Trivariate Spline Approximations of 3D Navier-Stokes Equations

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Overview

- The Navier-Stokes Equations
- Features of the Spline Method
- Trivariate Splines
- Discretization of the Stokes Equations
- Discretization of the Navier-Stokes Equations
- Iterative Method for Solving the Discrete Problem
- Computational Experiments
- Work in Progress

The Navier-Stokes equations

$$\begin{cases} -\nu\,\Delta\mathbf{u} + \sum_{j=1}^{3} u_j \frac{\partial \mathbf{u}}{\partial x_j} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div}\,\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{1}$$

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$$V_0 = \{\mathbf{v} \in H_0^1(\Omega)^3, \text{div } \mathbf{v} = 0\}$$

$$L_0^2(\Omega) = \{u \in L^2(\Omega), \int_\Omega u = 0\} \quad \text{and}$$

$$H^{\frac{1}{2}}(\partial\Omega) = \{\tau(u), u \in H^1(\Omega)\},$$

Existence and Uniqueness

Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary. For $\mathbf{f} \in H^{-1}(\Omega)^3$ and $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^3$ satisfying $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$, the problem: find $(\mathbf{u}, p) \in H^1(\Omega)^3 \times L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{rcll} -\nu\, \Delta \mathbf{u} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} \mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \text{div}\, \mathbf{u} & = & 0 & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{g} & \text{on } \partial\Omega, \end{array} \right.$$

has a solution which is unique provided that ν is sufficiently large.

Features of the spline method

- We shall assume that Ω is a polygonal domain of $\mathbb{R}^3$ with a tetrahedral partition $\mathcal{T}$ and use the spline space

$$S_d^r(\mathcal{T}) = \{s \in C^r(\Omega),\ s|_t \in \mathbb{P}_d,\ \forall t \in \mathcal{T}\},$$

where $\mathbb{P}^d$ is the space of polynomials of total degree d.
We use the B-form of splines and associate to each component u_i of $\mathbf{u}$, $i = 1, \ldots, 3$ a vector of coefficients c_i.

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  We use the B-form of splines and associate to each component u_i of **u**, $i = 1, \ldots, 3$ a vector of coefficients c_i.
- Smoothness requirements on **c**. In general, smoothness can be imposed in a flexible way across the domain.

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$$\nu \overline{K}\mathbf{c} + \overline{B}(\mathbf{c})\mathbf{c} + L^T \lambda = \overline{M}\mathbf{F}$$

$$L\mathbf{c} = \overline{\mathbf{G}}$$

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- The mass and stiffness matrices can be assembled easily and these processes can be done in parallel.
- The pressure is computed by solving a Poisson equation with Neumann boundary conditions.

Trivariate Splines

Let $d \geq 1$ and $r \geq 0$ be two fixed integers. Given a bounded domain Ω of $\mathbf{R}^3$ with piecewise planar boundary, let $\mathcal{T}$ be a tetrahedral partition of Ω.

$$S_d^r(\Omega) = \{p \in C^r(\Omega),\ p_{|t} \in P_d,\ \forall t \in \mathcal{T}\}.$$

$$p(x,y,z) = \sum_{0 \leq i+j+k \leq d} \alpha_{ijk} x^i y^j z^k,$$

Barycentric coordinates

Given a non-degenerate tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$, any point $v = (x,y,z)$ can be written uniquely in the form $v = b_1 v_1 + b_2 v_2 + b_3 v_3 + v_4 b_4$ with $b_1 + b_2 + b_3 + b_4 = 1$.

B-form of splines

Bernstein polynomials of degree d

$$B^d_{ijkl}(v) = \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \qquad i+j+k+l = d.$$

They are polynomials of degree d since each b_i is a linear polynomial.

The set $\mathcal{B}^d = \{B^d_{ijkl}(x,y,z), i+j+k+l = d\}$ is a basis for the space of polynomials P_d.

We recall that the dimension of P_d is $\binom{d+3}{3}$.

As a consequence any spline s in S^r_d can be written uniquely

$$s_{|T} = \sum_{i+j+k+l=d} c^T_{ijkl} B^d_{ijkl},$$

since $s_{|T}$ is a polynomial of degree d.

nth = number of tetrahedra, $m = \dim P_d$ and $N = m * nth$.

Bivariate splines

The restriction of a trivariate polynomial of degree d on a face of a tetrahedron is a bivariate polynomial and can be written in B-form

$$\sum_{i+j+k=d} \widetilde{c}_{ijk} \widetilde{B}^d_{ijk}(v),$$

where

$$\widetilde{B}^d_{ijk} = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k.$$

For example, given the trivariate spline on a tetrahedron T $p = \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl}$, $q = \sum_{i+j+k=d} c_{ijk0} B^d_{ijk0}$ can be considered as a bivariate polynomial.

Interpolation

There is a unique polynomial p of degree d that interpolates any given function f on a tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$ at the domain points $\xi_{ijkl} = \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}$.

This gives rise to an interpolation operator Π_d. $\Pi_d(f)$ will denote both the spline interpolant and its B-net.

We can also define a boundary interpolation operator Π_d^b since a bivariate polynomial p of degree d is uniquely determined on a triangle $\langle v_1, v_2, v_3 \rangle$ by its values at the domain points $\xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d}$. Note that here the domain points have three indices.

We have for a spline s with B-net c according to our notations

$$Rc = \Pi_d^b(s)$$

Derivatives

We want to give formulas for the directional derivatives of p in a direction defined by a vector $\mathbf{u}$ joining the points v_1 and v_2. Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ with components the difference of the barycentric coordinates of v_1 and v_2. $D_{\mathbf{u}} p$ can be written in B-form as a polynomial of degree $d-1$.

$$D_{\mathbf{u}} p = d \sum_{i+j+k+l=d-1} c^{(1)}_{ijkl}(\mathbf{a})\; B^{d-1}_{ijkl}, \qquad \text{where}$$

$$c^{(1)}_{ijkl}(\mathbf{a}) = a_1 c_{i+1,j,k,l} + a_2 c_{i,j+1,k,l} + a_3 c_{i,j,k+1,l} + a_4 c_{i+1,j,k,l+1}.$$

It's not difficult to see that there are matrices D_1, D_2 and D_3 such that if c encodes the B-net of s, $D_i c$, $i = 1, \ldots, 3$ encode respectively the B-net of $\frac{\partial s}{\partial x_i}$.

Integration

There's a matrix G such that if p and q have B-nets c and d,

$$\int_{\Omega} pq = c^T G d$$

Smoothness conditions

Let $t = \langle v_1, v_2, v_3, v_4 \rangle$ and $t' = \langle v_1, v_2, v_3, v_5 \rangle$ be two tetrahedra with common face $\langle v_1, v_2, v_3 \rangle$. Then s is of class C^r on $t \cup t'$ if and only if

$$c^{t'}_{ijkm} = \sum_{\mu+\nu+\kappa+\delta=m} c^t_{i+\mu,j+\nu,\gamma+\kappa,\delta} B^l_{\mu,\nu,\kappa,\delta}(v_5), \; m = 0, \ldots, r, \; i+j+k = d-m$$

This suggests that there's a (l, N) matrix H such that s is in $C^r(\Omega)$ if and only if

$$Hc = 0.$$

Discretization

The weak form of the Navier-Stokes equations is: Find $\mathbf{u} \in H^1(\Omega)^3$ such that

$$\nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \sum_{j=1}^{3} \int_\Omega u_j \frac{\partial \mathbf{u}}{\partial x_j} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V_0$$

$$\text{div } \mathbf{u} = 0 \qquad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega$$

where

$$V_0 = \{\mathbf{v} \in H_0^1(\Omega)^3, \text{div } \mathbf{v} = 0\}.$$

We now consider spline approximations of the velocity vector field $\mathbf{u}$. Let $d \geq 1$ and $r \geq 0$ be two given integers.

Let also $\mathcal{S} \subset S_d^0(\mathcal{T})$ be a spline subspace over a tetrahedral partition $\mathcal{T}$ of Ω consisting of spline functions which are C^r inside Ω and C^0 near the boundary $\partial\Omega$.

- Recall that there is a matrix H such that if $s \in \mathcal{S}$ with B-coefficient vector $\mathbf{c}$, then

$$H\mathbf{c} = 0.$$

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- Also recall that there is a matrix R which maps $\mathbf{c}$ to the B-coefficients of s on the boundary of Ω and $R\mathbf{c} = G$ represents the boundary condition, i.e., $s = g$ on the boundary approximately.

- Finally there are matrices D_1, D_2 and D_3 such that if c encodes the B-net of s, $D_i c$, $i = 1, \ldots, 3$ encode respectively the B-net of $\frac{\partial s}{\partial x_i}$.

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- $\mathbf{u} = (u_1, u_2, u_3)$ velocity vector
  $\mathbf{s_u} = (s_1, s_2, s_3)$ spline approximating vector
  $s_i \in \mathcal{S}$ satisfying $H\mathbf{c}_i = 0, R(\mathbf{c}_i) = G(g_i)$ for $i = 1, 2, 3$.
  div $\mathbf{u} = 0$ is discretized as $D_1\mathbf{c}_1 + D_2\mathbf{c}_2 + D_3\mathbf{c}_3 = 0$

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- Let

$$\overline{H} = \begin{pmatrix} H & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{pmatrix}, \quad \overline{R} = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix},$$

- $$\mathbf{G} = (G(g_1), G(g_2), G(g_3))^T \qquad \overline{D} = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}.$$

  $\overline{H}\mathbf{c} = 0$, $\overline{R}\mathbf{c} = \mathbf{G}$ and $\overline{D}\mathbf{c} = 0$

  $$L = \begin{bmatrix} \overline{H}^T & \overline{R}^T & \overline{D}^T \end{bmatrix}^T \text{ and } \overline{\mathbf{G}} = \begin{bmatrix} 0 & \mathbf{G}^T & 0 \end{bmatrix}^T,$$

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- In other words, if we let

  $$S_g = \{\mathbf{c} \in (\mathbb{R}^N)^3, L\mathbf{c} = \overline{\mathbf{G}}\},$$

  we are seeking for a solution in S_g. We approximate elements of V_0 by vectors $\mathbf{d} = (d_1, d_2, d_3)$ in

  $$S_0 = \{\mathbf{d} \in (\mathbb{R}^N)^3, L\mathbf{d} = 0\}.$$

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}$$

$$b(\mathbf{w};\mathbf{u},\mathbf{v}) = \sum_{j=1}^{3}\int_\Omega w_j \frac{\partial \mathbf{u}}{\partial x_j}\cdot\mathbf{v}.$$

$$\int_\Omega \mathbf{f}\mathbf{v}$$

$M^t = \left(\int_t B_\alpha^d B_\beta^d \right)_{|\alpha|=d,|\beta|=d}$, local mass matrix

M is the global mass matrix

$K^t = \left(\int_t \nabla B_\alpha^d \nabla B_\beta^d \right)_{|\alpha|=d,|\beta|=d}$, local stiffness matrix

K global stiffness matrix

Continuous	Discrete
$\mathbf{u} = (u_1, u_2, u_3)$	$\mathbf{c} = (c_1, c_2, c_3)$
$\mathbf{v} = (v_1, v_2, v_3)$	$\mathbf{d} = (d_1, d_2, d_3)$
$\mathbf{u} \in \mathcal{S} \subset S_d^r(\Omega)$	$\overline{H}\mathbf{c} = 0$
$\text{div}\, \mathbf{u} = 0$	$\overline{D}\mathbf{c} = 0$
$\mathbf{u} = \mathbf{g}$ on $\partial\Omega$	$\overline{R}\mathbf{c} = \mathbf{G}$
$\mathbf{u} \in \mathcal{S}_g$	$L\mathbf{c} = \overline{\mathbf{G}}$
$\int_\Omega f_i v_j$	$d_j^T M F_i$
$\int_\Omega \mathbf{f}\mathbf{v}$	$\mathbf{d}^T \overline{M} \mathbf{F}$
$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}$	$\mathbf{c}^T \overline{K} \mathbf{d}$
$b(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \sum_{j=1}^3 \int_\Omega w_j \frac{\partial \mathbf{u}}{\partial x_j} \cdot \mathbf{v}$	$\mathbf{d}^T \overline{B}(\mathbf{e})\mathbf{c}$

Discretization of the Stokes equations

Corollary of the Lax-Milgram lemma

Let V be a real Hilbert space with norm denoted by $||.||_V$, $(u,v) \longrightarrow a(u,v)$ a real bilinear form on $V \times V$, l an element of the dual of V and let us denote the duality pairing between V and its dual V' by $<,>$. If a is continuous, symmetric and is elliptic on V i.e. there is $\alpha > 0$ such that $a(v,v) \geq \alpha ||v||_V^2$ for all $v \in V$, then, the problem: Find $u \in V$ such that

$$a(u,v) =< l, v >,$$

has one an only one solution which minimizes the following functional over V

$$J(v) = \frac{1}{2} a(v,v) - < l, v > .$$

Under the same hypotheses as in the theorem on the Navier-Stokes equations, the Stokes equations:

$$\left\{ \begin{array}{rcl} -\nu\Delta\mathbf{u} + \nabla p & = & \mathbf{f} \text{ in } \Omega \\ \text{div } \mathbf{u} & = & 0 \text{ in } \Omega \\ \mathbf{u} & = & \mathbf{g} \text{ on } \partial\Omega \end{array} \right.$$

have a unique solution $\mathbf{u}$ in $H^1(\Omega)^3$ and a pressure p in $L^2(\Omega)$ unique up to an additive constant. These equations are derived under the assumption that the velocity is sufficiently small to ignore the nonlinear term $\mathbf{u} \cdot \nabla\mathbf{u}(\mathbf{x}, t)$.

The weak form of the equations is: Find $\mathbf{u}$ in $H^1(\Omega)^3$ such that div $\mathbf{u} = 0$ and $\nu \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \qquad \forall\mathbf{v} \in V_0,$

$$V_0 = \{\mathbf{v} \in H_0^1(\Omega)^3 \text{ such that } \operatorname{div} \mathbf{v} = 0\}.$$

In this case, the velocity vector $\mathbf{u}$ is is the unique minimizer in

$$V = \{\mathbf{v} \in H^1(\Omega)^3 \text{ such that } \operatorname{div} \mathbf{v} = 0\}$$

of the functional

$$J(\mathbf{u}) = \frac{\nu}{2} \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{u} - \int_\Omega \mathbf{f} \cdot \mathbf{u}.$$

If we let $\mathbf{c}$ encode the B-net of the approximant, the discrete problem is: Minimize

$$J(\mathbf{c}) = \frac{\nu}{2} \mathbf{c}^T \overline{K} \mathbf{c} + \mathbf{F}^T \overline{M} \mathbf{c}$$

over $(\mathbb{R}^N)^3$ under the constraint $L\mathbf{c} = \overline{\mathbf{G}}$.

By the theory of Lagrange multipliers, there is a vector of Lagrange multipliers λ such that

$$\begin{cases} \nu \overline{K}\mathbf{c} + L^T\lambda &= \overline{M}\mathbf{F}, \\ L\mathbf{c} &= \overline{\mathbf{G}}. \end{cases}$$

Computation of the pressure term

Assuming that **u** is smooth and taking the divergence of the equation

$$-\nu\Delta\mathbf{u} + \nabla p = \quad \mathbf{f} \text{ in } \Omega$$

we get

$$-\Delta p = -\text{div}\,\mathbf{f}$$

since div $\mathbf{u} = 0$. Here, the pressure is the minimizer over

$$L^2_0(\Omega) = \{p \in L^2(\Omega), \int_\Omega p = 0\}$$

of

$$Q(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega (-\textbf{div}\ \mathbf{f})v - \int_{\partial\Omega} (\mathbf{f}\cdot\mathbf{n} + \nu(\Delta\mathbf{u})\cdot\mathbf{n})v.$$

Discretization for the pressure in Navier-Stokes equations is similar.

Discretization of the Navier-Stokes equations

Find $\mathbf{c}$ in $\mathbf{R}^{3N}$ satisfying $L\mathbf{c} = \overline{\mathbf{G}}$ with $\overline{\mathbf{G}}$ encoding the side conditions and

$$\nu \mathbf{c}^T \overline{K} \mathbf{d} + (\overline{B}(\mathbf{c})\mathbf{c})^T \mathbf{d} = \mathbf{d}^T \overline{M} \mathbf{F}$$

for all $\mathbf{d}$ in $\mathbf{R}^{3N}$ with constraints $L\mathbf{d} = 0$.
Here, $\overline{K}$ and $\overline{M}$ are the stiffness and mass matrices respectively; $(\overline{B}\mathbf{c})\mathbf{d}$ encodes the nonlinear term. If one considers the following linear functional in $\mathbf{d}$,

$$J(\mathbf{d}) = (\nu \mathbf{c}^T \overline{K} + (\overline{B}(\mathbf{c})\mathbf{c})^T + \mathbf{F}^T \overline{M})\mathbf{d},$$

we have $J(\mathbf{d}) = 0$ for all $\mathbf{d}$ satisfying $L\mathbf{d} = 0$.

This implies the existence of a Lagrange multiplier λ such that $J(\mathbf{d}) + \lambda^T L\mathbf{d} = 0$.

$$\nu \mathbf{c}^T \overline{K} + (\overline{B}(\mathbf{c})\mathbf{c})^T + \lambda^T L = \mathbf{F}^T \overline{M}$$

In summary, the discrete solution $\mathbf{c}$ must satisfy

$$\begin{aligned} \nu \mathbf{c}^T \overline{K} + (\overline{B}(\mathbf{c})\mathbf{c})^T + \lambda^T L &= \mathbf{F}^T \overline{M} \\ L\mathbf{c} &= \overline{\mathbf{G}} \end{aligned}$$

This can be written

$$\begin{aligned} \nu \overline{K}\mathbf{c} + \overline{B}(\mathbf{c})\mathbf{c} + L^T \lambda &= \overline{M}\mathbf{F} \\ L\mathbf{c} &= \overline{\mathbf{G}}. \end{aligned}$$

This has a unique solution $\mathbf{c}$ provided the viscosity ν is sufficiently large.

Linearization

A simple iteration algorithm Starting with an initial guess $\mathbf{c}^{(0)}$ which can be computed by solving the Stokes equations, we consider the sequence of problems

$$\begin{aligned} \nu \overline{K}\mathbf{c}^{(n+1)} + \overline{B}(\mathbf{c}^{(n)})\mathbf{c}^{(n+1)} + L^T\lambda^{(n+1)} &= \overline{M}\mathbf{F} \\ L\mathbf{c}^{n+1} &= \overline{\mathbf{G}}, \end{aligned}$$

The following convergence result is similar to one of the convergence results of [Karakashian'82].
The previous system has a unique solution $\mathbf{c}^{(n+1)}$ and the unique solution $\mathbf{c}$ is such that

$$||\mathbf{c}^{(n+1)} - \mathbf{c}||_{H^1(\Omega)^3} \leq \gamma_1 ||\mathbf{c}^{(n)} - \mathbf{c}||_{H^1(\Omega)^3}$$

for a constant $\gamma_1 < 1$. As a consequence $\mathbf{c}^{(n+1)}$ converges to $\mathbf{c}$.

Newton's iterations We are interested in the sequence $\mathbf{c}^{(n+1)}$ defined by

$$\nu \overline{K}\mathbf{c}^{(n+1)} + \overline{B}(\mathbf{c}^{(n)})\mathbf{c}^{(n+1)} + \widetilde{B}(\mathbf{c}^{(n)})\mathbf{c}^{(n+1)} + L^T\lambda^{(n+1)} = \overline{M}\mathbf{F} + \overline{B}(\mathbf{c}^{(n)})\mathbf{c}^{(n)}$$

$$L\mathbf{c}^{(n+1)} = \overline{\mathbf{G}}.$$

$\widetilde{B}$ is defined such that $\widetilde{B}(\mathbf{c})\mathbf{d} = \overline{B}(\mathbf{d})\mathbf{c}$. We have the following convergence result

There exists $r > 0$ such that if $||\mathbf{c} - \mathbf{c}^{(0)}||_{H^1(\Omega)^3} < r$, there is a unique $\mathbf{c}^{(n+1)}$ solution of the system and $||\mathbf{c} - \mathbf{c}^{(n)}||_{H^1(\Omega)^3} < r$ for all n with $||\mathbf{c} - \mathbf{c}^{(n+1)}||_{H^1(\Omega)^3} \leq \frac{1}{r}||\mathbf{c} - \mathbf{c}^{(n)}||_{H^1(\Omega)^3}$. Moreover, if there's $\eta < 1$ such that $||\mathbf{c} - \mathbf{c}^{(0)}||_{H^1(\Omega)^3} = r\eta$, then $\mathbf{c}^{(n)}$ converges to $\mathbf{c}$ as

$$||\mathbf{c} - \mathbf{c}^{(n)}||_{H^1(\Omega)^3} \leq \frac{1}{r^{2^{n-1}}}||\mathbf{c} - \mathbf{c}^{(0)}||^{2^n}_{H^1(\Omega)^3}, \quad n = 1, 2, \ldots$$

Practical computation of c The previous methods all involve to find $\mathbf{c}$ solution of a singular system of type

$$\begin{pmatrix} A & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

with A non symmetric.

Under the hypothese that ν is sufficiently large or $||\mathbf{F}||_{L^2(\Omega)^3}$ is sufficiently small, the symmetric part $(A)_s$ of A is positive definite with respect to L in the sense that $x^T(A)_s x \geq 0$ and $x^T(A)_s x = 0$ with $Lx = 0$ implies $x = 0$.

We show that the later condition is sufficient for the solution $\mathbf{c}$ to be unique. Indeed if $(\mathbf{d}, \beta)$ is another solution we have

$$A(\mathbf{c} - \mathbf{d}) + L^T(\lambda - \beta) = 0.$$

So, with $\mathbf{e} = \mathbf{c} - \mathbf{d}$,

$$\begin{aligned} \mathbf{e}^T\big((A)_s\mathbf{e} + (A)_{as}\mathbf{e} + L^T(\lambda - \beta)\big) &= 0 \\ L\mathbf{e} &= 0. \end{aligned}$$

Here $(A)_{as}$ denotes the antisymmetric part of A. We have $\mathbf{e}^T((A)_{as}\mathbf{e} = 0$ and $\mathbf{e}^T L = 0$. Therefore $\mathbf{e}^T(A)_s\mathbf{e} = 0$ with $L\mathbf{e} = 0$. Thus $\mathbf{c} = \mathbf{d}$. This suggests that we can retrieve the solution $\mathbf{c}$ by computing any least squares solution of the system.

We consider for l$= 0, 1, 2, \ldots,$ the sequence of problems

$$\begin{pmatrix} A & L^T \\ L & -\epsilon I \end{pmatrix} \begin{bmatrix} \mathbf{c}^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} F \\ G - \epsilon\lambda^{(l)} \end{bmatrix}, \tag{1}$$

where $\lambda^{(0)}$ is a suitable initial guess for example $\lambda^{(0)} = 0$, and I is the identity matrix. Let also assume that A is a matrix of size $n \times n$; $c, F \in \mathbb{R}^n$; L is a matrix of size $m \times n$ and $\lambda, G \in \mathbb{R}^m$.

- Theorem

  Suppose that the linear system (of the discrete problem) has a unique solution c. Assume that $A_s = \frac{1}{2}(A + A^T)$ the symmetric part of A is positive definite with respect to L, i.e., $x^T A_s x \geq 0$ and $x^T A_s x = 0$ with $Lx = 0$ implies $x = 0$. Then, the sequence $(c^{(l+1)})$ defined by the iterative method converges to the solution c for any $\epsilon > 0$. Furthermore,

  $$\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|$$

  for some constant C independent of ϵ and l.

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- Proof

  We first show that $c^{(l+1)}$ and $\lambda^{(l+1)}$ are well-defined. Let us first rewrite the iterative method system as follows.

$$Ac^{(l+1)} + L^T\lambda^{(l+1)} = F \quad \text{and (1)}$$
$$Lc^{(l+1)} - \epsilon\lambda^{(l+1)} = G - \epsilon\lambda^{(l)} \quad (2).$$

Multiplying (2) on the left by L^T and substituing $L^T\lambda^{(l+1)}$ into (1) and rewriting (2), we get

$$(A + \frac{1}{\epsilon}L^T L)c^{(l+1)} = -L^T\lambda^{(l)} + F + \frac{1}{\epsilon}L^T G \quad (3)$$
$$\lambda^{(l+1)} + \frac{1}{\epsilon}Lc^{(l+1)} = \lambda^{(l)} + \frac{1}{\epsilon}G.$$

To show that the iterative method system is solvable under the hypotheses of the theorem, we need only to show that $A + \frac{1}{\epsilon}L^T L$ is invertible.

Since A is a square matrix, it is enough to show that

$$(A + \frac{1}{\epsilon} L^T L)x = 0 \Rightarrow x = 0.$$

That is,

$$0 = x^T(A + \frac{1}{\epsilon} L^T L)x = x^T(A_s + \frac{1}{\epsilon} L^T L)x = x^T A_s x + \frac{1}{\epsilon}(Lx)^T(Lx)$$

since $x^T A_a x = 0$. It follows that

$$x^T A_s x = 0 \text{ and } (Lx)^T(Lx) = 0.$$

By the assumptions on A, i.e., A_s is assumed to be symmetric positive definite with respect to L, we get $x = 0$. Hence, the new iterative linear system is invertible and $c^{(l+1)}$ and $\lambda^{(l+1)}$ are well-defined.

We now show that $c^{(l+1)}$ converges to c. Let also $u^{(l+1)} = c^{(l+1)} - c$ and $p^{(l+1)} = \lambda^{(l+1)} - \lambda$. We have

$$\left\{ \begin{array}{rcl} (A + \frac{1}{\epsilon} L^T L) u^{(l+1)} + L^T p^{(l)} & = & 0 \\ p^{(l+1)} & = & p^{(l)} + \frac{1}{\epsilon} L u^{(l+1)}. \end{array} \right.$$

$$\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 = \frac{2}{\epsilon}(A_s u^{(l+1)}, u^{(l+1)}) + \frac{1}{\epsilon^2}\|Lu^{(l+1)}\|^2.$$

We conclude that since A_s is nonnegative,

$$\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 \geq 0,$$

and the sequence $\{\|p^{(l)\|}\}$ is seen to be decreasing.

Being bounded below by 0, it converges; hence $\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2$ converges to 0 which implies that $(A_s u^{(l+1)}, u^{(l+1)})$ and $\|Lu^{(l+1)}\|^2$ converge to 0. Since $A_s + \frac{1}{\epsilon} L^T L$ is positive definite, it follows that $u^{(l+1)}$ converges to 0 and finally $c^{(l+1)}$ converges to c.

Sketch of proof of convergence rate

We prove that

$$\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|,$$

Recall that $u^{(l+1)} = c^{(l+1)} - c$ and $p^{(l+1)} = \lambda^{(l+1)} - \lambda$. We showed that

$$\|p^{(l+1)}\| \leq \|p^{(l)}\|, \quad \text{for all } l$$

i.e. that $(p^{(l)})$ is a decreasing sequence. We also have

$$\begin{cases} (A + \frac{1}{\epsilon} L^T L) u^{(l+1)} + L^T p^{(l)} & = & 0 \\ p^{(l+1)} & = & p^{(l)} + \frac{1}{\epsilon} L u^{(l+1)}, \end{cases}$$

from which it follows that

$$A u^{(l+1)} + L^T p^{(l+1)} = 0$$

We write $u^{(l+1)} = \hat{u}^{(l+1)} + \overline{u}^{(l+1)}$ with $\hat{u}^{(l+1)} \in \text{Ker}(L)$ and $\overline{u}^{(l+1)} \in \text{Im}(L^T)$. Note that $L : \text{Im}(L^T) \to \text{Im}(L)$ has a bounded inverse, so there exists $k_0 > 0$ such that

$$\|\overline{u}^{(l+1)}\| \leq \frac{1}{k_0} \|Lu^{(l+1)}\|,$$

from which it follows that

$$\|\overline{u}^{(l+1)}\| \leq \frac{2\epsilon}{k_0} \|p^{(l)}\|$$

To get a bound on $\|\hat{u}^{(l+1)}\|$, we notice that A is invertible on $\text{Ker}(L)$ since $A + \frac{1}{\epsilon} L^T L$ is invertible. This gives for some $\alpha_0 > 0$,

$$\|\hat{u}^{(l+1)}\| \leq \frac{1}{\alpha_0} \sup_{v_0 \in \text{Ker}(L)} \frac{(v_0, A\hat{u}^{(l+1)})}{\|v_0\|} = \sup_{v_0 \in \text{Ker}(L)} \frac{-v_0^T A \overline{u}^{(l+1)}}{\|v_0\|} \leq \|A\| \, \|\overline{u}^{(l+1)}\|$$

Putting together, we obtain

$$\|u^{(l+1)}\| \leq C\epsilon\|p^{(l)}\|, \quad \text{for some constant C} > 0$$

To finish, we need a bound on $\|p^{(l)}\|$ in terms of $\|u^{(l)}\|$. It can be shown that one can choose λ_0 such that $p^{(l)} \in$ Im(L) and since $L^T : \mathrm{Im}(L) \rightarrow \mathrm{Im}(L^T)$ has a bounded inverse,

$$\|p^{(l)}\| \leq \frac{1}{k_0}\|L^T p^{(l)}\|.$$

This completes the proof since $L^T p^{(l)} = -Au^{(l)}$.

Computational Experiments on the 3D Stokes Equations

Let $\Omega \subset \mathbb{R}^3$ be a cube with sides of length 1. We consider the vector field $\mathbf{u} = (u_1, u_2, u_3)$ with a pressure p.

$$u_1 = -\exp(x + 2y + 3z)$$

$$u_2 = 2\ \exp(x + 2y + 3z)$$

$$u_3 = -\exp(x + 2y + 3z)$$

$$p = x(1-x)z(1-z).$$

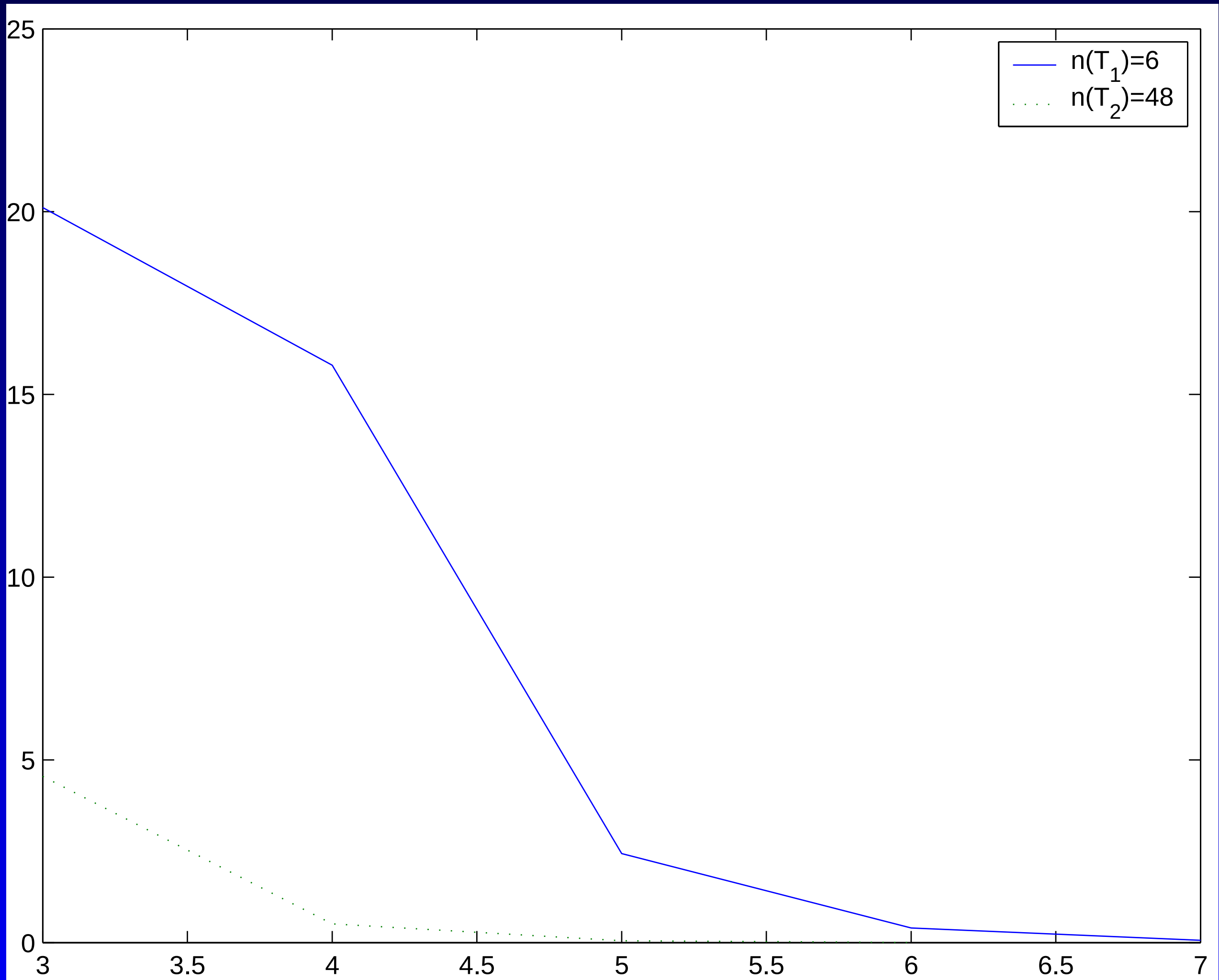
Table 1 Approximation Errors from Trivariate Spline Spaces on $\mathcal{I}_1$

degrees	u_1	u_2	u_3	p
3	3.3633×10	5.9431×10	4.0397×10	1.3466×10^3
4	1.7010×10	4.4374×10	3.5368×10	3.8562×10^2
5	2.3804	7.3711	5.9629	9.8470×10^1
6	3.9620×10^{-1}	1.2238	1.0311	2.7404×10^1
7	6.7456×10^{-2}	1.9789×10^{-1}	1.6260×10^{-1}	6.8411
Rate	$1.56\times10^7\ d^{-9.8294}$	$3.22\times10^7\ d^{-9.6203}$	$2.32\times10^7\ d^{-9.5463}$	$8.50\times10^6\ d^{-7.13}$

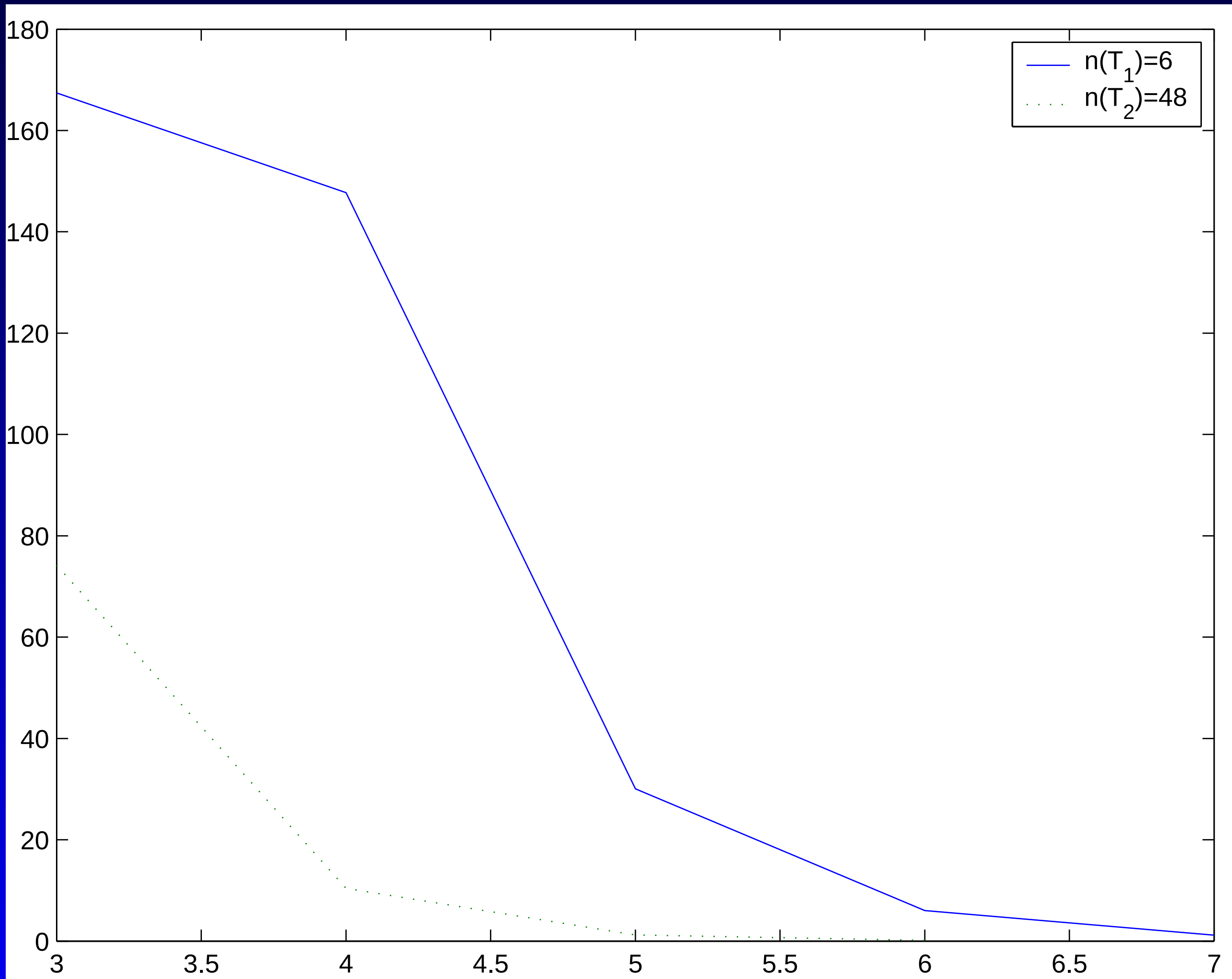
Table 2 Approximation Errors from Trivariate Spline Spaces on $\mathcal{I}_2$

degrees	u_1	u_2	u_3	p
3	1.5083×10	1.8709×10	1.5222×10	4.4382×10^2
4	9.4142×10^{-1}	2.2094	1.8373	3.5278×10^1
5	9.1619×10^{-2}	2.2322×10^{-1}	2.0176×10^{-1}	5.8199
6	8.5128×10^{-3}	2.3520×10^{-2}	1.9276×10^{-2}	7.1884×10^{-1}
Rate	$9.31\times10^6\ d^{-11.5631}$	$1.24\times10^7\ d^{-11.1692}$	$1.09\times10^7\ d^{-11.1901}$	$1.05\times10^7\ d^{-9}$

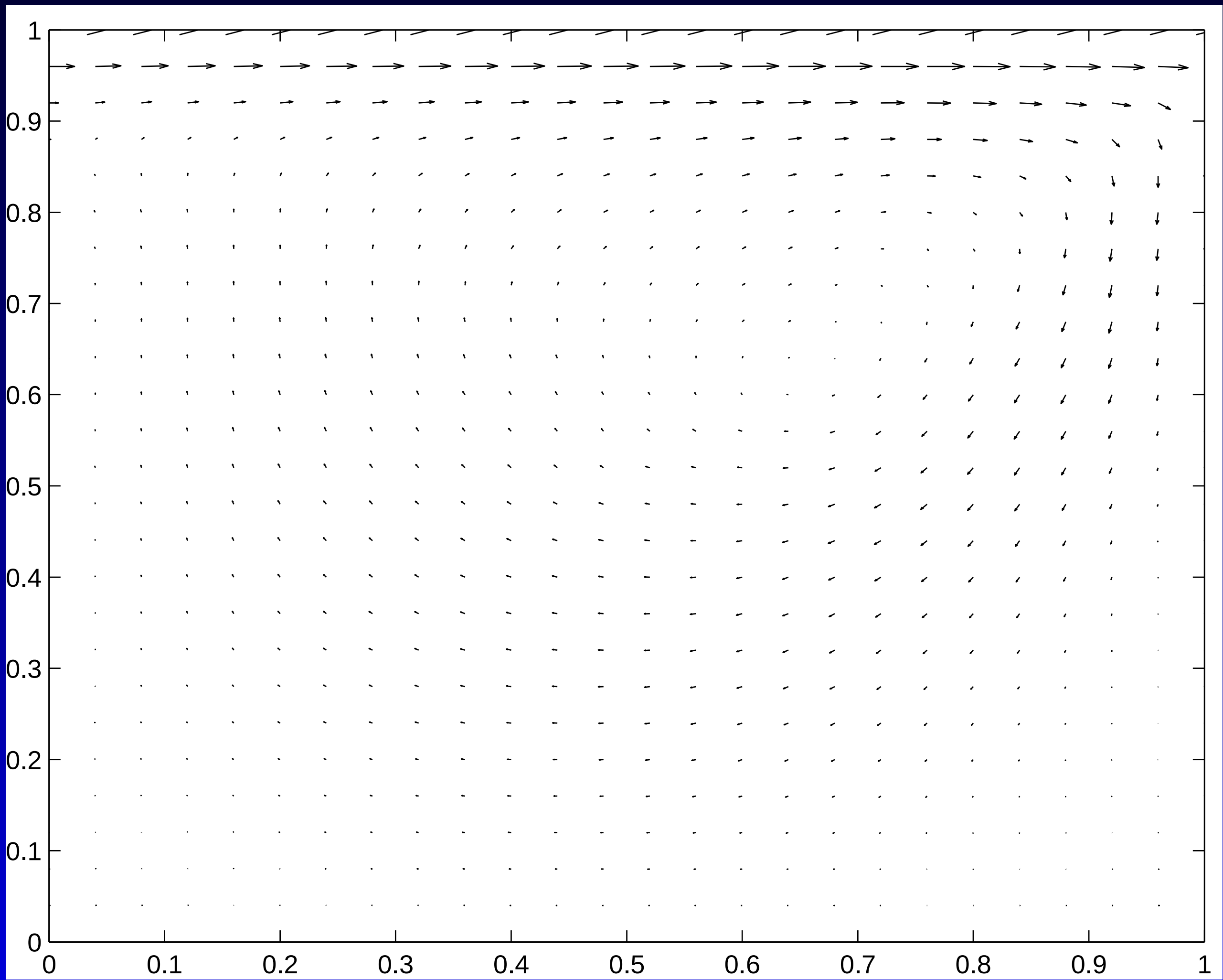
L^2 norm of the error versus degree on $\mathcal{T}_1$ (rate 1.6777×10^7 $d^{-9.8962}$) and $\mathcal{T}_2$ (rate 7.7013×10^6 $d^{-11.8503}$)



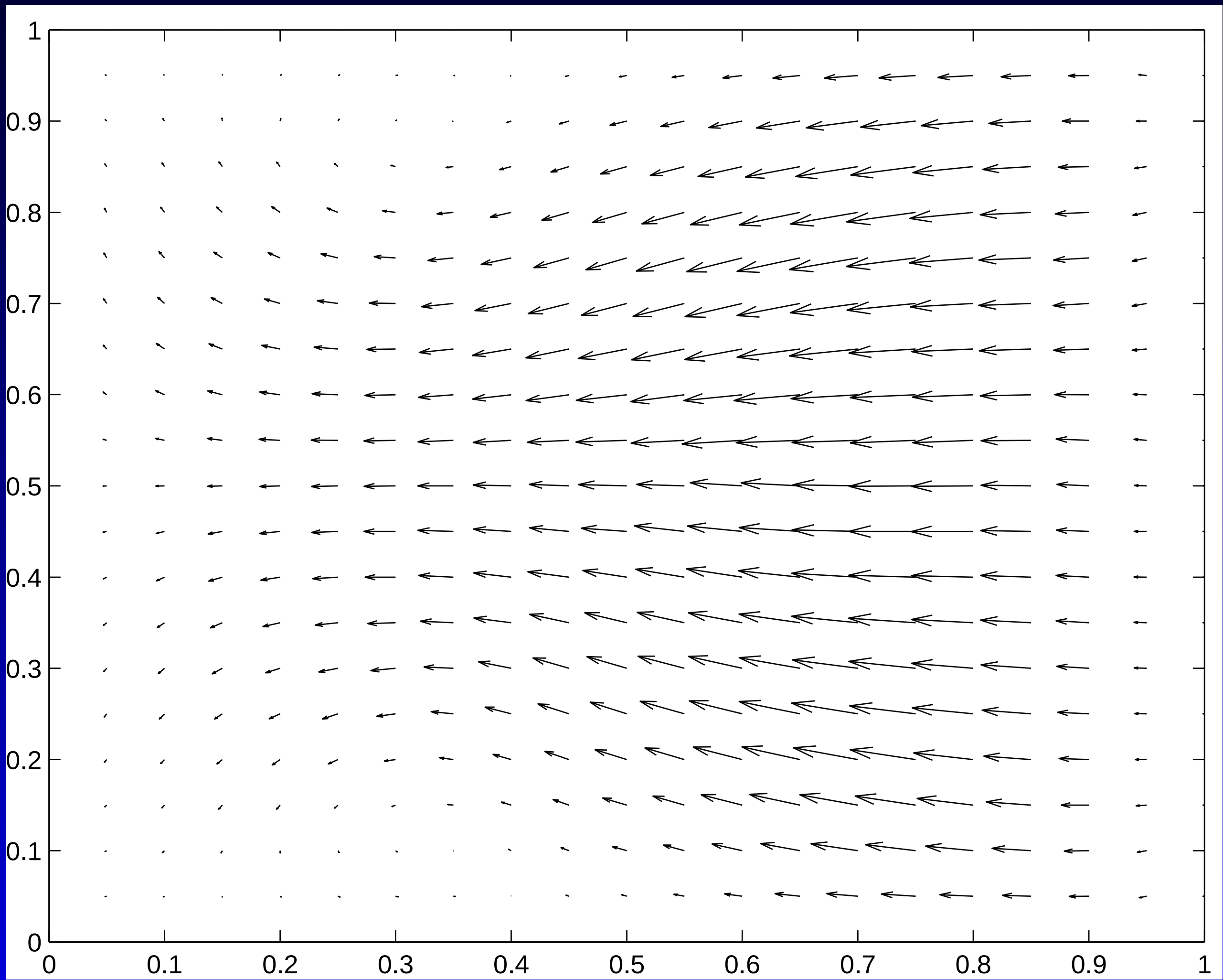
H^1 norm of the error versus degree on $\mathcal{T}_1$ (rate 1.6777×10^7 $d^{-9.8962}$) and $\mathcal{T}_2$ (rate 7.7013×10^6 $d^{-11.8503}$)



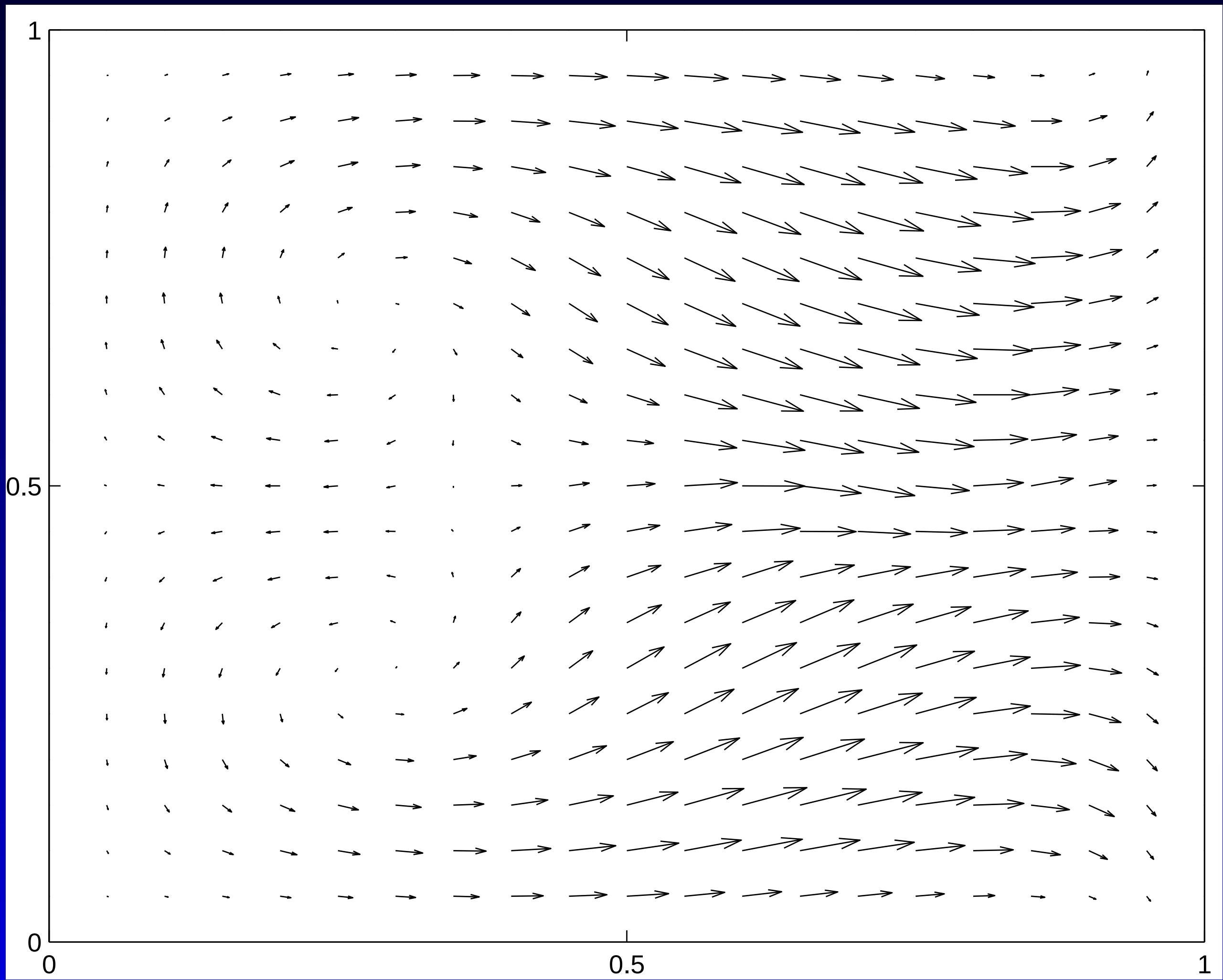
Lid Driven Cavity Flow Problem



3D fluid profile in the $x - y$ plane



3D fluid profile in the $y - z$ plane



3D fluid profile in the $x - z$ plane

Work in Progress

- Time dependent Navier-Stokes

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- Extension to variational inequalities

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- Thank You!