a 2-dimensional face is then identified with the normal component

trace of a 1-form on an edge is the tangential component of the proxy vector and its trace on a 2-dimensional face is identified with $u \times n$

$H\Lambda^k(\Omega) = \{ \omega \in L^2\Lambda^k(\Omega), \, d\omega \in L^2\Lambda^{k+1}(\Omega) \}$

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Define $L^2\Lambda^k(\Omega)$ to be the space of forms ω for which $\omega_\sigma \in L^2(\Omega)$ for all $\sigma \in \Sigma_k$. We take as the base complex

$$\mathbb{R} \to L^2\Lambda^0(\Omega) \xrightarrow{d} L^2\Lambda^1(\Omega) \xrightarrow{d} \dots \xrightarrow{d} L^2\Lambda^{n-1}(\Omega) \xrightarrow{d} L^2\Lambda^n(\Omega) \to 0,$$

and for the domain complex, one may choose

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where

$$H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega), dw \in L^2\Lambda^{k+1}(\Omega) \,\}.$$

Now, identifying the unit outward normal vector field $\widehat{n}$ with a one form $\widehat{n}^\flat = \sum_{i=1}^n \widehat{n}_i \, \mathrm{d}x^i$, we have for $v \in \Lambda^k(\Omega)$, a decomposition

$$v|_{\partial\Omega} = v^{\mathrm{tan}} + \widehat{n}^\flat \wedge v^{\mathrm{nor}},$$

where the tangential trace $v^{\mathrm{tan}} \in \Lambda^k(\partial\Omega)$ and the normal trace $v^{\mathrm{nor}} \in \Lambda^{k-1}(\partial\Omega)$.

k	proxy field	tangential trace	normal trace
0	$\varphi \in C^\infty(\Omega)$	$\varphi\|_{\partial\Omega}$	0
1	$v \in C^\infty(\Omega, \mathbb{R}^3)$	$v\|_{\partial\Omega} - (v \cdot \widehat{n})\widehat{n}$	$v \cdot \widehat{n}$
2	$w \in C^\infty(\Omega, \mathbb{R}^3)$	$(w \cdot \widehat{n})\widehat{n}$	$w \times \widehat{n}$
3	$\psi \in C^\infty(\Omega)$	0	$\psi\widehat{n}$

TABLE – Tangential and normal traces of differential forms on $\Omega \subset \mathbb{R}^3$, in terms of scalar and vector proxy fields.

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$$\text{Let } \widehat{H}\Lambda^{k,\mathrm{tan}}(\partial\Omega) = \{\, v^{\mathrm{tan}}, v \in H\Lambda^k(\Omega) \,\}.$$

It turns out that there is a duality pairing between $\widehat{H}\Lambda^{k,\mathrm{tan}}(\partial\Omega)$ and

$$\widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial\Omega) = \{\, v^{\mathrm{nor}}, v \in H\Lambda^k(\Omega) \,\}.$$

Both are subspaces of $H^{1/2}\Lambda^k(\partial\Omega)$. The following integration by parts formula holds : For $\tau \in \Lambda^{k-1}(\Omega)$ and $v \in \Lambda^k(\Omega)$

$$(\mathrm{d}\tau, v)_\Omega = (\tau, \delta v)_\Omega + \langle \tau^{\mathrm{tan}}, v^{\mathrm{nor}} \rangle_{\partial\Omega},$$

where $\delta : \Lambda^{k-1}(\Omega) \to \Lambda^k(\Omega)$ is the coderivative operator.

$$\text{Let } H^*\Lambda^k(\Omega) = \{\, v \in L^2\Lambda^k(\Omega), \delta v \in L^2\Lambda^{k-1}(\Omega) \,\}.$$

Observe that δ is the adjoint of d with domain $H\Lambda^{k-1}(\Omega)$.

$$H_0^*\Lambda^k(\Omega) := \{\, v \in H^*\Lambda^k(\Omega), \langle \tau^{\mathrm{tan}}, v^{\mathrm{nor}} \rangle_{\partial\Omega} = 0, \forall \tau \in H\Lambda^{k-1}(\Omega) \,\}$$
$$= \{\, v \in H^*\Lambda^k(\Omega), v^{\mathrm{nor}} = 0 \,\}.$$

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We define, analogously to the definition of $H_0^*\Lambda^k(\Omega)$

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In summary, for $\tau \in H\Lambda^{k-1}(\Omega)$ and $v \in H^*\Lambda^k(\Omega)$

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The domain complex

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then has a dual complex

$$0 \leftarrow H_0^*\Lambda^0(\Omega) \xleftarrow{\delta} H_0^*\Lambda^1(\Omega) \xleftarrow{\delta} \dots \xleftarrow{\delta} H_0^*\Lambda^{n-1}(\Omega) \xleftarrow{\delta} H_0^*\Lambda^n(\Omega) \leftarrow 0,$$

Let $L = d\delta + \delta d$. The Hodge Laplace problem consists in finding $u \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$ such that $du \in H_0^*\Lambda^k(\Omega), \delta u \in H\Lambda^{k-1}(\Omega)$ and $Lu = f$ for $f \in L^2\Lambda^k(\Omega)$.

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k-forms solutions of $Lu = 0$ are called harmonic forms and the space of harmonic forms is denoted $\mathfrak{H}^k$. The following problem is well-posed : find u in the domain of L such that $d\delta u + \delta du = f - P_{\mathfrak{H}} f$ and $u \perp \mathfrak{H}^k$.

In the mixed formulation with $p = P_{\mathfrak{H}} f$, $\sigma = \delta u$, the conditions $u^{\mathrm{nor}} = 0$ and $(du)^{\mathrm{nor}} = 0$ are imposed naturally. Find $(\sigma, u, p) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega) \times \mathfrak{H}^k$ such that

$$(\sigma, \tau)_\Omega - (u, \mathrm{d}\tau)_\Omega = 0, \qquad \forall \tau \in H\Lambda^{k-1}(\Omega), \tag{1a}$$

$$(\mathrm{d}\sigma, v)_\Omega + (\mathrm{d}u, \mathrm{d}v)_\Omega + (p, v)_\Omega = (f, v)_\Omega, \quad \forall v \in H\Lambda^k(\Omega), \tag{1b}$$

$$(u, q)_\Omega = 0, \qquad \forall q \in \mathfrak{H}^k. \tag{1c}$$

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The analogue of finite element spaces are piecewise polynomials spaces of differential forms $V_h^k \subset H\Lambda^k(\Omega)$ which are required to satisfy an approximation property, a subcomplex property i.e. $dV_h^k \subset V_h^{k+1}$ and the existence of bounded projections $\pi_h^k : H\Lambda^k(\Omega) \to V_h^k$ with some commutativity property.
One can then define the space of discrete harmonic forms to be

$$\mathfrak{H}_h^k = \{\, v \in V_h^k, dv = 0, v \perp dw, w \in V_h^{k+1} \,\}.$$

An analogous definition holds at the continuous level as well.

The mixed formulation is then : find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$(\sigma_h, \tau_h)_\Omega - (u_h, \mathrm{d}\tau_h)_\Omega = 0, \quad \forall \tau_h \in V_h^{k-1},$$
$$(\mathrm{d}\sigma_h, v_h)_\Omega + (\mathrm{d}u_h, \mathrm{d}v_h)_\Omega + (p_h, v_h)_\Omega = (f, v_h)_\Omega, \quad \forall v_h \in V_h^k,$$
$$(u_h, q_h)_\Omega = 0, \quad \forall q_h \in \mathfrak{H}_h^k.$$

To express the continuity property for $V_h^k \subset H\Lambda^k(\Omega)$, we define the broken spaces

$$H\Lambda^k(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H\Lambda^k(K) \text{ and } H^*\Lambda^k(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^*\Lambda^k(K),$$

with $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K}$.
We have

$$H\Lambda^k(\Omega) = \left\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \ \forall \eta \in H_0^*\Lambda^{k+1}(\Omega) \right\},$$
$$H_0\Lambda^k(\Omega) = \left\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \ \forall \eta \in H^*\Lambda^{k+1}(\Omega) \right\}.$$

The mixed formulation is then : find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$\begin{aligned}(\sigma_h, \tau_h)_\Omega - (u_h, \mathrm{d}\tau_h)_\Omega = 0, \quad \forall \tau_h \in V_h^{k-1},\\ (\mathrm{d}\sigma_h, v_h)_\Omega + (\mathrm{d}u_h, \mathrm{d}v_h)_\Omega + (p_h, v_h)_\Omega = (f, v_h)_\Omega, \quad \forall v_h \in V_h^k,\\ (u_h, q_h)_\Omega = 0, \quad \forall q_h \in \mathfrak{H}_h^k.\end{aligned}$$

To express the continuity property for $V_h^k \subset H\Lambda^k(\Omega)$, we define the broken spaces

$$H\Lambda^k(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H\Lambda^k(K) \text{ and } H^*\Lambda^k(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^*\Lambda^k(K),$$

with $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K}$.

We have

$$\begin{aligned}H\Lambda^k(\Omega) = \left\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \ \forall \eta \in H_0^*\Lambda^{k+1}(\Omega) \right\},\\ H_0\Lambda^k(\Omega) = \left\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \ \forall \eta \in H^*\Lambda^{k+1}(\Omega) \right\}.\end{aligned}$$

The mixed formulation is then : find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$
\begin{aligned}
(\sigma_h, \tau_h)_\Omega - (u_h, \mathrm{d}\tau_h)_\Omega = 0, \quad \forall \tau_h \in V_h^{k-1}, \\
(\mathrm{d}\sigma_h, v_h)_\Omega + (\mathrm{d}u_h, \mathrm{d}v_h)_\Omega + (p_h, v_h)_\Omega = (f, v_h)_\Omega, \quad \forall v_h \in V_h^k, \\
(u_h, q_h)_\Omega = 0, \quad \forall q_h \in \mathfrak{H}_h^k.
\end{aligned}
$$

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$$
\begin{aligned}
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H_0\Lambda^k(\Omega) = \left\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\mathrm{tan}}, \eta^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \ \forall \eta \in H^*\Lambda^{k+1}(\Omega) \right\}.
\end{aligned}
$$

The process of hybridization consists in solving local problems in terms of data which solve a global problem obtained from a transmission condition on the interfaces $\partial\mathcal{T}_h := \bigsqcup_{K\in\mathcal{T}_h} \partial K$.

On each element $K \in \mathcal{T}_h$, we aim to solve a mixed formulation of the local problem

$$d\delta u + \delta du = f - p, u \perp \mathfrak{H}^k(K).$$

Put $\sigma = \delta u$ and $\rho = du$. Integration by parts on each element gives

$$(\sigma, \tau)_K = (\delta u, \tau)_K = (u, d\tau)_K - \langle \tau^{\tan}, \widehat{u}^{\mathrm{nor}} \rangle_{\partial K}$$
$$(\rho, \eta)_K = (du, \eta)_K = (u, \delta\eta)_K + \langle \widehat{u}^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial K}.$$

From $d\sigma + \delta\rho = f - p$, we obtain the weak formulation

$$(d\sigma+\delta\rho, v)_K = (\sigma, \delta v)+(\rho, dv)+\langle \widehat{\sigma}^{\tan}, v^{\mathrm{nor}} \rangle_{\partial K}-\langle v^{\tan}, \widehat{\rho}^{\mathrm{nor}} \rangle_{\partial K} = (f-p, v)$$

The process of hybridization consists in solving local problems in terms of data which solve a global problem obtained from a transmission condition on the interfaces $\partial\mathcal{T}_h := \bigsqcup_{K\in\mathcal{T}_h}\partial K$.
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$$\begin{aligned}(\sigma,\tau)_K = (\delta u,\tau)_K = (u, d\tau)_K - \langle \tau^{\mathrm{tan}}, \widehat{u}^{\mathrm{nor}}\rangle_{\partial K}\\ (\rho,\eta)_K = (du,\eta)_K = (u, \delta\eta)_K + \langle \widehat{u}^{\mathrm{tan}}, \eta^{\mathrm{nor}}\rangle_{\partial K}.\end{aligned}$$

From $d\sigma + \delta\rho = f - p$, we obtain the weak formulation

$$(d\sigma+\delta\rho, v)_K = (\sigma,\delta v)+(\rho, dv)+\langle\widehat{\sigma}^{\mathrm{tan}}, v^{\mathrm{nor}}\rangle_{\partial K}-\langle v^{\mathrm{tan}},\widehat{\rho}^{\mathrm{nor}}\rangle_{\partial K} = (f-p, v)$$

Given global variables $\widehat{\sigma}_h^{\tan}$ and $\widehat{u}_h^{\tan}$ which are traces of forms in $H\Lambda^k(\Omega)$, i.e. $\widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan} := \{v_h^{\tan} : v_h \in V_h^k\}$ and $p_h \in \mathfrak{H}_h^k$, we seek $\sigma_h \in W_h^{k-1}(K), u_h \in W_h^k(K), \rho_h \in W_h^{k+1}(K)$ and approximate traces $\widehat{u}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K)$ and $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k+1,\mathrm{nor}}(\partial K)$ such that

$$
\begin{aligned}
(\sigma_h, \tau_h)_K - (u_h, \mathrm{d}\tau_h)_K + \langle \tau_h^{\tan}, \widehat{u}_h^{\mathrm{nor}} \rangle_{\partial K} = 0, &\quad \forall \tau_h \in W_h^{k-1}(K), \\
(\rho_h, \eta_h)_K - (u_h, \delta\eta_h)_K - \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K} = 0, &\quad \forall \eta_h \in W_h^{k+1}(K), \\
(\sigma_h, \delta v_h)_K + (\rho_h, \mathrm{d}v_h)_K + \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K} & \\
- \langle v_h^{\tan}, \widehat{\rho}_h^{\mathrm{nor}} \rangle_{\partial K} = (f - p_h, v_h)_K, &\quad \forall v_h \in W_h^k(K), \\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial K} = 0, &\quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K), \\
\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial K} = 0, &\quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K).
\end{aligned}
$$

(2^{nd}) and (5^{th}) give $\rho_h = du_h$. This, with (4^{th}) can be used in (3^{rd}) to obtain $(d\sigma_h, v_h)_K + (du_h, \mathrm{d}v_h)_K - \langle v_h^{\tan}, \widehat{\rho}_h^{\mathrm{nor}} \rangle_{\partial K} = (f - p_h, v_h)_K$.

However, the problem

$$\begin{aligned}
(\sigma_h, \tau_h)_K - (u_h, \mathrm{d}\tau_h)_K + \langle \tau_h^{\tan}, \widehat{u}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \tau_h \in W_h^{k-1}(K), \\
(d\sigma_h, v_h)_K + (du_h, \mathrm{d}v_h)_K - \langle v_h^{\tan}, \widehat{\rho}_h^{\mathrm{nor}} \rangle_{\partial K} &= (f - p_h, v_h)_K, \forall v_h \in W_h^k(K) \\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{H}^* \Lambda^{k,\mathrm{nor}}(\partial K), \\
\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{H}^* \Lambda^{k,\mathrm{nor}}(\partial K),
\end{aligned}$$

is not well-posed. The null space is contained in

$$\{\, v_h \in W_h^k(K), v_h^{\tan} = 0, (v_h, dw_h) = 0, \forall w_h \in H_0\Lambda^{k-1}(K) \,\}.$$

This space is denoted $\mathring{\mathfrak{H}}^k(K)$ in FEEC literature. As it is the space of harmonic forms for the de Rham complex with boundary conditions.

We enforce the condition $u_h \perp \mathring{\mathfrak{H}}^k(K)$ using a Lagrange multiplier $\overline{p}_h$. And introduce a new global unknown $\overline{u}_h$ on each element K which approximates the projection $\overline{u}$ of u onto $\mathring{\mathfrak{H}}^k(K)$.
The local problems then read :
Given $\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}$, $\widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan}$, $p_h \in \mathfrak{H}_h^k$ and $\overline{u}_h \in \mathring{\mathfrak{H}}^k(K)$, find $\sigma_h \in W_h^{k-1}(K), u_h \in W_h^k(K), \rho_h \in W_h^{k+1}(K)$, $\overline{p}_h \in \mathring{\mathfrak{H}}^k(K)$ and approximate traces $\widehat{u}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K)$ and $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k+1,\mathrm{nor}}(\partial K)$ such that

$$
\begin{aligned}
(\sigma_h, \tau_h)_K - (u_h, \mathrm{d}\tau_h)_K + \langle \tau_h^{\tan}, \widehat{u}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \tau_h \in W_h^{k-1}(K),\\
(d\sigma_h, v_h)_K + (du_h, \mathrm{d}v_h)_K - \langle v_h^{\tan}, \widehat{\rho}_h^{\mathrm{nor}} \rangle_{\partial K} + {\color{red}(\overline{p}_h, v_h)} &= (f - p_h, v_h)_K, \forall v_h \in W_h^k(K)\\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K),\\
\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial K} &= 0, \quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{H}^*\Lambda^{k,\mathrm{nor}}(\partial K),\\
{\color{red}(u_h, \overline{q}_h)_K} &= (\overline{u}_h, \overline{q}_h) \quad \forall \overline{q}_h \in \mathring{\mathfrak{H}}^k(K).
\end{aligned}
$$

The local problems are connected by "transmission conditions" which are the global equations needed to solve for the global variables. We require that the normal traces are single-valued. Define $\widehat{W}_h^{k,\mathrm{nor}} := (\widehat{W}_h^{k,\mathrm{tan}})^*$. We require $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}$ satisfy

$$\begin{aligned} \langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\mathrm{tan}} \rangle_{\partial\mathcal{T}_h} &= 0, \quad \forall \widehat{\tau}_h^{\mathrm{tan}} \in \widehat{V}_h^{k-1,\mathrm{tan}}, \\ \langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\mathrm{tan}} \rangle_{\partial\mathcal{T}_h} &= 0, \quad \forall \widehat{v}_h^{\mathrm{tan}} \in \widehat{V}_h^{k,\mathrm{tan}}. \end{aligned}$$

Static condensation

Encode the local variables into x_h and the global variables into y_h.

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} F_h \\ G_h \end{bmatrix}.$$

Then

$$-BA^{-1}B^T y_h = G_h - BA^{-1}F_h,$$

and

$$x_h = A^{-1}F_h - A^{-1}B^T y_h.$$

Hybrid method for the vector Poisson equation

On a contractible domain.

$$
\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{grad} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^0, \\
(\operatorname{grad} \sigma_h, v_h)_{\mathcal{T}_h} + (\operatorname{curl} u_h, \operatorname{curl} v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^1, \\
\langle \widehat{\sigma}_h^{\mathrm{tan}} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{0,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\mathrm{tan}} - u_h, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\mathrm{tan}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\mathrm{tan}} \in \widehat{V}_h^{0,\mathrm{tan}}, \\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\mathrm{tan}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{tan}} \in \widehat{V}_h^{1,\mathrm{tan}}.
\end{aligned}
$$

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