A VARIATIONAL METHOD FOR COMPUTING NUMERICAL SOLUTIONS OF THE MONGE-AMPÈRE EQUATION

GERARD AWANOU AND LEOPOLD MATAMBA MESSI

Abstract. We present a numerical method for solving the Monge-Ampère equation based on the characterization of the solution of the Dirichlet problem as the minimizer of a convex functional of the gradient and under convexity and nonlinear constraints. When the equation is discretized with a certain monotone scheme, we prove that the unique minimizer of the discrete problem solves the finite difference equation. For the numerical results we use both the standard finite difference discretization and the monotone scheme. Results with standard tests confirm that the numerical approximations converge to the Aleksandrov solution.

Introduction

We are interested in the numerical resolution of the Monge-Ampère equation

$$\det \nabla^2 u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{0.1}$$

where the domain $\Omega \subset \mathbb{R}^n$ is assumed to be bounded and convex with boundary $\partial\Omega$. For a smooth function u, $\nabla^2 u$ is the Hessian of u with entries $(\partial^2 u)/(\partial x_i \partial x_j), i, j = 1, \ldots, n$. In general the expression $\det \nabla^2 u$ should be interpreted in the sense of Aleksandrov. The functions f and g are given with $f \geq 0$. We make the assumption that $f \in L^1(\Omega)$ and $g \in C^{0,1}(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex on Ω.

We present a numerical method based on a remark of P.L. Lions [16] which asserts that the unique $C^{0,1}(\overline{\Omega})$ solution of (0.1) in the sense of Aleksandrov is the unique minimizer of

$$J(w) = \int_\Omega \Phi(\nabla w)\, dx,$$

over the convex set

$$S = \big\{ v \in C(\overline{\Omega}),\ v = g \text{ on } \partial\Omega, v \text{ convex on } \Omega,\ (\det \nabla^2 v)^{\frac{1}{n}} \geq f^{\frac{1}{n}} \text{ in } \Omega \big\}, \tag{0.2}$$

where Φ is a nonnegative strictly convex function satisfying (1.1).

Our strategy consists in reproducing the above principle by solving discrete versions of the convex program

$$\arg \min_{v \in S} J(v). \tag{0.3}$$

We consider in this paper two notions of discrete convexity: the first one which we will refer to as local discrete convexity requires a certain discrete Hessian to be positive. The second one, recently used in [11], will be referred to as wide stencil convexity and requires to enforce the convexity of the mesh function approximately.

Given a discretization of the functional J, we consider two possible discrete counterparts of the convex set S corresponding to different approximations of $\det \nabla^2 u$. If the determinant

This research was supported in part by NSF grant DMS-1319640 to Gerard Awanou and NSF grant DMS-0931642 to the Mathematical Biosciences Institute.

operator is discretized with the monotone scheme of [11], and if one uses wide stencil convexity, we prove that the unique minimizer of the corresponding discrete optimization problem solves the associated finite difference version of (0.1). However, uniqueness of the solution of the latter finite difference problem remains an open question. We present numerical evidence of convergence to the Aleksandrov solution for a test case for which solving directly the nonlinear finite difference equations does not give satisfactory results. We do not address in this paper the convergence of the discretization proposed in [11] to the Aleksandrov solution and refer to [5].

We also consider the discrete version of the convex set S obtained from standard finite difference approximations of $\det \nabla^2 v$ and local discrete convexity. We prove the existence of the solution of the resulting discrete convex program under the assumption that a solution of the standard finite difference equations exists. The existence of a solution to the finite difference equations in this case is obtained in [2] under the assumption that (0.1) has a smooth solution. Furthermore, using a weak convergence result proved in [5], we prove convergence of minimizers when (0.1) has a smooth solution. We present numerical evidence of convergence to the Aleksandrov solution. Our results can be combined with the approach in [4] to give a convergence result for Aleksandrov solutions.

The unknown u in (0.1) is a convex function which may not be smooth even if the data are smooth. Approximating the appropriate weak solution and preserving numerically the convexity property had posed great challenges for the numerical resolution of (0.1) in the context of standard discretizations. In this paper, we take a direct approach by including explicitly the convexity constraints in an optimization framework. The notion of viscosity solution and that of Aleksandrov solution are the best known notions of weak solution for (0.1). They are equivalent for $f > 0$ and continuous [14]. We refer to [17] and the references therein for the precise definition of the concept of Aleksandrov solution for (0.1).

The convexity of the set S in (0.2) follows from Minkowski determinant theorem. See for example [18, Theorem G, p. 205] for smooth functions and [17, Proposition 3.3] for a procedure for the general case using approximation by smooth functions. We recall that for a convex function v on a convex domain Ω, Rademeister theorem [19, Theorem 19 p. 13] states that the set of points at which v fails to be differentiable has zero Lebesgue measure and that its gradient is continuous on the set of points where it exists. Thus the functional J is well defined on S. In (0.2), $\det \nabla^2 v$ denotes the Monge-Ampère measure associated to the convex function v, [14]. We use the notation $C^{0,1}(\overline{\Omega})$ for the set of Lipschitz continuous functions on $\overline{\Omega}$. Under our assumptions on f and g, it can be shown that the Aleksandrov solution of (0.1) is the maximal element of S when the domain is convex and not necessarily strictly convex [15]. Then the arguments in [16] extend to the case where Ω is assumed to be only convex.

Relation with a method of Dean and Glowinski. The optimization approach we take has some similarity with an optimization approach proposed by Dean and Glowinski [10]. See also [8]. They proposed in [10] to solve (0.1) by minimizing

$$L_1(w) = \int_\Omega (\Delta w)^2 \, dx,$$

over

$$E = \{ \, v \in H^2(\Omega), \, v = g \text{ on } \partial\Omega, \det \nabla^2 v = f \text{ in } \Omega \}.$$

An equivalent mixed formulation was used with additional unknown $q = \nabla^2 u$ and then solved by an augmented Lagrangian algorithm. In the mixed formulation, the functional

$L(w,q)=\int_\Omega |\Delta w|^2$ is minimized over the set

$$E'=\{(v,q)\in H^2(\Omega)\times L^2(\Omega,\mathbb{S}), v=g \text{ on } \partial\Omega, q=\nabla^2 v, \det q = f\},$$

where $\mathbb{S}$ is the space of 2×2 symmetric matrices and $L^2(\Omega,\mathbb{S})$ is the space of symmetric matrix fields with components in $L^2(\Omega)$. In the case f is unbounded the results were not satisfactory. As remarked in [7], the convergence of their method, even for smooth solutions, is still an open problem.

It will be seen that if one replaces the functional L_1 by

$$L_2(w)=\int_\Omega |\nabla w|^2 \, dx,$$

and the set E by the convex set S defined in (0.2), one obtains a method of the type discussed in this paper and the difficulties indicated for the method of Dean and Glowinski disappear. One of the main features of the method we propose is that the convexity constraint in (0.2) has been carefully taken into account.

Organization of the paper. The paper is organized as follows: in the next section, we introduce some notation and analyze the discrete optimization problem with a standard discretization of $\det \nabla^2 u$ and local discrete convexity. We then prove that when the equation is discretized with the monotone scheme of [11], the unique minimizer of the discrete problem solves the finite difference equation. Section 2 is devoted to numerical results.

1. Discrete optimization problem

We start with some notations needed to state the discrete optimization problem. All the functions we consider take finite values. We make the usual convention of denoting constants by the letter C. We make the following assumptions on the nonnegative convex function Φ

(1.1a) $$\Phi\in C^2(\mathbb{R}^n)$$

(1.1b) $$|\nabla\Phi(p)|\leq C+C|p|$$

(1.1c) $$\exists\, \nu>0 \text{ such that } \langle \nabla\Phi(p)-\nabla\Phi(q), p-q\rangle \geq \nu |p-q|^2, \quad \forall p,q\in\mathbb{R}^n$$

where $\langle , \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$ and $|.|$ denotes the associated norm. Without loss of generality, we will assume, for the analysis in this paper, that Φ is strictly convex. For example, one may take $\Phi(p)=|p|^2$. The results in [16] hold for more general convex functions Φ and we also give numerical results for functions Φ not satisfying the above assumptions.

1.1. **Notation and preliminaries.** Most of the notation is taken from [1]. Let $\Omega = (0,1)^n, h=1/N, N\in\mathbb{N}$ and let $\mathcal{M}_h$ denote the mesh which consists of points $x=hz\in \mathbb{R}^n\cap\overline{\Omega}, z\in\mathbb{Z}^n$. Put

$$\mathcal{M}_h^\circ=\mathcal{M}_h\cap\Omega, \quad \partial\mathcal{M}_h=\mathcal{M}_h\cap\partial\Omega,$$

and denote by $\mathcal{U}_h$ the set of real valued functions defined on $\mathcal{M}_h$. Without loss of generality, for a function v defined on $\overline{\Omega}$, we use the notation v for the restriction of v to $\mathcal{M}_h$.

Definition 1.1. *If u is defined on Ω and $u_h\in\mathcal{U}_h$, we say that u_h converges uniformly to u on a compact subset K of Ω if and only if*

$$\max_{x\in\mathcal{M}_h^\circ\cap K} |u_h(x)-u(x)|\to 0, \ \textit{as } h\to 0.$$

We denote by $e_i, i = 1, \ldots, n$ the i-th coordinate unit vector of $\mathbb{R}^n$ and consider first order difference operators defined on $\mathcal{M}_h^\circ$ by

$$\partial_+^i v_h(x) := \frac{v_h(x + he_i) - v_h(x)}{h}$$
$$\partial_-^i v_h(x) := \frac{v_h(x) - v_h(x - he_i)}{h}$$
$$\partial_h^i v_h(x) := \frac{v_h(x + he_i) - v_h(x - he_i)}{2h}.$$

We consider the following second-order difference operators on mesh functions

$$\partial_+^i \partial_-^i v_h(x) = \frac{v_h(x + he_i) - 2v_h(x) + v_h(x - he_i)}{h^2}, \tag{1.2}$$

and

$$\begin{aligned} \partial^i \partial^j v_h(x) = \frac{1}{4h^2} \Big\{ & v_h(x + he^i + he^j) + v_h(x - he^i - he^j) \\ & - v_h(x + he^i - he^j) - v_h(x - he^i + he^j) \Big\}, i \neq j. \end{aligned} \tag{1.3}$$

We use the notation $A = (a_{ij})_{1 \leq i,j \leq n}$ to denote the matrix A with entries a_{ij}. The Hessian of a mesh function v_h is the discrete matrix field $H[v_h] = \big(H[v_h]_{ij}\big)_{1 \leq i,j \leq n}$ with entries the mesh functions

$$H[v_h]_{ij}(x) = \begin{cases} \partial_+^i \partial_-^i v_h(x) & \text{if } i = j, \\ \partial^i \partial^j v_h(x) & \text{otherwise.} \end{cases}$$

Definition 1.2. *We say that a mesh function v_h is (locally) discrete convex if $H[v_h](x)$ is positive semidefinite for all $x \in \mathcal{M}_h^\circ$. The mesh function v_h is (locally) strictly discrete convex if $H[v_h](x)$ is positive definite for all $x \in \mathcal{M}_h^\circ$.*

We endow the space of mesh functions defined on $\mathcal{M}_h$ with the following norms and semi-norms.

$$\begin{aligned} |v_h|_{0,\infty} &= \max_{x \in \mathcal{M}_h^\circ} |v_h(x)|, \\ |v_h|_{1,\infty} &= \max\{ |\partial_+^i v_h|_{0,\infty}, i = 1, \ldots, n \}, \\ |v_h|_{2,\infty} &= \max\{ |\partial_+^i \partial_-^i v_h|_{1,\infty}, |\partial^i \partial^j v_h|_{1,\infty}, i, j = 1, \ldots, n \}, \\ ||v_h||_{2,\infty} &= \max\{ |v_h|_{0,\infty}, |v_h|_{1,\infty}, |v_h|_{2,\infty} \}. \end{aligned}$$

Analogues of the Sobolev norms and semi-norms can be defined on $\mathcal{M}_h$. For $v_h \in \mathcal{U}_h$ we define

$$||v_h||_0 = \Big(h^n \sum_{x \in \mathcal{M}_h^\circ} v_h(x)^2 \Big)^{\frac{1}{2}},$$

and

$$|v_h|_1 = \Big(\sum_{i=1}^n ||\partial_+^i v_h||_0^2 \Big)^{\frac{1}{2}}.$$

We have the discrete Poincare's inequality, see for example [9, Lemma 3.1]

Lemma 1.3. *There exists a constant $C > 0$ independent of h such that*

$$|v_h|_1 \geq C ||v_h||_0,$$

for $v_h = 0$ on $\partial\mathcal{M}_h$.

1.2. **Discretization of the functional J.** We recall that for a convex function v on Ω, the one-sided partial derivatives are defined everywhere and denote by $\nabla_- v$ the left-hand side gradient of v. Since the convex function v is differentiable almost everywhere, we have

$$J(w) = \int_\Omega \Phi(\nabla_- w)\, dx. \tag{1.4}$$

We now use the latter expression of $J(w)$ to construct a discrete analogue of $J(w)$ using Riemann sums.

For $x = (x_1, \ldots, x_n) \in \mathcal{M}_h$, we define

$$P_x = \{\, y \in \bar{\Omega}, x_i - h \leq y_i \leq x_i \,\}.$$

It is easy to see that P_x is nonempty if and only if either $\operatorname{dist}(x, \partial\Omega) < h\sqrt{n}$ or $\operatorname{dist}(x - h, \partial\Omega) < h\sqrt{n}$. Furthermore, we have $\cup_{x\in\mathcal{M}_h} P_x = \bar{\Omega}$ and for $x \neq y, P_x \cap P_y$ is a set of Lebesgue measure 0. For $x \in \mathcal{M}_h$, we denote by $|P_x|$ the Lebesgue measure of P_x, i.e. $|P_x| = h^n$ if $P_x \cap \Omega \neq \emptyset$. We use the notation $x - h/2$ to denote the center of P_x that is, the point obtained by subtracting $h/2$ from each coordinate x_i of x.

We define for $v_h \in \mathcal{U}_h$ and a convex real-valued function Φ on $\mathbb{R}^n$

$$J_h(v_h) = \sum_{x\in\mathcal{M}_h} |P_x| \Phi(\nabla_h v_h(x)),$$

where $\nabla_h v_h$ is the vector of backward finite differences of the mesh function v_h, i.e.

$$\nabla_h v_h(x) = (\partial_-^i v_h(x))_{i=1,\ldots,n}.$$

We assume that the sum in the definition of $J_h(v_h)$ is over the set of mesh points at which $\nabla_h v_h(x)$ is defined. We note that for all i, $\partial_-^i v_h$ extends to a piecewise constant function on Ω, denoted also by $\partial_-^i v_h$ and taking the constant value $\partial_-^i v_h(x)$ on P_x for $x \in \mathcal{M}_h$.

Lemma 1.4. *Let $v \in C^2(\overline{\Omega})$ and v_h a family of mesh functions. We have*

$$\max_{x\in\mathcal{M}_h} \left| \nabla_h v_h(x) - \nabla_- v(x) \right| \to 0 \; as\, h \to 0 \; implies\, J_h(v_h) \to J(v) \; as\, h \to 0. \tag{1.5}$$

Proof. For $v \in C^2(\overline{\Omega})$, $\nabla v(x)$ is defined, is equal to $\nabla_- v(x)$ and is uniformly bounded on $\overline{\Omega}$. We have by definition of the integral

$$J(v) = \lim_{h\to 0} \sum_{x\in\mathcal{M}_h} |P_x| \Phi\Big(\nabla_- v\big(x - \frac{h}{2}\big) \Big). \tag{1.6}$$

Moreover

$$\sum_{x\in\mathcal{M}_h} |P_x| \Phi\Big(\nabla_- v\big(x - \frac{h}{2}\big) \Big) - J_h(v_h) = h^n \sum_{x\in\mathcal{M}_h} \Big(\Phi\Big(\nabla_- v\big(x - \frac{h}{2}\big) \Big) - \Phi(\nabla_h v_h(x)) \Big).$$

And so by the C^1 continuity of Φ and the mean value theorem

$$\begin{aligned}\left| h^n \sum_{x\in\mathcal{M}_h} \Phi\Big(\nabla_- v\big(x - \frac{h}{2}\big) \Big) - J_h(v_h) \right| &\leq C \max_{x\in\mathcal{M}_h} \left| \Phi\Big(\nabla_- v\big(x - \frac{h}{2}\big) \Big) - \Phi(\nabla_h v_h(x)) \right| \\ &\leq C \max_{x\in\mathcal{M}_h} \left| \nabla_- v\big(x - \frac{h}{2}\big) - \nabla_h v_h(x) \right|.\end{aligned}$$

On the other hand

$$\Big|\nabla_- v\big(x - \frac{h}{2}\big) - \nabla_h v_h(x)\Big| \leq \Big|\nabla_- v\big(x - \frac{h}{2}\big) - \nabla_- v(x)\Big| + \Big|\nabla_- v(x) - \nabla_h v_h(x)\Big|.$$

By assumption $\max_{x \in \mathcal{M}_h} |\nabla_h v_h(x) - \nabla_- v(x)| \to 0 \, \text{as} \, h \to 0$. By the C^1 continuity of v, $\Big|\nabla_- v\big(x - \frac{h}{2}\big) - \nabla_- v(x)\Big| \to 0$ as $h \to 0$. Thus using (1.4) and (1.6), we obtain $J_h(v_h) \to J(v)$ as $h \to 0$. □

Remark 1.5. *If $v \in C^2(\overline{\Omega})$, we have for all $i = 1, \ldots, n$ and $x \in \mathcal{M}_h$ by a Taylor series expansion*

$$\Big|\frac{\partial v}{\partial x_i}(x) - \partial_-^i v(x)\Big| \leq Ch,$$

where C depends only on the maximum of $\partial^2 v/(\partial x_i^2)$ on $\overline{\Omega}$. Moreover

$$\Big|\frac{\partial v}{\partial x_i}(x - \frac{h}{2}) - \partial_-^i v(x)\Big| \leq \Big|\partial_-^i v(x) - \frac{\partial v}{\partial x_i}(x)\Big| + \Big|\frac{\partial v}{\partial x_i}(x) - \frac{\partial v}{\partial x_i}(x - \frac{h}{2})\Big|.$$

Using the C^1 continuity of $\partial v/\partial x_i$, we obtain that the condition $\max_{x \in \mathcal{M}_h} |\nabla_h v(x) - \nabla_- v(x - h/2)| \to 0$ holds for $v \in C^2(\overline{\Omega})$ and v_h the mesh function obtained by restriction of v onto the set $\mathcal{M}_h$.

1.3. **Standard discretization for** $\det \nabla^2 u$ **and local discrete convexity.** We seek to minimize J_h over the following discrete counterpart of the set S defined in (0.2)

(1.7)

$$S_h = \{\, v_h \in \mathcal{U}_h, v_h = g \text{ on } \partial\mathcal{M}_h, \ H[v_h](x) \geq 0 \text{ and } (\det H[v_h](x))^{\frac{1}{n}} \geq f(x)^{\frac{1}{n}} \text{ for all } x \in \mathcal{M}_h^\circ \,\}.$$

This amounts to minimizing a strictly convex functional over the convex set S_h. Thus the main difficulty we face is to show that the set S_h is nonempty. Our approach for proving that the set S_h is nonempty is to prove the existence of a solution of the finite difference system

$$\begin{aligned} \det H[\hat{u}_h](x) &= f(x), x \in \mathcal{M}_h^\circ \\ H[\hat{u}_h](x) &\geq 0, x \in \mathcal{M}_h^\circ \\ \hat{u}_h &= g \, \text{on} \, \partial\mathcal{M}_h. \end{aligned} \tag{1.8}$$

The idea has been used in [3]. Let us assume that $0 < c_0 \leq f \leq c_1$ for constants c_0 and c_1. It is shown in [2, Theorem 3.4] that if $u \in C^4(\overline{\Omega})$ is a solution of (0.1), then the finite difference system (1.8) has a unique solution $\hat{u}_h$ in the ball

$$B_\rho(u) = \{\, v_h \in \mathcal{U}_h, ||v_h - u||_{2,\infty} \leq \rho \,\},$$

for

$$\rho = C||u||_{C^4(\Omega)} h^2,$$

and a constant C.

Proposition 1.6. *Suppose that the convex solution u of* (0.1) *is in $C^4(\overline{\Omega})$ and strictly convex. Then the functional J_h has a unique minimizer u_h in S_h and $\inf_h J_h(u_h) \leq J(u)$.*

Proof. It is shown in [2, Theorem 3.4] that there exists $\hat{u}_h \in \mathcal{U}_h$ which solves (1.8). It follows that $\hat{u}_h \in S_h$ and thus S_h is nonempty. By the strict convexity of J_h we conclude that J_h has a unique minimizer u_h in S_h. Since $||\hat{u}_h - u||_{2,\infty} \leq Ch^2$, we have $\max_{x \in \mathcal{M}_h^\circ} |\nabla_h \hat{u}_h(x) - \nabla u(x)| \to 0$ as $h \to 0$. Recall that $\hat{u}_h(x) = u(x)$ on $\partial\mathcal{M}_h$; therefore for $x \in \partial\mathcal{M}_h$ with

$\nabla_h \hat{u}_h(x)$ defined, we have

$$\begin{aligned} |\nabla_h \hat{u}_h(x) - \nabla u(x)| &\leq C \max_j \Big| \frac{\hat{u}_h(x) - \hat{u}_h(x - he_j)}{h} - \frac{\partial u}{\partial x_j}(x) \Big| \\ &= C \max_j \Big| \frac{u(x) - \hat{u}_h(x - he_j)}{h} - \frac{\partial u}{\partial x_j}(x) \Big| \\ &\leq C \max_j \Big| \frac{u(x) - u(x - he_j)}{h} - \frac{\partial u}{\partial x_j}(x) \Big| + C \max_j \Big| \frac{u(x - he_j) - \hat{u}_h(x - he_j)}{h} \Big| \\ &\leq C \max_j \sup_{x \in \bar{\Omega}} \Big| \frac{\partial^2 u}{\partial^2 x_j} \Big| h + C \big| \hat{u}_h - u \big|_{0,\infty}. \end{aligned}$$

Since $|\hat{u}_h - u|_{0,\infty} \leq Ch^2$ and u is C^2 with uniformly bounded second derivatives, we conclude that

$$\max_{x \in \mathcal{M}_h} |\nabla_h \hat{u}_h(x) - \nabla u(x)| \to 0, \text{ as } h \to 0.$$

Thus by Lemma 1.4 we have $J_h(\hat{u}_h) \to J(u)$.

We now prove, using a contradiction argument, that

$$\inf_h J_h(\hat{u}_h) = J(u).$$

If $J(u) < \inf_h J_h(\hat{u}_h)$, then there is number a such that $J(u) < a < \inf_h J_h(\hat{u}_h) \leq J_h(\hat{u}_h)$ for all h. Thus $\lim_{h \to 0} J_h(\hat{u}_h) \geq a > J(u)$ a contradiction.

On the other hand, if $\inf_h J_h(\hat{u}_h) < J(u)$, then there is number b such that $\inf_h J_h(\hat{u}_h) < b < J(u)$. By definition of the infimum, we can find a subsequence h_k such that $\inf_h J_h(\hat{u}_h) \leq J_{h_k}(\hat{u}_{h_k}) \leq b < J(u)$. This also leads to a contradiction. Finally, since u_h is a minimizer of J_h it follows that $J_h(u_h) \leq J_h(\hat{u}_h)$ and $\inf_h J_h(u_h) \leq J(u)$. $\square$

To $v_h \in S_h$, we associate a Borel measure $M[v_h]$ defined by

$$M[v_h](B) = h^n \sum_{x \in B \cap \mathcal{M}_h} \det H[v_h](x).$$

The following lemma is proved in [5].

Lemma 1.7. *Let $v_h \in S_h$ converge uniformly on compact subsets to $v \in C(\overline{\Omega})$ and convex. Then $M[v_h]$ converges weakly to* $\det \nabla^2 v$.

The following lemma is the only place in the paper where we use the result of Lions which gives a variational approach to the Aleksandrov solution of (0.1). Arguing as in [1], we have

Lemma 1.8. *Under the assumptions of Proposition 1.6, we have*

$$\inf_h J_h(u_h) = J(u).$$

Proof. It remains to prove that $J(u) \leq \inf_h J_h(u_h)$. The technique to prove such a result was given in [1, Section 5]. We make an essential use of Lemma 1.7 and the result of Lions [16].

Let us define

$$\begin{aligned} |v_h|'_{0,\infty} &= \max_{x \in \mathcal{M}_h} |v_h(x)|, \\ |v_h|'_{1,\infty} &= \max\{ |\partial^i_+ v_h|'_{0,\infty}, i = 1, \ldots, n \}, \\ |v_h|'_{2,\infty} &= \max\{ |\partial^i_+ \partial^i_- v_h|'_{1,\infty}, |\partial^i \partial^j v_h|'_{1,\infty}, i, j = 1, \ldots, n \}, \\ ||v_h||'_{2,\infty} &= \max\{ |v_h|'_{0,\infty}, |v_h|'_{1,\infty}, |v_h|'_{2,\infty} \}. \end{aligned}$$

We make the assumption that the terms appearing in the definition of the above norms and semi norms are those for which the indicated discrete derivatives are defined.

For $K > 0$, put

$$\Lambda^2_{h,K} = \{\, v_h \in \mathcal{U}_h, ||v_h||'_{2,\infty} \leq K \,\}.$$

Moreover, put $S_{h,K} = S_h \cap \Lambda^2_{h,K}$.

Since $u \in C^4(\bar{\Omega})$, there exist $K > 0$ and $h_0 > 0$ such that $\hat{u}_h \in \mathcal{S}_h \cap \Lambda^2_{h,K}$, where $\hat{u}_h$ is the solution of (1.8). This proves that there exists $K > 0$ and $h_0 > 0$ such that $S_{h,K}$ is nonempty for all $0 < h < h_0$.

Let u^K_h denote the unique minimizer of J_h in $S_{h,K}$. By [1, Theorem 4.5], there exists a subsequence $u^K_{h'}$ and a C^2 convex function u^K such that $|u^K_{h'} - u^K|'_{1,\infty} \to 0$ and $|u^K_{h'} - u^K|'_{0,\infty} \to 0$ as $h' \to 0$.

We prove that $u^K = g$ on $\partial\Omega$. For $x \in \partial\Omega$, there is a family $x_{h'} \in \partial\mathcal{M}_{h'}$ such that $x_{h'}$ converges to x. Thus

$$\begin{aligned}
|u^K(x) - g(x)| &\leq |u^K(x) - u^K(x_{h'})| + |u^K(x_{h'}) - u_{h'}(x_{h'})| + |u_{h'}(x_h) - g(x)| \\
&\leq |u^K(x) - u^K(x_{h'})| + |u^K(x_{h'}) - u_{h'}(x_{h'})| + |g(x_{h'}) - g(x)| \\
&\leq |u^K(x) - u^K(x_{h'})| + |u^K_{h'} - u^K|'_{0,\infty} + |g(x_{h'}) - g(x)|
\end{aligned}$$

Passing to the limit as $h' \to 0$ yields $u^K(x) = g(x)$ by continuity of g and u^K and the fact that $|u^K_{h'} - u^K|'_{0,\infty}$ as $h' \to 0$.

By Lemma 1.4 $J_{h'}(u^K_{h'}) \to J(u^K)$ and by Lemma 1.7 we have $\det D^2 u^K \geq f$ in the sense of measures. Thus $u^K \in S$.

Since $u^K \in S$ and $J(u) = \inf_{v \in S} J(v)$, we have

$$J(u) \leq J(u^K) = \inf_{h'} J_{h'}(u^K_{h'}).$$

We may assume that $\inf_{h'} J_{h'}(u^K_{h'}) = \inf_h J_h(u^K_h)$. This is because, using the definition of infimum, the sequence $u^K_{h'}$ can be chosen to satisfy that property before passing to a subsequence converging to u^K. We conclude that

$$J(u) \leq \inf_{h,K} J_h(u^K_h).$$

For a fixed h, we can find K such that u_h is in $S_{h,K}$. Here K depends on h. To see this, note that since h is fixed, the number of mesh points is finite and u_h takes real values. Thus K can simply be taken as any umber greater than $||u_h||'_{2,\infty}$. We conclude that $\inf_{h,K} J_h(u^K_h) = \inf_h J_h(u_h)$. This implies that

$$J(u) \leq \inf_h J_h(u_h),$$

and concludes the proof. $\square$

For $\Phi(p) = |p|^2$ we can give a more precise result.

Theorem 1.9. *Suppose that the convex solution u of* (0.1) *is in $C^4(\overline{\Omega})$ and strictly convex. Then for $\Phi(p) = |p|^2$, the unique minimizer u_h in S_h of the functional J_h satisfies*

$$||u_h - u||_0 \to 0 \text{ as } h \to 0.$$

Proof. Let $\mathcal{M}^-_h$ denote the subset of $\mathcal{M}_h$ of mesh points x at which $\nabla_h v_h(x)$ is defined for $v_h \in \mathcal{U}_h$.

We first establish that for $v_h \in \mathcal{U}_h$, we have

$$\sum_{x\in\mathcal{M}_h^-} \langle \nabla_h u_h(x), \nabla_h v_h(x)\rangle \geq \sum_{x\in\mathcal{M}_h^-} |\nabla_h u_h(x)|^2. \tag{1.9}$$

For $t \in [0,1]$, define

$$\phi(t) = J_h(u_h + t(v_h - u_h)) = \frac{h^n}{2} \sum_{x\in\mathcal{M}_h^-} |\nabla_h u_h(x) + t\nabla_h(v_h - u_h)(x)|^2.$$

Clearly $\phi(t) \geq 0$ and ϕ is continuous. It is not difficult to check that for two vectors r and s,

$$\frac{d}{dt}|r + ts|^2 = 2t|s|^2 + 2\langle r, s\rangle.$$

This implies that ϕ is C^1. Since $\phi(t)$ achieves its minimum at $t = 0$, we have

$$0 \leq \phi'(0) = h^n \sum_{x\in\mathcal{M}_h^-} \langle \nabla_h u_h(x), \nabla_h(v_h - u_h)(x)\rangle,$$

from which (1.9) follows.

By (1.9) we have

$$\begin{aligned}
|u_h - u|_1^2 &= h^n \sum_{x\in\mathcal{M}_h^-} |\nabla_h(u_h - u)(x)|^2 \\
&= h^n \sum_{x\in\mathcal{M}_h^-} |\nabla_h u_h(x)|^2 + |\nabla_h u(x)|^2 - 2\langle \nabla_h u_h(x), \nabla_h u(x)\rangle \\
&\leq h^n \sum_{x\in\mathcal{M}_h^-} -|\nabla_h u_h(x)|^2 + |\nabla_h u(x)|^2 \\
&\leq J_h(u) - J_h(u_h) \\
&\leq \big|J_h(u) - J(u)\big| + \big|J(u) - J_h(u_h)\big|
\end{aligned}$$

It then follows from Lemma 1.6 and Lemma 1.8 that $|u_h - u|_1 \to 0$ as $h \to 0$. Since $u_h - u = 0$ on $\partial\mathcal{M}_h$ the result follows from Poincare's inequality, Lemma 1.3.

□

1.4. **Monotone discretization of** $\det \nabla^2 u$ **and wide stencil convexity.** We prove in this section that if one uses the monotone scheme introduced in [11], the minimizer of the discrete optimization problem is a solution of the corresponding finite difference equations. Following [11], we define a monotone Monge-Ampère operator by

$$M[v_h](x) = \inf_{(\alpha_1,\ldots,\alpha_n)\in W_h(x)} \prod_{i=1}^n \frac{v_h(x+\alpha_i) - 2v_h(x) + v_h(x-\alpha_i)}{|\alpha_i|^2} \quad \text{for } x \in \mathcal{M}_h^\circ, \tag{1.10}$$

where $W_h(x)$ denotes the set of orthogonal bases of $\mathbb{R}^n$ such that for $(\alpha_1,\ldots,\alpha_n) \in W_h(x)$, $x \pm \alpha_i \in \mathcal{M}_h^\circ, \forall i$.

Definition 1.10. *We say that a mesh function v_h is wide stencil convex if and only if $\Delta_e v_h(x) = v_h(x+e) - 2v_h(x) + v_h(x-e) \geq 0$ for all $x \in \Omega_h$ and $e \in \mathbb{Z}^n$ for which $\Delta_e v_h(x)$ is defined.*

We recall that the discrete Laplacian takes the form

$$\Delta_h v_h(x) = \sum_{i=1}^{d} \Delta_{he_i} v_h(x),$$

where $\{ e_i, i = 1, \ldots, d \}$ denotes the canonical basis of $\mathbb{R}^d$.

Let $\mathcal{C}_h$ denote the cone of wide stencil convex mesh functions. In this section, we will refer to a wide stencil convex function simply as a discrete convex function. We also make the slight abuse of notation of denoting by S_h the discrete version of the set S using the notion of wide stencil convexity and the monotone discretization of $\det \nabla^2 u$, i.e.

$$S_h = \{ v_h \in \mathcal{C}_h, v_h = g \,\text{on}\, \partial \mathcal{M}_h, \text{ and } (M[v_h](x))^{\frac{1}{n}} \geq f(x)^{\frac{1}{n}}, x \in \mathcal{M}_h^\circ \}. \tag{1.11}$$

We seek a minimizer of J_h over S_h. As in the previous section we consider the problem: find $u_h \in \mathcal{C}_h$ such that

$$\begin{aligned} M[u_h](x) &= f(x), x \in \mathcal{M}_h^\circ \\ u_h &= g \,\text{on}\, \partial \mathcal{M}_h. \end{aligned} \tag{1.12}$$

When $f \in C(\Omega)$, it is shown in [11] that u_h converges uniformly on compact subsets to the so-called viscosity solution of (0.1) when (0.1) has a unique one. One can add a perturbation ϵu_h to the operator to force uniqueness. But (1.12) may have more than one solution. Here we prove that there is no uniqueness issue with the variational framework proposed in this paper.

Theorem 1.11. *For $\Phi(p) = |p|^2$, the functional J_h has a unique minimizer u_h in S_h and u_h solves the finite difference equation* (1.12).

Proof. We first prove that the set S_h is convex. We note that for $\lambda \geq 0$

$$(M[\lambda v_h](x))^{\frac{1}{n}} = \lambda (M[v_h](x))^{\frac{1}{n}}.$$

It is therefore enough to prove that for $v_h, w_h \in \mathcal{C}_h$, we have

$$(M[v_h + w_h](x))^{\frac{1}{n}} \geq (M[v_h](x))^{\frac{1}{n}} + (M[w_h](x))^{\frac{1}{n}}. \tag{1.13}$$

Let $(\alpha_1, \ldots, \alpha_n) \in W_h(x)$ be such that

$$M[v_h + w_h](x) = \prod_{i=1}^{n} (\lambda_1^i + \lambda_2^i)$$

with

$$\lambda_1^i = \frac{v_h(x+\alpha_i) - 2v_h(x) + v_h(x-\alpha_i)}{|\alpha_i|^2} \text{ and } \lambda_2^i = \frac{w_h(x+\alpha_i) - 2w_h(x) + w_h(x-\alpha_i)}{|\alpha_i|^2}.$$

Since $v_h, w_h \in \mathcal{C}_h$, $\lambda_1^i, \lambda_2^i \geq 0$ for all i. By Minkowski's determinant theorem,

$$\left(\prod_{i=1}^{n} (\lambda_1^i + \lambda_2^i) \right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n} \lambda_1^i \right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \lambda_2^i \right)^{\frac{1}{n}},$$

from which (1.13) follows.

We recall that Problem (1.12) was shown in [11] to have a solution. Thus the set S_h is nonempty. Since Φ is strictly convex by assumption, it follows that the functional J_h has a unique minimizer u_h on the convex set S_h.

We now show that u_h solves the finite difference system (1.12). To this end, it suffices to show that

$$M[u_h] = f \text{ on } \mathcal{M}_h^\circ.$$

Let us assume to the contrary that there exists $x_0 \in \mathcal{M}_h^\circ$ such that

$$M[u_h](x_0) > f(x_0) \geq 0.$$

This implies that for all $e \in \mathbb{Z}^d$, $\Delta_e u_h(x_0) > 0$. Let $\epsilon_0 = \inf\{\Delta_e u_h(x_0), e \in \mathbb{Z}^d\}$ and $\epsilon_1 = M[u_h](x_0) - f(x_0)$. Finally, put $\epsilon = \min(\epsilon_0, \epsilon_1)$. We define w_h by

$$w_h(x) = u_h(x), x \neq x_0, w_h(x_0) = u_h(x_0) + \frac{\epsilon}{4}.$$

By construction $w_h = g$ on $\partial\mathcal{M}_h$. For $x \neq x_0$, $\Delta_e w_h(x) = \Delta_e u_h(x)$ or $\Delta_e w_h(x) = \Delta_e u_h(x) + \epsilon/4$. Moroever $\Delta_e w_h(x_0) = \Delta_e u_h(x_0) - \epsilon/2 \geq \epsilon_0 - \epsilon/2 \geq \epsilon/2 > 0$ by the definition of ϵ. We conclude that $w_h \in \mathcal{C}_h$.

For $x \neq x_0$, $M[w_h](x) \geq M[u_h](x)$ and $M[w_h](x_0) = M[u_h](x_0) - \epsilon/2$. Thus $M[w_h](x) \geq f(x)$ for all $x \in \mathcal{M}_h^\circ$. It remains to show that $J_h(w_h) < J_h(u_h)$. Let $\mathcal{M}_{x_0}$ denotes the subset of $\mathcal{M}_h$ consisting in x_0 and the points $x_0 + he_j, j = 1, \ldots, d$ at which $\nabla_h u_h$ is defined. We have

$$J_h(w_h) = h^d \sum_{x \notin \mathcal{M}_{x_0}} |\nabla_h u_h(x)|^2 + h^d |\nabla_h u_h(x_0)|^2 + \sum_{j=1}^d |\nabla_h u_h(x_0 + he_j)|^2$$

$$\begin{aligned}
J_h(w_h) &= h^d \sum_{x \notin \mathcal{M}_{x_0}} |\nabla_h u_h(x)|^2 + h^{d-2} \sum_{i=1}^d (w_h(x_0) - w_h(x_0 - he_i))^2 \\
&\quad + h^{d-2} \sum_{j=1}^d \sum_{i=1}^d (w_h(x_0 + he_j) - w_h(x_0 + he_j - he_i))^2 \\
&= h^d \sum_{x \notin \mathcal{M}_{x_0}} |\nabla_h u_h(x)|^2 + h^{d-2} \sum_{j=1}^d \sum_{\substack{i=1 \\ i \neq j}}^d (w_h(x_0 + he_j) - w_h(x_0 + he_j - he_i))^2 \\
&\quad + h^{d-2} \sum_{i=1}^d (w_h(x_0) - w_h(x_0 - he_i))^2 + (w_h(x_0 + he_i) - w_h(x_0))^2.
\end{aligned}$$

However

$$\begin{aligned}
&\sum_{i=1}^d (w_h(x_0) - w_h(x_0 - he_i))^2 + (w_h(x_0 + he_i) - w_h(x_0))^2 = \\
&\qquad \sum_{i=1}^d (u_h(x_0) - u_h(x_0 - he_i))^2 + (u_h(x_0 + he_i) - u_h(x_0))^2 + \frac{d\epsilon^2}{8} - \frac{\epsilon}{2} \Delta_h u_h(x_0).
\end{aligned}$$

Thus, by our choice of ϵ

$$\begin{aligned}
J_h(w_h) &= J_h(u_h) + \frac{d\epsilon^2}{8} - \frac{\epsilon}{2} \Delta_h u_h(x_0) = J_h(u_h) + \frac{\epsilon}{2}\left(\frac{d\epsilon}{4} - \Delta_h u_h(x_0)\right) \\
&< J_h(u_h),
\end{aligned}$$

Indeed, since $\Delta_e u_h(x_0) \geq \epsilon_0$, we get $\Delta_h u_h(x_0)) \geq d\epsilon_0 \geq d\epsilon > d\epsilon/4$ which contradicts the assumption that u_h is a minimizer and concludes the proof. $\square$

2. Numerical results

In this section, we report numerical results for the variational framework proposed in the paper for the 2D problem. For most of the numerical results we use a standard discretization for $\det \nabla^2 u$ and local discrete convexity. We also include numerical results for a more general situation where the right hand side of (0.1) is a measure. Previously published results on the monotone scheme are not satisfactory for this case [6].

Throughout this section $\bar{\Omega}$ is the unit square $[0,1] \times [0,1]$. We recall that the optimization problem of interest to us, in the case of the standard finite difference discretization, is the following:

$$
\begin{aligned}
&\text{Minimize} \quad && h^2 \sum_{x \in \mathcal{M}_h} \Phi(\nabla_h u_h(x)) && \\
&\text{subject to} \quad && u_h = g && \text{on } \partial \mathcal{M}_h \\
& && \lambda_{min}(H[u_h](x)) \geq 0 && \text{for } x \in \mathcal{M}_h^\circ \\
& && \sqrt{\det(H[u_h](x))} \geq \sqrt{f(x)} && \text{for } x \in \mathcal{M}_h^\circ,
\end{aligned}
\tag{2.1}
$$

where u_h is the unknown variable, λ_{min} is the smallest eigenvalue of the matrix $H[u_h](x)$ and we recall that $H[u_h](x)$ denotes the discrete Hessian.

2.1. **Solvability and implementation.** Under the convexity assumption on Φ, since the operators $\sqrt{\det(\cdot)}$ and $\lambda_{min}(\cdot)$ are concave, it is easily verified that (2.1) is a convex optimization program. Therefore, we are guaranteed that any algorithm that finds a local minimizer recovers a global minimizer. Furthermore, the global minimizer will be unique if we choose Φ to be strictly convex. A variety of methods and algorithms to solve convex programs like (2.1), including primal-dual and barrier methods, are readily available in the literature. It is not our goal in this section to develop or identify the most efficient method for solving (2.1). Instead, we aim to provide numerical evidence supporting the analysis done in section 1. For rapid prototyping, we use MATLAB and take advantage of the fact that our convex program is a typical "disciplined convex program" as introduced in the convex optimization toolbox CVX [12, 13]. In CVX, the user has the choice between several solvers and we choose SDPT3 [20]. It implements an infeasible primal-dual path-following algorithm.

For computational expedience but at a cost of increased problem size, CVX internally converts problem (2.1) – see for example [12] and the references therein – to the canonical form

$$
\begin{aligned}
\text{Minimize } & c^T x + d \\
\text{subject to } & \boldsymbol{A} x = b \\
& x \in \mathcal{S},
\end{aligned}
\tag{2.2}
$$

where $\mathcal{S}$ is a convex set, $x \in \mathbb{R}^m$ is the unknown, and the parameters $\boldsymbol{A} \in \mathbb{R}^{k \times m}$, $b \in \mathbb{R}^k$, $c \in \mathbb{R}^m$ and $d \in \mathbb{R}$ are computed from the original problem. The canonical problem (2.2) is then solved using the solver SDPT3.

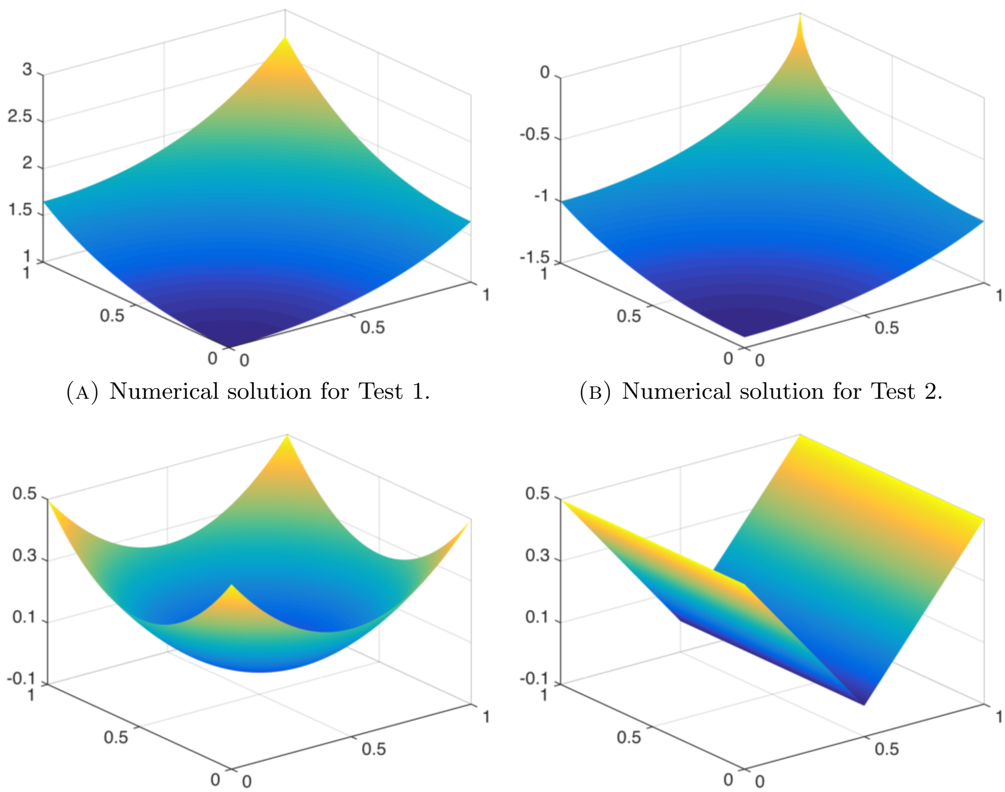
2.2. **Test cases and results.** We provide numerical evidence for four scenarios. The data of the corresponding Monge-Ampère Dirichlet problems are given in TABLE 1. The results are reported in TABLES 2, 3 and FIGURE 1.

We conclude this section with numerical experiments for the Dirichlet problem for $\det \nabla^2 u = \nu$ with $\nu = \pi/2\, \delta_{(1/4,1/2)} + \pi/2\, \delta_{(3/4,1/2)}$ using the standard finite difference discretization and the monotone scheme. The exact solution, taken from [6], is given by

$$
u(x,y) = \begin{cases} |y - \frac{1}{2}| & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ \min \left\{ \sqrt{(x-\frac{1}{4})^2 + (y-\frac{1}{2})^2}, \sqrt{(x-\frac{3}{4})^2 + (y-\frac{1}{2})^2} \right\} & \text{otherwise.} \end{cases}
$$

Test 1	$f(x,y) = (1+x^2+y^2)\mathrm{e}^{x^2+y^2}$	$g(x,y) = \mathrm{e}^{(x^2+y^2)/2}$
Test 2	$f(x,y) = \dfrac{2}{(2-x^2-y^2)^2}$	$g(x,y) = -\sqrt{2-x^2-y^2}$
Test 3	$f(x,y) = 4$	$g(x,y) = (x-1/2)^2 + (y-1/2)^2$
Test 4	$f(x,y) = 0$	$g(x,y) = \|x-1/2\|$

TABLE 1. Data for the numerical experiments with $\Omega = (0,1) \times (0\,1)$.



(A) Numerical solution for Test 1. (B) Numerical solution for Test 2.

(C) Numerical solution for Test 3. (D) Numerical solution for Test 4.

FIGURE 1. Solutions computed on a 65×65 grid with $\Phi(x,y) = \sqrt{x^2+y^2}$ using the disciplined convex programming toolbox CVX with the solver SDPT3.

Our results are reported on TABLES 4, 5 and FIGURE 2. This example was chosen because many existing methods fail to capture the solution.

Remark 2.1. *Our numerical method also provides a technique for computing the convex envelope of boundary data and the convex envelope of a function defined on* $\overline{\Omega}$. *We recall from* [16] *that given* g *on* $\partial\Omega$ *(satisfying the assumptions of this paper), the convex envelope of* g *on* $\overline{\Omega}$ *is the mimimum of* J *over*

$$S = \{v \in C(\overline{\Omega}), \quad v = g \text{ on } \partial\Omega,\ v \text{ convex on } \Omega\}.$$

Also given any function g *defined on* $\overline{\Omega}$, *the convex envelope of* g *is the minimum of* J *over*

$$S = \{v \in C(\overline{\Omega}), \quad v \leq g \text{ on } \partial\Omega,\ v \text{ convex on } \Omega\}.$$

h	Test 1	Test 2	Test 3	Test 4
2^{-2}	3.9093×10^{-3}	2.5104×10^{-2}	2.9143×10^{-16}	1.7233×10^{-5}
2^{-3}	1.0340×10^{-3}	2.6475×10^{-2}	1.1102×10^{-16}	3.8580×10^{-15}
2^{-4}	2.6643×10^{-4}	2.2113×10^{-2}	5.5511×10^{-17}	1.0963×10^{-14}
2^{-5}	6.6964×10^{-5}	1.6920×10^{-2}	3.0531×10^{-16}	1.5155×10^{-14}
2^{-6}	1.6781×10^{-5}	1.2440×10^{-2}	1.6098×10^{-15}	2.3870×10^{-15}

TABLE 2. L^∞ error for the function $\Phi(x,y) = \sqrt{1+x^2+y^2}$.

h	Test 1	Test 2	Test 3	Test 4
2^{-2}	2.2524×10^{-2}	2.7012×10^{-2}	6.3363×10^{-11}	7.3175×10^{-5}
2^{-3}	4.1574×10^{-3}	2.6801×10^{-2}	1.6653×10^{-16}	1.0270×10^{-15}
2^{-4}	1.1233×10^{-3}	2.2223×10^{-2}	5.5511×10^{-17}	3.1919×10^{-15}
2^{-5}	3.1368×10^{-4}	1.6967×10^{-2}	4.7184×10^{-16}	6.5781×10^{-15}
2^{-6}	1.3201×10^{-4}	1.2500×10^{-2}	2.1649×10^{-15}	1.0464×10^{-14}

TABLE 3. L^∞ error for the function $\Phi(x,y) = |x| + |y|$.

		h		
$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$
$1.40\ 10^{-2}$	$1.44\ 10^{-2}$	$1.31\ 10^{-2}$	$1.25\ 10^{-2}$	$1.17\ 10^{-2}$

TABLE 4. Dirac masses. $\Phi(x,y) = \sqrt{x^2+y^2}$, standard discretization of the determinant of the Hessian with discrete local convexity.

		h		
$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$
$4.31\ 10^{-3}$	$1.08\ 10^{-3}$	$2.70\ 10^{-4}$	$6.74\ 10^{-5}$	$1.68\ 10^{-5}$

TABLE 5. Dirac masses. $\Phi(x,y) = x^2+y^2$, monotone discretization of the determinant of the Hessian with wide stencil convexity on a 9 point scheme.

References

[1] N. E. Aguilera and P. Morin. Approximating optimization problems over convex functions. *Numer. Math.*, 111(1):1–34, 2008.

[2] G. Awanou. Iterative methods for k-Hessian equations. http://arxiv.org/abs/1406.5366, 2013.

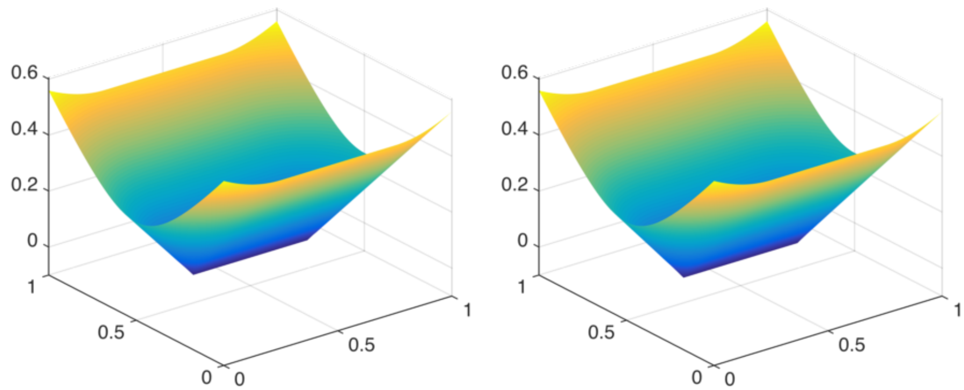
[3] G. Awanou. Isogeometric method for the elliptic Monge-Ampère equation. In *Approximation Theory, XIV (San Antonio, TX, 2013)*, pages 1–13. 2014.

[4] G. Awanou. On standard finite difference discretizations of the elliptic Monge-Ampère equation. http://arxiv.org/abs/1311.2812, 2014.

[5] G. Awanou and R. Awi. Convergence of stable and consistent schemes to the Aleksandrov solution of the Monge-Ampère equation. http://arxiv.org/abs/1507.08490, 2015.

[6] J.-D. Benamou and B. D. Froese. A viscosity framework for computing Pogorelov solutions of the Monge-Ampère equation. http://arxiv.org/pdf/1407.1300v2.pdf, 2014.

[7] S. C. Brenner, T. Gudi, M. Neilan, and L.-Y. Sung. C^0 penalty methods for the fully nonlinear Monge-Ampère equation. *Math. Comp.*, 80(276):1979–1995, 2011.



(A) Analytical solution. (B) Numerical solution.

FIGURE 2. Dirac masses. The numerical solution is computed on a 65×65 rectangular grid using the standard finite difference discretization.

[8] A. Caboussat, R. Glowinski, and D. C. Sorensen. A least-squares method for the numerical solution of the Dirichlet problem for the elliptic Monge-Ampère equation in dimension two. *ESAIM Control Optim. Calc. Var.*, 19(3):780–810, 2013.

[9] S. K. Chung, A. K. Pani, and M. G. Park. Convergence of finite difference method for the generalized solutions of Sobolev equations. *J. Korean Math. Soc.*, 34(3):515–531, 1997.

[10] E. J. Dean and R. Glowinski. Numerical methods for fully nonlinear elliptic equations of the Monge-Ampère type. *Comput. Methods Appl. Mech. Engrg.*, 195(13-16):1344–1386, 2006.

[11] B. Froese and A. Oberman. Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampère equation in dimensions two and higher. *SIAM J. Numer. Anal.*, 49(4):1692–1714, 2011.

[12] M. Grant. *Disciplined Convex Programming*. PhD thesis, Stanford University, 2004.

[13] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. `http://cvxr.com/cvx`, Mar. 2014.

[14] C. E. Gutiérrez. *The Monge-Ampère equation*. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA, 2001.

[15] D. Hartenstine. The Dirichlet problem for the Monge-Ampère equation in convex (but not strictly convex) domains. *Electron. J. Differential Equations*, pages No. 138, 9 pp. (electronic), 2006.

[16] P.-L. Lions. Two remarks on Monge-Ampère equations. *Ann. Mat. Pura Appl. (4)*, 142:263–275 (1986), 1985.

[17] J. Rauch and B. A. Taylor. The Dirichlet problem for the multidimensional Monge-Ampère equation. *Rocky Mountain J. Math.*, 7(2):345–364, 1977.

[18] A. W. Roberts and D. E. Varberg. *Convex functions*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Pure and Applied Mathematics, Vol. 57.

[19] N. Z. Shor. *Nondifferentiable optimization and polynomial problems*, volume 24 of *Nonconvex Optimization and its Applications*. Kluwer Academic Publishers, Dordrecht, 1998.

[20] K. C. Toh, M. J. Todd, and R. H. Tütüncü. A MATLAB software package for semidefinite programming, version 1.3. 11(2):205–267, 1999.

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, M/C 249. UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607-7045, USA

E-mail address: `awanou@uic.edu`

URL: `http://www.math.uic.edu/~awanou`

MATHEMATICAL BIOSCIENCES INSTITUTE, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA

E-mail address: `matambamessi.1@mbi.osu.edu`

URL: `http://people.mbi.ohio-state.edu`