CONVERGENCE OF A HYBRID SCHEME FOR THE ELLIPTIC MONGE-AMPIERE EQUATION

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Abstract. We prove the convergence of a hybrid discretization to the viscosity solution of the elliptic Monge-Ampère equation. The hybrid discretization uses a standard finite difference discretization in parts of the computational domain where the solution is expected to be smooth and a monotone scheme elsewhere. A motivation for the hybrid discretization is the lack of an appropriate Newton solver for the standard finite difference discretization on the whole domain.

1. Introduction

In this paper, we prove convergence to the viscosity solution of the elliptic Monge-Ampère equation of a hybrid discretization. The discretization we analyze is a variant of the hybrid method proposed by Froese and Oberman in [20]. The elliptic Monge-Ampère equation is a fully nonlinear equation, i.e. nonlinear in the highest order derivatives. Unless the domain is smooth and strictly convex and the data are smooth, the solution is not expected to be smooth. Nevertheless, numerical experiments [8, 2] indicate that the discrete equations obtained through standard finite difference discretizations have a discrete convex solution in the sense that a certain discrete Hessian is positive. The solution can be retrieved through appropriate iterative methods. But the convergence of the standard finite difference discretization in that sense to the viscosity solution remained obscure till [2] where a regularization approach is taken. However it is not known whether an appropriate Newton solver can be developed for the standard finite difference discretization. We note that the standard finite difference discretization is commonly used in science and engineering [25, 14, 13].

The hybrid discretization proposed in [20] uses a consistent, monotone and stable scheme in parts of the domain where the solution is not expected to be smooth and a standard discretization in parts of the domain where the solution is smooth. We will often refer to a consistent, monotone and stable scheme simply as a monotone scheme. The monotone discretization is known to converge to the viscosity solution when used on the whole domain with a convergent Newton’s method solver [19]. As pointed out in [21] the convergence of the hybrid discretization introduced in [20] is still an open problem and results with the hybrid discretization of [20] are comparable with the ones obtained with the filtered approach in [21].

In this paper, we build on the recent advances in [2] on the analysis of the standard finite difference discretization. Combined with the classical framework for convergence of monotone schemes to viscosity solutions, we obtain convergence to the viscosity solution of our hybrid discretization. A fixed point argument is used to show existence
of a discrete convex solution. The discretization we analyze differs from the one presented in [20] only by the choice of the standard finite difference discretization. With a monotone discretization one can transfer to the discrete level arguments for viscosity solutions for partial differential equations. But it does not allow to give, in general, results for the standard finite difference discretization for smooth solutions. In fact, the quadratic convergence rate of the latter was only known as "formally second-order accurate" [8]. Moreover, the theory of Barles and Souganidis [7] cannot be applied directly to a hybrid discretization. For the latter and a standard finite difference discretization, pure numerical analysis techniques seem required. The Banach fixed point theorem, which is ubiquitous in the finite element analysis of nonlinear problems, has been adapted to the Monge-Ampère equation in [9, 18, 11]. Combined with the continuity of the eigenvalues of a matrix as a function of its entries and an appropriate rescaling of the equation, the analysis of a standard finite difference discretization for smooth solutions of the equation was done in [2]. Another key argument used in [2] is that, in the context of the standard finite difference discretization, the discrete Hessian, for a mesh function near a strictly convex smooth solution, is positive definite. One of the main difficulties overcome in this paper is the choice of a suitable norm to measure the error resulting from a hybrid discretization. The impact of the results of this paper goes beyond the particular application considered. For example, the techniques used here may equally be applied to a hybrid scheme for the convex envelope presented in [33].

Numerical experiments reported in [20] indicate that Newton’s method can be applied to the nonlinear system resulting from a hybrid discretization. Although central difference approximations of the second order derivatives are used in [20], it is easily seen that the general methodology presented in this paper can be combined with new results on standard finite discretizations presented in [1] to yield the convergence of the original method used in [20].

The best results for the hybrid discretization are obtained when the subdomain where the solution is not smooth, the set of singular points, is known in advance. An adaptive mesh refinement scheme could make it easier to identify the set of singular points. As with [20], one can take a conservative approach and include a priori in the set of singular points, points where either $f(x)$ is not Hölder continuous, $f(x)$ is too small or $f(x)$ is too large. It is very likely that the approximation will deteriorate at points which are close to boundary points where $\partial \Omega$ is not $C^3$ or strictly convex and points close to boundary points where $g(x)$ cannot be extended to a $C^3$ function. They may be included in the set of singular points as well. The motivation to consider these points as singular points comes from the regularity theory of the Monge-Ampère equation. See for example Theorem 1.1 in [35].

We provide numerical experiments to illustrate the performance of the hybrid discretization introduced in this paper. We use a monotone scheme near the boundary and what we call a compatible standard discretization, [2], in the interior of the domain. The nonlinear system is solved by a time marching method (5.1) and Newton’s method. For smooth solutions and a standard compatible discretization on the whole domain, the time marching method (5.1) was shown in [2] to be faster than Newton’s method. Newton’s method with the hybrid discretization is faster than the time
marching method (5.1) for some non smooth solutions. The numerical experiments go beyond the theory presented since it appears not necessary, with the time marching method (5.1), to use a monotone discretization at interior points where \( f(x) = 0 \) or non smooth. In some cases, with (5.1) and the hybrid discretization introduced in this paper, the results we obtain are more accurate than the ones presented in [20] with Newton’s method and central finite difference discretizations at points where the solution is smooth.

A standard finite difference discretization of the Dirichlet problem for the Monge-Ampère equation was introduced in [17]. Finite element discretizations have also been proposed, e.g. [22, 9, 18, 11, 27, 16, 12]. In general, there is not yet a convergence theory to explain the performance of these methods for non smooth solutions. In [3], we presented a general framework which can be used as a blueprint to give an analysis of these methods. Our results essentially say that, results for smooth solutions can be extended to strictly convex non smooth solutions for standard discretizations in the interior of the domain and with regularization of the data. It is expected that the results obtained in [31] for linear problems in non-divergence form can be extended to the Monge-Ampère equation.

This paper is organized as follows. In the second section, we recall the notion of viscosity solution and present the hybrid discretization. We also define our notion of discrete convex function in the second section and the main notation of the paper. Existence and uniqueness of a discrete convex solution is given in the third section. In the fourth section we use the now classical arguments of [7] and new arguments for smooth solutions and the standard finite difference discretization of [2] to prove convergence to the viscosity solution of the hybrid discretization. We conclude with numerical experiments.

2. Viscosity solutions of the elliptic Monge-Ampère equation and the hybrid discretization

To avoid difficulties with a curved boundary, we assume in this paper that the domain \( \Omega \) is rectangular. We further make the assumption that \( \Omega = (0, 1)^n \subset \mathbb{R}^n \). For given \( f > 0 \) continuous on \( \overline{\Omega} \) and \( g \) continuous on \( \partial \Omega \), with a convex extension \( \tilde{g} \in C(\overline{\Omega}) \), we consider the Monge-Ampère equation

\[
\det D^2 u = f \text{ in } \Omega \\
u = g \text{ on } \partial \Omega.
\]

(2.1)

Let \( h > 0 \) denote the mesh size. We assume without loss of generality that \( 1/h \in \mathbb{Z} \). Put

\[
Z_h = \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : x_i/h \in \mathbb{Z}\} \\
\Omega_h^0 = \Omega \cap Z_h, \Omega_h = \overline{\Omega} \cap Z_h, \partial \Omega_h = \partial \Omega \cap Z_h = \Omega_h \setminus \Omega_h^0.
\]

For \( x \in \mathbb{R}^n \), we denote the maximum norm of \( x \) by \( |x| = \max_{i=1,\ldots,n} |x_i| \). The norm \( |.| \) is extended canonically to matrices.
Let $\mathcal{M}(\Omega^h)$ denote the set of real valued functions defined on $\Omega^h$, i.e. the set of mesh functions. For a subset $T_h$ of $\Omega^h$, and $v^h \in \mathcal{M}(\Omega^h)$ we define
\[ |v^h|_{T_h} = \max_{x \in T_h} |v^h(x)|. \]
The norm $|.|_{T_h}$ is extended canonically to matrix fields.
Let $v$ be a continuous function on $\Omega$ and let $r_h(v)$ denote the unique element of $\mathcal{M}(\Omega^h)$ defined by
\[ r_h(v)(x) = v(x), x \in \Omega^h. \]
We extend the operator $r_h$ canonically to vector fields and matrix fields. For a function $g$ defined on $\partial \Omega$, $r_h(g)$ defines the analogous restriction on $\partial \Omega^h$.

2.1. Viscosity solutions. A convex function $u \in C(\overline{\Omega})$ is a viscosity solution of (2.1) if $u = g$ on $\partial \Omega$ and for all $\phi \in C^2(\Omega)$ the following holds
- at each local maximum point $x_0$ of $u - \phi$, $f(x_0) \leq \det D^2 \phi(x_0)$
- at each local minimum point $x_0$ of $u - \phi$, $f(x_0) \geq \det D^2 \phi(x_0)$, if $D^2 \phi(x_0) \geq 0$.

As explained in [26], the requirement $D^2 \phi(x_0) \geq 0$ in the second condition above is natural at least in even dimensions. The space of test functions in the definition above can be restricted to the space of strictly convex quadratic polynomials [23, Remark 1.3.3].

An upper semi-continuous convex function $u$ is said to be a viscosity sub solution of $\det D^2 u(x) = f(x)$ if the first condition holds and a lower semi-continuous convex function is said to be a viscosity super solution when the second holds. A viscosity solution of (2.1) is a continuous function which satisfies the boundary condition and is both a viscosity sub solution and a viscosity super solution.

Note that the notion of viscosity solution is a pointwise notion. It is not very difficult to prove that if $u$ is $C^2$ at $x_0$, then $u$ is a viscosity solution at the point $x_0$ of $\det D^2 u = f$.

For further reference, we recall the comparison principle of sub and super solutions, [26, Theorem V. 2]. Let $u$ and $v$ be respectively sub and super solutions of $\det D^2 u(x) = f(x)$ in $\Omega$ and put
\[ u^* = \limsup_{y \to x, y \in \Omega} u(y) \quad \text{and} \quad v_* = \liminf_{y \to x, y \in \Omega} v(y). \]
Then if $\sup_{x \in \partial \Omega} \max(u^*(x) - v_*(x), 0) = M$, then $u(x) - v(x) \leq M$ in $\Omega$.

There are very few references which give an existence and uniqueness result for (2.1) in the degenerate case $f \geq 0$. In [26] it is required that one can find a sub solution and a super solution. The difficulty is that the Monge-Ampère equation is not often studied in convex but not necessarily strictly convex domains. Thus we assume in addition that $f > 0$. Since $f \in C(\overline{\Omega})$ it follows that there exists a constant $c_0 > 0$ such that
\[ f \geq c_0 > 0. \]
We also assume that $g$ can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$. Then by [24, Theorem 1.1], (2.1) has a unique Aleksandrov solution. We can then use the equivalence of viscosity and Aleksandrov solutions [23, Propositions 1.3.4 and 1.7.1].
2.2. A reformulation of convexity. We recall that a function $\phi \in C^2(\Omega)$ is convex on $\Omega$ if the Hessian matrix $D^2\phi$ is positive semidefinite or $\lambda_1[\phi] \geq 0$ where $\lambda_1[\phi]$ denotes the smallest eigenvalue of $D^2\phi$. This notion was extended to continuous functions in [33]. See also the remarks on [34, p. 226]. A continuous function $u$ is convex if and only if it is a viscosity solution of $-\lambda_1[u] \leq 0$ (or $\lambda_1[u] \geq 0$), that is, for all $\phi \in C^2(\Omega)$, whenever $x_0$ is a local maximum point of $u - \phi$, $-\lambda_1[\phi] \leq 0$ (resp. $\lambda_1[\phi] \geq 0$). This can also be written $\max(-\lambda_1[u], 0) = 0$ in $\Omega$, c.f. [33].

The Dirichlet problem for the Monge-Ampère equation (2.1) can then be written

\begin{align}
-\det D^2u + f &= 0 \text{ in } \Omega \\
\max(-\lambda_1[u], 0) &= 0 \text{ in } \Omega,
\end{align}

with boundary condition $u = g$ on $\partial \Omega$. We write (2.2) as $F(u) = 0$ and note that the form of the equation is chosen to be consistent with the definition of ellipticity used for example in [26].

Since we have now rewritten (2.2) convexity as an additional equation, sub solutions and super solutions of $-\det D^2u + f = 0$ do not need to be convex. It can be shown that a comparison principle still holds [29]. The definition in [29] for super solution does not require the test function to be convex. However the proof of the comparison principle given in [26, section 2] also holds with the convexity requirements. A viscosity solution of $-\det D^2u + f = 0$ is necessarily continuous. As a viscosity solution of $\max(-\lambda_1[u], 0) = 0$, it is convex in the viscosity sense and hence convex, see for example [28, Proposition 4.1]. We conclude that a viscosity solution of (2.2) is also a viscosity solution as defined in section 2.1.

2.3. Standard finite difference discretizations. The version of the standard finite difference discretization we consider was first introduced in [2]. Let $\Omega_r$ be a bounded convex domain of $\mathbb{R}^n$. Put $\Omega_r^h = \overline{\Omega}_r \cap \mathbb{Z}_h$.

Let $e^i, i = 1, \ldots, n$ denote the $i$-th unit vector. We define first order difference operators acting on functions defined on $\mathbb{Z}_h$. For $x \in \mathbb{Z}_h$

$$
\partial^+_i v^h(x) := \frac{v^h(x + he^i) - v^h(x)}{h},
\partial^-_i v^h(x) := \frac{v^h(x) - v^h(x - he^i)}{h}.
$$

We define discrete analogues of the gradient $D_h$ and $\overline{D}_h$ as:

$$
D_h v^h := (\partial^+_i v^h)_{i=1,\ldots,n},
\overline{D}_h v^h := (\partial^-_i v^h)_{i=1,\ldots,n}.
$$

We make the convenient abuse of notation of using $v^h$ for a scalar field and for a vector field. For $v^h = (v^{h,i})_{i=1,\ldots,d}$, $v^{h,i} \in \mathcal{M}(\Omega^h)$ for all $i$, we define

$$
\text{div}_h v^h = \sum_{i=1}^d \partial^+_i v^{h,i},
$$

with boundary condition $u = g$ on $\partial \Omega$. We write (2.2) as $F(u) = 0$ and note that the form of the equation is chosen to be consistent with the definition of ellipticity used for example in [26].

Since we have now rewritten (2.2) convexity as an additional equation, sub solutions and super solutions of $-\det D^2u + f = 0$ do not need to be convex. It can be shown that a comparison principle still holds [29]. The definition in [29] for super solution does not require the test function to be convex. However the proof of the comparison principle given in [26, section 2] also holds with the convexity requirements. A viscosity solution of $-\det D^2u + f = 0$ is necessarily continuous. As a viscosity solution of $\max(-\lambda_1[u], 0) = 0$, it is convex in the viscosity sense and hence convex, see for example [28, Proposition 4.1]. We conclude that a viscosity solution of (2.2) is also a viscosity solution as defined in section 2.1.
and
\[
\overline{D}_hv^h = (\partial^j_i v^h)_{i,j=1,\ldots,n}.
\]

The discrete Hessian is defined as
\[
\mathcal{H}_d(v^h) := \overline{D}_hD_hv^h.
\]

Put
\[
\Omega^h_{r,0} = \{ x \in \Omega^h_r, \mathcal{H}_d(v^h)(x) \text{ is defined for } v^h \in \mathcal{M}(\Omega^h) \}
\]
\[
\partial \Omega^h_r = \Omega^h_r \setminus \Omega^h_{r,0}.
\]

For a matrix \( A \), we recall that the cofactor matrix \( \text{cof} A \) is defined by
\[
(\text{cof} A)_{ij} = (-1)^{i+j}d(A)^j_i
\]
where \( d(A)^j_i \) is the determinant of the matrix obtained from \( A \) by deleting the \( i \)th row and the \( j \)th column. For two matrices \( A = (A_{ij})_{i,j=1,\ldots,n} \) and \( B = (B_{ij})_{i,j=1,\ldots,n} \) we recall the Frobenius inner product
\[
A : B = \sum_{i,j=1}^n A_{ij}B_{ij}.
\]

The discrete version of (2.1) on \( \Omega^h_r \) takes the form
\[
\frac{1}{n} \text{div}_h[(\text{cof sym} \mathcal{H}_d u^h)^T D_h u^h] = r_h(f) \text{ in } \Omega^h_{r,0}, u^h = r_h(\tilde{g}) \text{ on } \partial \Omega^h_r,
\]
where for a matrix \( A \), \( \text{sym} A = (A + A^T)/2 \) denotes the symmetric part of \( A \).

We recall that the motivation of the above form of the discretization is to be able to transfer to the discrete level arguments for smooth solutions of the Monge-Ampère equation. For simplicity, we define for a mesh function
\[
M_r[v^h] = \frac{1}{n} \text{div}_h[(\text{cof sym} \mathcal{H}_d u^h)^T D_h u^h].
\]

Let \( \mathcal{M}(\Omega^h_r) \) denote the set of real valued functions defined on \( \Omega^h_r \). We define an inner product on \( \mathcal{M}(\Omega^h_r) \) by
\[
\langle v^h, w^h \rangle = h^n \sum_{x \in \Omega^h_{r,0}} v^h(x)w^h(x), v^h, w^h \in \mathcal{M}(\Omega^h_r),
\]
and the following semi norms
\[
\|v^h\|_{0,h} = \sqrt{\langle v^h, v^h \rangle}, \|v^h\|_{1,h} = \left( \|v^h\|_{0,h}^2 + \sum_{i=1}^n (\|\partial^i_+ v^h\|_{0,h}^2) \right)^{\frac{1}{2}},
\]
\[
|v^h|_{1,h} = \left( \sum_{i=1}^n (\|\partial^i_+ v^h\|_{0,h}^2) \right)^{\frac{1}{2}}, |v^h|_{2,\infty,h} = \max \{ \partial^i_\nu \partial^j_\nu v^h(x), x \in \Omega^h_{r,0}, i, j = 1, \ldots, n \}.
\]

Put
\[
H^1_0(\Omega^h) = \{ v^h \in \mathcal{M}(\Omega^h), \|v^h\|_{1,h} < \infty, v^h = 0 \text{ on } \partial \Omega^h \}.
\]

We have the discrete analogue of Poincaré’s inequality, see for example [15, Lemma 3.1].

**Lemma 2.1** (Discrete Poincaré’s inequality). There exists a constant \( C_p > 0 \) independent of \( h \) such that for \( v^h \in H^1_0(\Omega^h) \),
\[
|v^h|_{1,h} \geq C_p \|v^h\|_{0,h}.
\]
2.4. Monotone schemes. Let us denote by $F_h(u^h) \equiv \hat{F}_h(u^h(x), u^h(y)|_{y \neq x})$ a discretization of $F(u)$. We recall the elements of the convergence theory of Barles and Souganidis [7] and how its conditions were met by the discretization introduced in [19].

The scheme $F_h(u^h) = 0$ is said to be monotone if for $v^h$ and $w^h$ in $\mathcal{M}(\Omega^h)$, $v^h(y) \geq w^h(y)$, $y \neq x$ implies $\hat{F}_h(v^h(x), v^h(y)|_{y \neq x}) \geq \hat{F}_h(w^h(x), v^h(y)|_{y \neq x})$. Here we use the partial ordering of $\mathbb{R}^2$, $(a_1, b_1) \geq (a_2, b_2)$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

The scheme is said to be consistent if for all $C^2$ functions $\phi$, and a sequence $x_h \rightarrow x \in \Omega$, $\lim_{h \rightarrow 0} F_h(r_h(\phi))(x_h) = F(\phi)(x)$.

Finally the scheme is said to be stable if $F_h(u^h) = 0$ has a solution $u^h$ which is bounded independently of $h$.

It follows from [7] that a consistent, stable and monotone scheme has a solution $u^h$ which converges locally uniformly to the unique viscosity solution of (2.2). Note that the convexity assumption on the exact solution is enforced through the definition of $F(u)$.

Next, we recall the notion of degenerate ellipticity of a scheme and that of proper scheme introduced in [32]. For $x \in \Omega^h$, let us denote by $N(x)$ a set of mesh points $y \neq x$ which are within a certain fixed distance from $x$. The choice of $N(x)$ introduces another discretization error, called directional error in [32]. Without loss of generality, we assume in this paper that $N(x) = \Omega^h$ for all $x$. We use the notation $\#N(x)$ to denote the cardinality of the set $N(x)$.

We now assume that the discretization takes the form

$$F_h(u^h)(x) \equiv \hat{F}_h(u^h(x), u^h(y) - u^h(x)|_{y \neq x, y \in N(x)})$$

where for $x \in \Omega^h$, $\hat{F}_h$ is a real valued map defined on $\mathbb{R} \times \mathbb{R}^{\#N(x)}$. For convenience we do not write explicitly the dependence of $\hat{F}_h$ on $x$.

The scheme is said to be degenerate elliptic if it is nondecreasing in each of the variables $u^h(x)$ and $u^h(y) - u^h(x), y \in N(x), y \neq x$.

The scheme is proper if there is $\delta > 0$ such that for $x \in \Omega^h$ and for all $a_0, a_1 \in \mathbb{R}$ and $b \in \mathbb{R}^{\#N(x)}$, $a_0 \leq a_1$ implies $\hat{F}_h(a_0, b) - \hat{F}_h(a_1, b) \leq \delta(a_0 - a_1)$.

The scheme $F_h(u^h) = 0$ is Lipschitz continuous if there is $K > 0$ such that for all $x \in \Omega^h$ and $a_0, b_0, (a_1, b_1) \in \mathbb{R}^{\#N(x) + 1}$

$$|\hat{F}_h((a_0, b_0)) - \hat{F}_h((a_1, b_1))| \leq K|((a_0, b_0)) - (a_1, b_1)|_{\infty}.$$ 

It is not very difficult to prove that a degenerate elliptic scheme is monotone. Moreover for a scheme which is proper, degenerate elliptic and Lipschitz continuous, the equation $F_h(u^h) = 0$ has a unique solution to which converges the iteration

$$(2.5) \quad u^h_{k+1} = u^h_k - \nu F_h(u^h_k),$$

for $\nu$ sufficiently large [32, Theorem 7].

We recall the expression of the consistent, monotone and stable discretization of $\lambda_1[u]$ introduced in [33]. For simplicity we consider only wide stencils as the theoretical
developments of this paper can be easily extended to discretizations with smaller
stencils. We have at an interior grid point \( x \)
\[
\lambda_1^h[u^h](x) = \min_{\alpha^h \in \mathbb{R}^n} \frac{u^h(x + \alpha^h) - 2u^h(x) + u^h(x - \alpha^h)}{|\alpha^h|^2},
\]
where by \( \alpha^h \in \mathbb{R}^n \) we mean vectors \( \alpha^h \) for which the above expression is well defined
for grid points.

We also recall the expression \( M_s[u^h] \) of the discretization of \( \det D^2u \) used in [19].
For \( x \in \Omega^h \) we denote by \( W_h(x) \) the set of orthogonal bases of \( \mathbb{R}^n \) such that for
\( (\alpha_1, \ldots, \alpha_n) \in W_h(x) \) \( x \pm \alpha_i \in \Omega^h, \forall i \). We have
\[
M_s[u^h](x) = \inf_{(\alpha_1, \ldots, \alpha_n) \in W_h(x)} \prod_{i=1}^n \frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2}.
\]

Now, the scheme introduced in [19] can be shown to be stable using the contraction
mapping principle used to show convergence of the iteration (2.5). See Section 4 for
a similar situation. It is degenerate elliptic and consistent [19]. Moreover it is also
Lipschitz continuous, see for example [32].

Without loss of generality we will make the assumption that it is proper by considering
the perturbation \( F_h(u^h) + \epsilon u^h \) where \( \epsilon > 0 \) is a small parameter close to machine
precision or of the size of the discretization parameter.

The monotone discretization of (2.2) can then be written
\[
M_s[u^h](x) - r_h(f)(x) = 0, x \in \Omega^h
\]
\[
\max(-\lambda_1^h[u^h](x), 0) = 0, x \in \Omega^h
\]
\[
u^h(x) = r_h(g)(x) \text{ on } \partial\Omega^h.
\]

It follows from the above discussion that (2.8) has a unique solution which converges
locally uniformly to the unique solution of (2.2).

As with [19], the first two equations of (2.8) are combined in a single equation. Recall
that \( x^+ = \max(x, 0) \) and define
\[
M_s^+[u^h](x) = \inf_{(\alpha_1, \ldots, \alpha_n) \in W_h(x)} \prod_{i=1}^n \max \left( \frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2}, 0 \right).
\]

Then (2.8) can be written
\[
M_s^+[u^h](x) - r_h(f)(x) = 0, x \in \Omega^h
\]
\[
u^h(x) = r_h(g)(x) \text{ on } \partial\Omega^h.
\]

2.5. The hybrid discretization.

**Definition 2.2.** We call a point \( x \in \Omega \) a regular point if the solution \( u \) of (2.1) is
\( C^2 \) in a neighborhood of \( x \). A point which is not a regular point is called a singular
point.
The above definition is natural if one considers the one dimensional Monge-Ampère equation $-u''(x) = f$ and a standard finite difference approximation. In particular, at a regular point $x$, by a Taylor series expansion,

$$
\lim_{h \to 0} \max_{i,j=1,\ldots,n} |\partial^2 v(x)/(\partial x_i \partial x_j) - \partial^2 \partial_i (r_h v)(x)| = 0.
$$

Next, for $v \in C^4(\Omega)$, and $x \in \Omega$

$$
\max_{i,j=1,\ldots,n} |\partial^2 v(x)/(\partial x_i \partial x_j) - \partial^2 \partial_i (r_h v)(x)| \leq C h^2 |v|_{4,\Omega},
$$

where for an integer $j$, $|v|_{j,\Omega} = \sup_{|\beta|=j} \sup_{\Omega} |D^\beta v(x)|$ for a multi-index $\beta$.

Let $\Omega_r$ denote an open subset of $\Omega$ such that at every point $x$ of $\Omega_r$ the exact solution $u$ is $C^2$ in a neighborhood of $x$. Using the notation of section 2.3 we define

$$
\Omega^h_s = \Omega^h_0 \setminus \Omega^h_{r,0}.
$$

**Definition 2.3.** By a discrete convex function, we mean a mesh function $v^h$ such that

$$
\Lambda^h_1[v^h] \geq 0 \text{ in } \Omega^h_s
$$

$$
\mathcal{H}_d v^h \geq 0 \text{ in } \Omega^h_{r,0}.
$$

Strictly discrete convex functions are defined analogously.

We note that a discrete convex function in the sense of the above definition is not necessarily convex on $\Omega^h_0$. See [33] for the case $\Omega^h_s = \Omega^h_0$ and [30] for a counterexample showing that the discrete Hessian $\mathcal{H}_d v^h$ can be positive without the mesh function $v^h$ being convex in the usual sense. The minor abuse of terminology we make is justified by Theorem 4.4 below which says in particular that the uniform limit of mesh functions which satisfy (2.12) and solve the discrete Monge-Ampère equation (2.15) below, is convex.

For a subset $T^h$ of $\Omega^h$, we denote by $\mathcal{C}^h(T^h)$ the cone of discrete convex functions on $T^h$ and by $\mathcal{C}^h_0(T^h)$ the cone of strictly discrete convex functions on $T^h$. We define on $\Omega^h_0$ for a mesh function $v^h$, $F^h(v^h)$ by

$$
F^h(v^h)(x) = M^+_r[v^h](x) - r_h(f)(x), \quad x \in \Omega^h_s
$$

$$
F^h(v^h)(x) = M_r[v^h](x) - r_h(f)(x), \quad x \in \Omega^h_{r,0}.
$$

Put $\mathcal{C}^h = \mathcal{C}^h(\Omega^h_0)$ and define a norm $|.|_h$ on $\mathcal{M}(\Omega^h)$ by

$$
|v^h|_h = \max\{ |v^h|_{\Omega^h_0}, \frac{h^{n/2}}{C_p} |v^h|_{1,h} \},
$$

where we denote by $C_p$ the constant in the Poincaré’s inequality for the domain $\Omega_r$ and we recall that the semi-norm $|.|_{1,h}$ takes only into account mesh points in $\Omega^h_{r,0}$. To verify that the above formula defines a norm, one uses the observation that $\partial \Omega^h_{r,0} \subset \Omega^h_s$ and Lemma 2.1.

The hybrid discretization of (2.2) can then be written: find $u^h \in \mathcal{C}^h$

$$
F^h(u^h)(x) = 0 \text{ in } \Omega^h_0, u^h(x) = r_h(g)(x) \text{ on } \partial \Omega^h.
$$
In [20], the authors use a weight function to write the hybrid discretization as a combination of the monotone scheme and the standard finite difference discretization. We omit it in this paper as it plays no role in our analysis.

3. Existence and uniqueness of a discrete convex solution

We make the usual convention of denoting constants by $C$ but will occasionally index some constants.

We first show that the problem (2.15) has a unique local solution. We define a linear operator $L_h$ on $\mathcal{M}(\Omega^h)$ as

$$L_h v^h(x) = v^h(x), \quad x \in \Omega^h_s$$

$$L_h v^h(x) = \Delta_h^{-1} v^h(x), \quad x \in \Omega^h_{r,0},$$

and where the operator $\Delta_h = \text{div}_h D_h v^h$ is considered only on $\Omega^h_{r,0}$, i.e. if $w^h = \Delta_h^{-1} v^h$, then

$$\Delta_h w^h = v^h \text{ on } \Omega^h_{r,0} \text{ and } w^h = 0 \text{ on } \partial \Omega^h_r.$$

Consider the ball

$$B_\rho(r_h(u)) = \{ v^h \in \mathcal{M}(\Omega^h), |v^h - r_h(u)|_{1,h} \leq \rho, v^h = r_h(g) \text{ on } \partial \Omega^h, v^h = r_h(u) \text{ on } \partial \Omega^h_{r,0} \}.$$

Recall that the semi norm $|.|_{1,h}$ takes only into account mesh points in $\Omega^h_{r,0}$. As with [2] and [4], it will be necessary to use a “rescaling argument”, i.e. solve a rescaled equation in the ball $\alpha B_\rho(r_h(u))$ for

$$\alpha = h^{\frac{3+2}{n-1}}.$$

Next we define the mapping

$$S : \mathcal{M}(\Omega^h) \to \mathcal{M}(\Omega^h)$$

$$S(\alpha v^h)(x) = \alpha v^h(x) - \nu_x \alpha^n L_h F_h(v^h)(x)$$

$$\nu_x = \nu_1, \quad x \in \Omega^h_s$$

$$\nu_x = \nu_2, \quad x \in \Omega^h_{r,0},$$

for $\nu_1, \nu_2 > 0$. Our goal is to show that for $\rho, h$ sufficiently small and an appropriate $\nu$, $S$ has a fixed point in $B_\rho(r_h(u))$. We have

**Lemma 3.1.** There exists a positive constant $a_1 < 1$ such that for all $v^h, w^h \in \mathcal{M}(\Omega^h)$, we have

$$|S(\alpha v^h) - S(\alpha w^h)|_{\Omega^h} \leq a_1 |\alpha v^h - \alpha w^h|_{\Omega^h_r},$$

for $C_0 \leq \nu_1 \leq C_1$ where $C_0$ and $C_1$ are positive constants.

**Proof.** The proof essentially follows the one for monotone schemes on the whole domain given in [32, Theorem 7].

We have

$$|\alpha v^h - \alpha w^h|_{\Omega^h_r} = \max\{|\alpha v^h - \alpha w^h|_{\Omega^h_s}, |\alpha v^h - \alpha w^h|_{\Omega^h_{r,0}}\}.$$
Since $|v_h|_{\Omega_{r,0}}^2 \leq \sum_{x \in \Omega_{r,0}} |v_h(x)|^2$, we obtain
\begin{equation}
|v_h|_{\Omega_{r,0}} \leq h^{-\frac{2}{n}} \|v_h\|_{0,h} \leq \frac{h^{-\frac{2}{n}}}{C_p} |v^h|_{1,h}.
\end{equation}

It follows from the results of [2] that the operator $S|_{\Omega_{r,0}}$ is a strict contraction in a rescaled ball and maps the rescaled ball in itself. In other words, there exists $\nu, \rho > 0$ and a constant $a_2, 0 < a_2 < 1$, which depends on $h$ such that for $v^h, w^h \in B_\rho(r_h(u))$,
\begin{equation}
|S(\alpha v^h) - S(\alpha w^h)|_{1,h} \leq a_2 |\alpha v^h - \alpha w^h|_{h},
\end{equation}
for $\rho = Ch^{2+n/2}$ and $h$ sufficiently small.
Moreover, for $v^h \in B_\rho(r_h(u))$ we have
\begin{equation}
|S(\alpha v^h) - \alpha r_h(u)|_{1,h} \leq \alpha \rho.
\end{equation}

Using the definition of the hybrid norm (2.14) on $M(\Omega^h)$, we easily obtain
\begin{equation}
|S(\alpha v^h) - S(\alpha w^h)|_h \leq \max\{a_1, a_2\} |\alpha v^h - \alpha w^h|_h.
\end{equation}
The existence of a unique local solution to (2.15) then follows from the Banach fixed point theorem. For the choice of an initial guess, we recall that the nine-point finite difference approximation to the solution of a Poisson equation produces an approximation of order 4.

4. Convergence to the viscosity solution of the hybrid discretization

We recall that we follow the usual convention of denoting constants by $C$ but will occasionally index some constants.

We first prove the stability of the hybrid discretization (2.15). Then we prove that the half-relaxed limits
\begin{align*}
u^*(x) &= \limsup_{y \to x, h \to 0} v^h(y) = \limsup_{\delta \to 0} \{ v^h(y), y \in \Omega^h_0, |y - x| \leq \delta, 0 < h \leq \delta \} \\
u_*(x) &= \liminf_{y \to x, h \to 0} v^h(y) = \liminf_{\delta \to 0} \{ v^h(y), y \in \Omega^h_0, |y - x| \leq \delta, 0 < h \leq \delta \},
\end{align*}
are respectively sub and super solutions of (2.2).

4.1. Stability on the set of singular points. Let $x \in \Omega^h_0$ and let $\nu$ be a vector such that both $x + \nu h$ and $x - \nu h$ are in $\Omega^h_0$. For $\phi(x) = 1/2|x|^2$, the second order directional derivative
\begin{equation}
D_{\nu\nu}\phi(x) = \frac{1}{|\nu|^2 h^2} (\phi(x + \nu h) - 2\phi(x) + \phi(x - \nu h)),
\end{equation}
is a constant equal to 1. It follows that $M^+(r_h(\phi))$ is a constant. And since we assumed that $f$ is continuous, $f$ is bounded on $\overline{\Omega}$ and hence by (2.13), $|F_h(r_h \phi)| = \cdots$
\[ | - M^+_s(r_h(\phi))(x) + r_h(f)(x) | \] is bounded by a constant which does not depend on \( h \) and \( x \). Since \( u^h \) is a fixed point of \( S \), by (3.1) and the boundedness of the domain

\[
|u^h|_{\Omega^h} = |S(u^h)|_{\Omega^h} \leq |S(u^h) - S(r_h(\phi))|_{\Omega^h} + |S(r_h(\phi))|_{\Omega^h} \\
\leq a_1|u^h - r_h(\phi)|_{\Omega^h} + |r_h(\phi)|_{\Omega^h} + \nu_1|F_h(r_h(\phi))|_{\Omega^h} \\
\leq a_1|u^h - r_h(\phi)|_{\Omega^h} + C,
\]

for a constant \( C \). But

\[
|u^h - r_h(\phi)|_{\Omega^h} = \max\{ |u^h - r_h(\phi)|_{\Omega^h_0}, |u^h - r_h(\phi)|_{\Omega^h_{r,0}} \}.
\]

Since \( 0 < a_1 < 1 \), by triangular inequality, we have for some constant \( C \)

\[
(4.1) \quad \text{either } |u^h|_{\Omega^h_0} \leq C \quad \text{or} \quad |u^h|_{\Omega^h} \leq |u^h|_{\Omega^h_{r,0}} + C.
\]

4.2. Stability on the set of regular points. Since \( \rho = Ch^{2+n/2} \), we have by (3.2) and assuming without loss of generality that \( h \leq 1 \)

\[
(4.2) \quad |u^h - r_h(u)|_{\Omega^h_{r,\alpha}} \leq Ch^{-\frac{n}{2}} |u^h - r_h(u)|_{1,h} \leq Ch^2 \leq C.
\]

Thus for \( h \) sufficiently small,

\[
(4.3) \quad |u^h|_{\Omega^h_{r,0}} \leq |u^h - r_h(u)|_{\Omega^h_{r,0}} + |r_h(u)|_{\Omega^h_{r,0}} \leq C,
\]

since by definition \( u \in C(\overline{\Omega}) \).

Inequalities (4.1) and (4.3) allow us to state the following theorem

**Theorem 4.1.** There is a constant \( C > 0 \) independent of \( h \) such that for \( h \) sufficiently small, the solution \( u^h \) of (2.15) satisfies \( |u^h|_{\Omega^h} \leq C \).

4.3. Sub and super solution property of the half-relaxed limits. Theorem 4.1 implies that the half-relaxed limits are well defined. We have

**Theorem 4.2.** The upper half-relaxed limit \( u^* \) is a viscosity sub solution of \( \det D^2 u(x) = f(x) \) and the lower half-relaxed limit \( u_* \) is a viscosity super solution of \( \det D^2 u(x) = f(x) \) at every point of \( \Omega \setminus \Omega_r \). In addition, \( u^* \) is a viscosity solution of \(-\lambda_1[u](x) \leq 0 \) at every point of \( \Omega \setminus \Omega_r \).

**Proof.** The result follows from the results of [7] and the stability, consistency and monotonicity of the scheme used in the "singular" part of the domain. For the convenience of the reader, we give a proof following [10].

We show that \( u_* \) is a viscosity super solution of \( \det D^2 u(x) = f(x) \) at every point of \( \Omega \setminus \Omega_r \). The corresponding result for \( u^* \) is proved similarly. This also proves that \( u^* \) is a viscosity solution of \(-\lambda_1[u](x) \leq 0 \)

It follows from the definitions that \( u_* \) is lower semi-continuous. Let \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) with \( D^2 \phi(x_0) \geq 0 \) such that \( u_* - \phi \) has a local minimum at \( x_0 \) with \((u_* - \phi)(x_0) = 0\). Without loss of generality, we may assume that \( x_0 \) is a strict local minimum.
Let $B_0$ denote a closed ball contained in $\Omega$ and containing $x_0$ in its interior. We let $x_l$ be a sequence in $B_0$ such that $x_l \to x_0$ and $u^{h_l}(x_l) \to u^*(x_0)$ and let $x_l'$ be defined by

$$c_l := (u^{h_l} - \phi)(x_l') = \min_{B_0} u^{h_l} - \phi.$$

Since the sequence $x_l'$ is bounded, it converges to some $x_1$ after possibly passing to a subsequence. Since $(u^{h_l} - \phi)(x_l') \leq (u^{h_l} - \phi)(x_l)$ we have

$$(u_\ast - \phi)(x_0) = \lim_{l \to \infty} (u^{h_l} - \phi)(x_l) \geq \lim inf_{l \to \infty} (u^{h_l} - \phi)(x_l') \geq (u_\ast - \phi)(x_1).$$

Since $x_0$ is a strict minimizer of the difference $u_\ast - \phi$, we conclude that $x_0 = x_1$ and $c_l \to 0$ as $l \to \infty$.

By definition

$$u^h(x) \geq \phi(x) + c_l, \forall x \in B_0,$$

and thus, by the monotonicity of the scheme

$$0 = \tilde{F}(u^h(x_0), u^h(y)|y \neq x_0) \geq \tilde{F}(u^h(x_0), (\phi(y) + c_l)|y \neq x_0),$$

which gives by the consistency of the scheme $\det D^2 \phi(x_0) - f(x_0) \leq 0$ and $-\lambda_1[\phi](x_0) \leq 0$.

Similarly one shows that if $\phi \in C^2(\Omega)$ and $u^* - \phi$ has a local maximum at $x_0$ with $(u_\ast - \phi)(x_0) = 0$, we have $\det D^2 \phi(x_0) - f(x_0) \geq 0$ and thus $-\lambda_1[\phi](x_0) \leq 0$. □

For the behavior at regular points, we have

**Theorem 4.3.** At every regular point $x \in \Omega$,

$$u_\ast(x) = u^*(x) = u(x).$$

And thus $u_\ast$ and $u^*$ are viscosity solutions of (2.2) at $x \in \Omega_r$.

**Proof.** By (4.2), $u^h$ converges to $u$ uniformly on compact subsets of $\Omega_r$. The result then follows since $u$ is $C^2$ at $x$. □

We close this section by stating the main result of this paper

**Theorem 4.4.** The solution $u^h$ of (2.15) converges uniformly on compact subsets to the unique solution of (2.2).

**Proof.** First using the definitions we have $u_\ast \leq u^*$. Second, by the comparison principle (recalled in section 2.2) and Theorems 4.2 and 4.3, we have $u_\ast \geq u^*$. Hence $u_\ast = u^*$ is the unique viscosity solution of (2.2). By uniqueness of the viscosity solution $u_\ast = u^* = u$ and hence $u^h$ converges uniformly on compact subsets to $u$ by [6, Lemma 1.9 p. 290]. □

**Remark 4.5.** As pointed out in the introduction, the analysis of this paper extends immediately to the case where one uses central difference approximations to approximate the Hessian. One only needs to modify the definition of the operator $M_r$, and replace $\Delta h$ in the definition of the operator $L_r$ by the linear elliptic operator used in the proof of the quadratic convergence rate of the discretization [1]. The analysis is easier since one can now use the maximum norm for the hybrid discretization.
5. Numerical experiments

We consider the two dimensional Monge-Ampère equation and take as computational domain the unit square $[0,1]^2$. We use the 9 point monotone scheme at interior mesh points within a distance $h$ of the boundary and the compatible standard discretization

$$\frac{1}{2} \text{div}_h[(\text{cof sym } H^h_d u^h)^T D_h u^h] = r_h(f),$$

in the interior of the domain.

To illustrate the performance of our hybrid discretization, we consider two test functions, for viscosity solutions, which are both discussed in [19].

**Test 1**: $C^1(\Omega)$ solution not in $C^2(\Omega)$ with $f(x) = 0$ at some points on the boundary and at some interior points which do not align with the grid.

$$u(x,y) = \frac{1}{2}(\max(\sqrt{x^2 + y^2} - 1, 0))^2 \text{ and } f(x,y) = \max\left(1 - \frac{1}{\sqrt{x^2 + y^2}}, 0\right).$$

**Test 2**: Solution with a singularity on the boundary, $u(x,y) = -\sqrt{2 - x^2 - y^2}$ and $f(x,y) = 2/(2 - x^2 - y^2)^2$.

5.1. Results with a time marching method. The nonlinear system is first solved iteratively by the time marching method

$$-\nu \Delta u_{k+1}^h = -\nu \Delta u_k^h + F_h(u_k^h)(x) \text{ in } \Omega^h_0, u_{k+1}^h(x) = r_h(g)(x) \text{ on } \partial \Omega^h,$$

with $\nu > 0$ given and $F_h$ defined in (2.13).

The iterative method (5.1) corresponds to an explicit forward Euler discretization of the pseudo time equation

$$\frac{\partial \Delta u^h(x)}{\partial t} = F_h(u^h)(x),$$

hence the name time marching method. The iterative method (2.5) is also a time marching method but without the Laplacian term and is significantly slower than (5.1).

For $F_h$ encoding the compatible standard discretization in the whole domain, the convergence of the iterative method (5.1) is given in [2] and was used in the paper, combined with (2.5) for the monotone scheme, as part of the proof of convergence of the hybrid discretization. For $F_h$ encoding a monotone discretization in the whole domain, the convergence of the iterative method (5.1) is given in [5] where a strict contraction property is proven. Its convergence for the hybrid discretization then follows from the arguments of this paper. One only need to modify the definition of the operator $L_h$ to have $L_h v^h(x) = \Delta_h^{-1} v^h(x), x \in \Omega^h_s$. The corresponding strict contraction property for the proof of Lemma 3.1 is given in [5].

Let $H_d v^h$ denote the discrete version of the Hessian matrix with central finite difference approximations. For the choice of the initial guess $u^h_0$, we first get the finite difference approximation $v^h_0$ of the solution of $\Delta u = 2\sqrt{f}$ with boundary condition $u = g$. We then use the time marching method

$$-\nu \Delta v_{k+1}^h = -\nu \Delta v_k^h + \det H_d v_k^h(x) - r_h(f)(x) \text{ in } \Omega^h_0, v_{k+1}^h(x) = r_h(g)(x) \text{ on } \partial \Omega^h.$$

The maximum number of iterations for the latter was set at 10 000. All iterations were stopped when the $L^\infty$ norm of the difference between two iterates is less than
\[
\begin{array}{c|cccc}
\nu & 1/2^5 & 1/2^6 & 1/2^7 & 1/2^8 \\
150 & 3.60 \times 10^{-3} & 2.86 \times 10^{-3} & 5.92 \times 10^{-4} & 9.30 \times 10^{-5} \\
\end{array}
\]

Table 1. Maximum errors for Test 1 with nonlinear system solved by (5.1)

\[
\begin{array}{c|cccc}
\nu & 1/2^2 & 1/2^3 & 1/2^4 & 1/2^5 & 1/2^6 & 1/2^7 & 1/2^8 \\
150 & 1.34 \times 10^{-2} & 7.57 \times 10^{-3} & 4.55 \times 10^{-3} & 2.94 \times 10^{-3} & 1.91 \times 10^{-3} & 1.27 \times 10^{-3} & 8.57 \times 10^{-4} \\
850 & & & & & & & \\
2050 & & & & & & & \\
\end{array}
\]

Table 2. Maximum errors for Test 2 with nonlinear system solved by (5.1)

\[
\begin{array}{c|cccc}
\nu & 1/2^5 & 1/2^6 & 1/2^7 & 1/2^8 \\
150 & 3034 & 7945 & 6002 & 2442 \\
\end{array}
\]

Table 3. Iteration counts for Test 1 with nonlinear system solved by (5.1)

\[
\begin{array}{c|cccc}
\nu & 1/2^2 & 1/2^3 & 1/2^4 & 1/2^5 & 1/2^6 & 1/2^7 & 1/2^8 \\
150 & 2060 & 45 & 22 & 11 & 5 & & \\
850 & & & & & & & \\
2050 & & & & & & & 27 & 10 \\
\end{array}
\]

Table 4. Iteration counts for Test 2 with nonlinear system solved by (5.1)

10^{-10} or when that value is bigger than the one previously computed. The parameter \( \nu \) is chosen adaptively. Starting with the value \( \nu = 50 \), it is increased if the accuracy of the numerical solution is deemed not good. Numerical errors are in the maximum norm.

Computation times may be reduced by reducing the number of iterations in the preprocessing step with (5.2), decreasing the value of \( \nu \) and by stopping the iterations when the \( L^\infty \) norm of the difference between two iterates is less than \( 10^{-a} \) for a much smaller than 10. Although, for the non smooth solutions considered here, (5.1) is slower than Newton’s method, the time marching method (5.1) is significantly easier to implement. Moreover for both test functions, the results we obtain are more accurate than the ones presented in [20] (the test function of Test 1 belongs to a two-parameter family and different parameters were used in [20]. We obtained similar results with the parameters used there). As pointed out in the introduction, the numerical experiments go beyond the theory presented since it appears not necessary, with the time marching method (5.1), to use a monotone discretization at interior points where \( f(x) = 0 \) or non smooth. Also, no interpolation or wide stencil is used. The accuracy of the discretization for the test function of Test 2 may be increased by using the standard compatible discretization on the whole domain as in [2], or by
using a wide stencil only near the point $(1, 1)$ of the boundary as in [20]. The latter requires the use of interpolation, see [20].

5.2. Results with Newton’s method. To solve (2.15) with a Newton type algorithm requires a significant amount of regularization. First, as in [20], we replace the operator $\max(x, 0)$ in the definition of $M^+_h$ by $\max(x, 10^{-2})$. Second, we regularize the operator $M^+_h$ using the following regularizations of the max and min operators

$$
\delta \max(a, b) = \frac{1}{2}(a + b + \sqrt{(a - b)^2 + \delta^2})
$$

$$
\delta \min(a, b) = \frac{1}{2}(a + b - \sqrt{(a - b)^2 + \delta^2}),
$$

where $\delta > 0$ is a small parameter. For Test 1 we take $\delta = 10^{-1/2}$ and for Test 2, $\delta = 10^{-1/4}$. The initial guesses for both test cases are chosen as in section 5.1 with the same values of $\nu$. Finally we had to use the 9-point stencil monotone scheme at interior mesh points $x$ where $|f(x)| \leq 10^{-6}$ or $|f(x)| \geq 10^b$. For Test 1 we take $b = 2$ and for Test 2, $b = 1$.

The nonlinear system (2.15) is solved by a damped Newton’s method

$$
F'_h(u^h_k)(u^h_{k+1} - u^h_k) = -\frac{1}{\beta_k} F_h(u^h_k).
$$

For Test 1 we took $\beta_k = 2^{r_k}$ where $r_k$ is the smallest integer such that the $||F_h(u^h_k + 1/2^{r_k} \gamma_k)||_2 < ||F_h(u^h_k)||_2$. Here $\gamma_k$ solves $F'_h(u^h_k)\gamma_k = -F_h(u^h_k)$ and $||.||_2$ denotes the Euclidian norm. For Test 2 we simply took $\beta_k = 10^2$.

In Tables 7 and 8, $n_{it}$ refers to the iteration counts.

Computations were performed in MATLAB on a 2.5 GHz MacBook Pro. The finite difference code used to produce the results is available upon request.

Acknowledgements

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Table 7. Newton’s method Test 1

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Table 8. Newton’s method Test 2

REFERENCES