Trivariate Spline Approximations of 3D Navier-Stokes Equations

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Overview

The Navier-Stokes Equations

Features of the Spline Method

Trivariate Splines

Discretization of the Stokes Equations

Discretization of the Navier-Stokes Equations

Iterative Method for Solving the Discrete Problem

Computational Experiments

Work in Progress
The Navier-Stokes equations

\[
\begin{aligned}
-\nu \ \Delta u + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega.
\end{aligned}
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\end{aligned}
\]  

(1)

\[
V_0 = \{ v \in H^1_0(\Omega)^3, \text{div } v = 0 \}
\]

\[
L^2_0(\Omega) = \{ u \in L^2(\Omega), \int_{\Omega} u = 0 \} \quad \text{and}
\]

\[
H^{1/2}(\partial \Omega) = \{ \tau(u), u \in H^1(\Omega) \},
\]
Existence and Uniqueness

Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary. For $f \in H^{-1}(\Omega)^3$ and $g \in H^{1/2}(\partial\Omega)^3$ satisfying $\int_{\partial\Omega} g \cdot n = 0$, the problem: find $(u, p) \in H^1(\Omega)^3 \times L^2_0(\Omega)$ such that

$$
\begin{cases}
-\nu \Delta u + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u + \nabla p &= f & \text{in } \Omega \\
\text{div } u &= 0 & \text{in } \Omega \\
u \begin{array}{c}
\end{array}u &= g & \text{on } \partial\Omega,
\end{cases}
$$

has a solution which is unique provided that $\nu$ is sufficiently large.
Features of the spline method

- We shall assume that $\Omega$ is a polygonal domain of $\mathbb{R}^3$ with a tetrahedral partition $\mathcal{T}$ and use the spline space

$$S^r_d(\mathcal{T}) = \{ s \in C^r(\Omega), \ s|_t \in \mathbb{P}_d, \ \forall t \in \mathcal{T} \} ,$$

where $\mathbb{P}^d$ is the space of polynomials of total degree $d$.

We use the $B$-form of splines and associate to each component $u_i$ of $u$, $i = 1, \ldots, 3$ a vector of coefficients $c_i$. 
Features of the spline method

- We shall assume that $\Omega$ is a polygonal domain of $\mathbb{R}^3$ with a tetrahedral partition $T$ and use the spline space

\[ S_d^r(T) = \{ s \in C^r(\Omega), \ s|_t \in \mathbb{P}_d, \ \forall t \in T \}, \]

where $\mathbb{P}_d$ is the space of polynomials of total degree $d$.

We use the $B$-form of splines and associate to each component $u_i$ of $u$, $i = 1, \ldots, 3$ a vector of coefficients $c_i$.

- Smoothness requirements on $c$. In general, smoothness can be imposed in a flexible way across the domain.
Features of the spline method

- Polynomials of high degrees can be easily used locally to get better approximation properties.
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- Polynomials of high degrees can be easily used locally to get better approximation properties.

- Weak formulation: Find $u \in H^1(\Omega)^3$ such that

\[
\nu \int_{\Omega} \nabla u \cdot \nabla v + \sum_{j=1}^{3} \int_{\Omega} u_j \frac{\partial u}{\partial x_j} \cdot v = \int_{\Omega} f \cdot v \quad \forall v \in V_0
\]

\[
\text{div } u = 0 \quad \text{in } \Omega
\]

\[
u u = g \quad \text{on } \partial \Omega
\]
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\text{div } u = 0 \quad \text{in } \Omega
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\[
u \bar{K} c + \bar{B}(c)c + L^T \lambda = \bar{M} \bar{F}
\]

\[
Lc = \bar{G}
\]
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- The mass and stiffness matrices can be assembled easily and these processes can be done in parallel.
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• The mass and stiffness matrices can be assembled easily and these processes can be done in parallel.

• The pressure is computed by solving a Poisson equation with Neumann boundary conditions.
Trivariate Splines

Let \( d \geq 1 \) and \( r \geq 0 \) be two fixed integers. Given a bounded domain \( \Omega \) of \( \mathbb{R}^3 \) with piecewise planar boundary, let \( \mathcal{T} \) be a tetrahedral partition of \( \Omega \).

\[
S_{d}^{r}(\Omega) = \{ p \in C^{r}(\Omega), \ p|_{t} \in P_{d}, \ \forall t \in \mathcal{T} \}.
\]

\[
p(x, y, z) = \sum_{0 \leq i+j+k \leq d} \alpha_{ijk} x^{i} y^{j} z^{k},
\]

Barycentric coordinates

Given a non-degenerate tetrahedron \( T = \langle v_1, v_2, v_3, v_4 \rangle \), any point \( v = (x, y, z) \) can be written uniquely in the form

\[
v = b_1 v_1 + b_2 v_2 + b_3 v_3 + v_4 b_4 \text{ with } b_1 + b_2 + b_3 + b_4 = 1.
\]
B-form of splines

Bernstein polynomials of degree $d$

$$B_{ijkl}^d(v) = \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \quad i + j + k + l = d.$$

They are polynomials of degree $d$ since each $b_i$ is a linear polynomial. The set $\mathcal{B}^d = \{B_{ijkl}^d(x, y, z), i + j + k + l = d\}$ is a basis for the space of polynomials $P_d$.

We recall that the dimension of $P_d$ is $\binom{d+3}{3}$.

As a consequence any spline $s$ in $S^r_d$ can be written uniquely

$$s|_T = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^d,$$

since $s|_T$ is a polynomial of degree $d$.

$n_{th} = \text{number of tetrahedra}, \ m = \dim P_d$ and $N = m \ast n_{th}$. 

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Bivariate splines

The restriction of a trivariate polynomial of degree $d$ on a face of a tetrahedron is a bivariate polynomial and can be written in $B$-form

$$
\sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{B}^d_{ijk}(v),
$$

where

$$
\tilde{B}^d_{ijk} = \frac{d!}{i!j!k!} b^i_1 b^j_2 b^k_3.
$$

For example, given the trivariate spline on a tetrahedron $T$

$$
p = \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl}, \quad q = \sum_{i+j+k=d} c_{ijk0} B^d_{ijk0}
$$

can be considered as a bivariate polynomial.
Interpolation

There is a unique polynomial \( p \) of degree \( d \) that interpolates any given function \( f \) on a tetrahedron \( T = \langle v_1, v_2, v_3, v_4 \rangle \) at the domain points

\[
\xi_{ijkl} = \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}.
\]

This gives rise to an interpolation operator \( \Pi_d \). \( \Pi_d(f) \) will denote both the spline interpolant and its \( B \)-net.

We can also define a boundary interpolation operator \( \Pi_d^b \) since a bivariate polynomial \( p \) of degree \( d \) is uniquely determined on a triangle \( \langle v_1, v_2, v_3 \rangle \) by its values at the domain points \( \xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} \). Note that here the domain points have three indices.

We have for a spline \( s \) with \( B \)-net \( c \) according to our notations

\[
Rc = \Pi_d^b(s)
\]
Derivatives

We want to give formulas for the directional derivatives of \( p \) in a direction defined by a vector \( \mathbf{u} \) joining the points \( v_1 \) and \( v_2 \). Let \( \mathbf{a} = (a_1, a_2, a_3, a_4) \) with components the difference of the barycentric coordinates of \( v_1 \) and \( v_2 \). \( D_{\mathbf{u}} p \) can be written in \( B \)-form as a polynomial of degree \( d - 1 \).

\[
D_{\mathbf{u}} p = d \sum_{i+j+k+l=d-1} c^{(1)}_{ijkl}(\mathbf{a}) B^{d-1}_{ijkl}, \quad \text{where}
\]

\[
c^{(1)}_{ijkl}(\mathbf{a}) = a_1 c_{i+1,j,k,l} + a_2 c_{i,j+1,k,l} + a_3 c_{i,j,k+1,l} + a_4 c_{i+1,j,k,l+1}.
\]

It’s not difficult to see that there are matrices \( D_1, D_2 \) and \( D_3 \) such that if \( c \) encodes the \( B \)-net of \( s \), \( D_i c, i = 1, \ldots, 3 \) encode respectively the \( B \)-net of \( \frac{\partial s}{\partial x_i} \).
Integration
There’s a matrix $G$ such that if $p$ and $q$ have $B$-nets $c$ and $d$,

$$\int_{\Omega} pq = c^T G d$$

Smoothness conditions
Let $t = \langle v_1, v_2, v_3, v_4 \rangle$ and $t' = \langle v_1, v_2, v_3, v_5 \rangle$ be two tetrahedra with common face $\langle v_1, v_2, v_3 \rangle$. Then $s$ is of class $C^r$ on $t \cup t'$ if and only if

$$c_{ijkm}^t = \sum_{\mu + \nu + \kappa + \delta = m} c_{i+\mu, j+\nu, \gamma+\kappa, \delta}^t B^l_{\mu, \nu, \kappa, \delta}(v_5), \ m = 0, \ldots, r, \ i + j + k = d - m$$

This suggests that there’s a $(l, N)$ matrix $H$ such that $s$ is in $C^r(\Omega)$ if and only if

$$H c = 0.$$
Discretization

The weak form of the Navier-Stokes equations is: Find \( u \in H^1(\Omega)^3 \) such that

\[
\nu \int_{\Omega} \nabla u \cdot \nabla v + \sum_{j=1}^{3} \int_{\Omega} u_j \frac{\partial u}{\partial x_j} \cdot v = \int_{\Omega} f \cdot v \quad \forall v \in V_0
\]

\[
\text{div } u = 0 \quad \text{in } \Omega
\]

\[
u \int_{\partial \Omega} u \cdot n = \int_{\partial \Omega} g \quad \text{on } \partial \Omega
\]

where

\[
V_0 = \{ v \in H^1_0(\Omega)^3, \text{div } v = 0 \}.
\]

We now consider spline approximations of the velocity vector field \( u \). Let \( d \geq 1 \) and \( r \geq 0 \) be two given integers.
Let also $S \subset S^0_d(\mathcal{T})$ be a spline subspace over a tetrahedral partition $\mathcal{T}$ of $\Omega$ consisting of spline functions which are $C^r$ inside $\Omega$ and $C^0$ near the boundary $\partial \Omega$.

- Recall that there is a matrix $H$ such that if $s \in S$ with B-coefficient vector $c$, then

$$Hc = 0.$$
Let also $S \subset S_0^d(\mathcal{T})$ be a spline subspace over a tetrahedral partition $\mathcal{T}$ of $\Omega$ consisting of spline functions which are $C^r$ inside $\Omega$ and $C^0$ near the boundary $\partial \Omega$.

- Recall that there is a matrix $H$ such that if $s \in S$ with B-coefficient vector $c$, then

$$Hc = 0.$$ 

- Also recall that there is a matrix $R$ which maps $c$ to the $B$-coefficients of $s$ on the boundary of $\Omega$ and $Rc = G$ represents the boundary condition, i.e., $s = g$ on the boundary approximately.
Finally there are matrices $D_1$, $D_2$ and $D_3$ such that if $c$ encodes the $B$-net of $s$, $D_i c$, $i = 1, \ldots, 3$ encode respectively the $B$-net of $\frac{\partial s}{\partial x_i}$. 
• Finally there are matrices $D_1$, $D_2$ and $D_3$ such that if $c$ encodes the $B$-net of $s$, $D_i c$, $i = 1, \ldots, 3$ encode respectively the $B$-net of $\frac{\partial s}{\partial x_i}$.

• $\mathbf{u} = (u_1, u_2, u_3)$ velocity vector
  $s_{\mathbf{u}} = (s_1, s_2, s_3)$ spline approximating vector
  $s_i \in S$ satisfying $H \mathbf{c}_i = 0, R(\mathbf{c}_i) = G(g_i)$ for $i = 1, 2, 3$.

  $\text{div} \mathbf{u} = 0$ is discretized as $D_1 \mathbf{c}_1 + D_2 \mathbf{c}_2 + D_3 \mathbf{c}_3 = 0$
• Finally there are matrices \( D_1, D_2 \) and \( D_3 \) such that if \( c \) encodes the \( B \)-net of \( s \), \( D_i c, i = 1, \ldots, 3 \) encode respectively the \( B \)-net of \( \frac{\partial s}{\partial x_i} \).

• \( u = (u_1, u_2, u_3) \) velocity vector

• \( s_u = (s_1, s_2, s_3) \) spline approximating vector

• \( s_i \in S \) satisfying \( Hc_i = 0, R(c_i) = G(g_i) \) for \( i = 1, 2, 3 \).

• \( \text{div} \ u = 0 \) is discretized as \( D_1 c_1 + D_2 c_2 + D_3 c_3 = 0 \)

• Let

\[
\overline{H} = \begin{pmatrix}
H & 0 & 0 \\
0 & H & 0 \\
0 & 0 & H
\end{pmatrix}, \quad \overline{R} = \begin{pmatrix}
R & 0 & 0 \\
0 & R & 0 \\
0 & 0 & R
\end{pmatrix},
\]
\[ \mathbf{G} = (G(g_1), G(g_2), G(g_3))^T \quad \overline{\mathbf{D}} = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}. \]

\[ \overline{H} \mathbf{c} = 0, \overline{R} \mathbf{c} = \mathbf{G} \text{ and } \overline{D} \mathbf{c} = 0 \]

\[ L = \left[ \begin{array}{ccc} \overline{H}^T & \overline{R}^T & \overline{D}^T \end{array} \right]^T \quad \text{and} \quad \overline{\mathbf{G}} = \left[ \begin{array}{ccc} 0 & \mathbf{G}^T & 0 \end{array} \right]^T. \]
\[
G = (G(g_1), G(g_2), G(g_3))^T \quad \overline{D} = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}.
\]

\[
\overline{H}c = 0, \overline{R}c = G \text{ and } \overline{D}c = 0
\]

\[
L = \begin{bmatrix} \overline{H}^T & \overline{R}^T & \overline{D}^T \end{bmatrix}^T \text{ and } \overline{G} = \begin{bmatrix} 0 & G^T & 0 \end{bmatrix}^T,
\]

In other words, if we let

\[
S_g = \{ c \in (\mathbb{R}^N)^3, Lc = \overline{G} \},
\]

we are seeking for a solution in \(S_g\). We approximate elements of \(V_0\) by vectors \(d = (d_1, d_2, d_3)\) in

\[
S_0 = \{ d \in (\mathbb{R}^N)^3, Ld = 0 \}.
\]
\[ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \]

\[ b(w; u, v) = \sum_{j=1}^{3} \int_{\Omega} w_j \frac{\partial u}{\partial x_j} \cdot v. \]

\[ \int_{\Omega} fv \]

\[ M^t = \left( \int_{t} B^d_{\alpha} B^d_{\beta} \right)_{|\alpha|=d,|\beta|=d} \text{, local mass matrix} \]

\[ M \text{ is the global mass matrix} \]

\[ K^t = \left( \int_{t} \nabla B^d_{\alpha} \nabla B^d_{\beta} \right)_{|\alpha|=d,|\beta|=d} \text{, local stiffness matrix} \]

\[ K \text{ global stiffness matrix} \]
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<th>Discrete</th>
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<td>( \mathbf{u} = (u_1, u_2, u_3) )</td>
<td>( \mathbf{c} = (c_1, c_2, c_3) )</td>
</tr>
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Discretization of the Stokes equations

Corollary of the Lax-Milgram lemma
Let $V$ be a real Hilbert space with norm denoted by $\| \cdot \|_V$, $(u, v) \mapsto a(u, v)$ a real bilinear form on $V \times V$, $l$ an element of the dual of $V$ and let us denote the duality pairing between $V$ and its dual $V'$ by $<, >$. If $a$ is continuous, symmetric and is elliptic on $V$ i.e. there is $\alpha > 0$ such that $a(v, v) \geq \alpha \|v\|^2_V$ for all $v \in V$, then, the problem:

Find $u \in V$ such that

$$a(u, v) = < l, v >,$$

has one an only one solution which minimizes the following functional over $V$

$$J(v) = \frac{1}{2}a(v, v) - < l, v >.$$
Under the same hypotheses as in the theorem on the Navier-Stokes equations, the Stokes equations:

\[
\begin{aligned}
-\nu \Delta u + \nabla p &= f \text{ in } \Omega \\
\text{div } u &= 0 \text{ in } \Omega \\
u u &= g \text{ on } \partial \Omega
\end{aligned}
\]

have a unique solution \( u \) in \( H^1(\Omega)^3 \) and a pressure \( p \) in \( L^2(\Omega) \) unique up to an additive constant. These equations are derived under the assumption that the velocity is sufficiently small to ignore the nonlinear term \( u \cdot \nabla u(x, t) \).

The weak form of the equations is: Find \( u \) in \( H^1(\Omega)^3 \) such that \( \text{div } u = 0 \) and \( \nu \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f \cdot v, \quad \forall v \in V_0 \),
\[ V_0 = \{ \mathbf{v} \in H^1_0(\Omega)^3 \text{ such that } \text{div } \mathbf{v} = 0 \}. \]

In this case, the velocity vector \( \mathbf{u} \) is the unique minimizer in

\[ V = \{ \mathbf{v} \in H^1(\Omega)^3 \text{ such that } \text{div } \mathbf{v} = 0 \} \]

of the functional

\[
J(\mathbf{u}) = \frac{\nu}{2} \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{u} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u}.
\]

If we let \( \mathbf{c} \) encode the \( B \)-net of the approximant, the discrete problem is: Minimize

\[
J(\mathbf{c}) = \frac{\nu}{2} \mathbf{c}^T \mathbf{K} \mathbf{c} + \mathbf{F}^T \mathbf{M} \mathbf{c}
\]

over \( (\mathbb{R}^N)^3 \) under the constraint \( L \mathbf{c} = \mathbf{G} \).
By the theory of Lagrange multipliers, there is a vector of Lagrange multipliers $\lambda$ such that

$$\begin{cases}
\nu K c + L^T \lambda &= \overline{M} F, \\
L c &= \overline{G}.
\end{cases}$$
Computation of the pressure term

Assuming that $u$ is smooth and taking the divergence of the equation

$$-\nu \Delta u + \nabla p = f \text{ in } \Omega$$

we get

$$-\Delta p = -\text{div } f$$

since $\text{div } u = 0$. Here, the pressure is the minimizer over

$$L^2_0(\Omega) = \{ p \in L^2(\Omega), \int_\Omega p = 0 \}$$

of

$$Q(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega (-\text{div } f)v - \int_{\partial \Omega} (f \cdot n + \nu (\Delta u) \cdot n)v.$$  

Discretization for the pressure in Navier-Stokes equations is similar.
Discretization of the Navier-Stokes equations

Find \( c \) in \( \mathbb{R}^{3N} \) satisfying \( Lc = \overline{G} \) with \( \overline{G} \) encoding the side conditions and

\[
\nu c^T K d + (\overline{B}(c)c)^T d = d^T M F
\]

for all \( d \) in \( \mathbb{R}^{3N} \) with constraints \( Ld = 0 \).

Here, \( K \) and \( M \) are the stiffness and mass matrices respectively; \((\overline{B}c)d\) encodes the nonlinear term. If one considers the following linear functional in \( d \),

\[
J(d) = (\nu c^T K + (\overline{B}(c)c)^T + F^T M) d,
\]

we have \( J(d) = 0 \) for all \( d \) satisfying \( Ld = 0 \).
This implies the existence of a Lagrange multiplier $\lambda$ such that
\[ J(d) + \lambda^T Ld = 0. \]

\[ \nu c^T K + (\overline{B}(c)c)^T + \lambda^T L = F^T \overline{M} \]

In summary, the discrete solution $c$ must satisfy
\[ \nu c^T K + (\overline{B}(c)c)^T + \lambda^T L = F^T \overline{M} \]
\[ Lc = \overline{G} \]

This can be written
\[ \nu Kc + \overline{B}(c)c + L^T \lambda = \overline{M} F \]
\[ Lc = \overline{G}. \]

This has a unique solution $c$ provided the viscosity $\nu$ is sufficiently large.
Linearization

A simple iteration algorithm Starting with an initial guess \( c^{(0)} \) which can be computed by solving the Stokes equations, we consider the sequence of problems

\[
\nu K c^{(n+1)} + B(c^{(n)})c^{(n+1)} + LT \lambda^{(n+1)} = MF \\
Lc^{n+1} = \overline{G},
\]

The following convergence result is similar to one of the convergence results of [Karakashian’82].

The previous system has a unique solution \( c^{(n+1)} \) and the unique solution \( c \) is such that

\[
\|c^{(n+1)} - c\|_{H^1(\Omega)^3} \leq \gamma_1 \|c^{(n)} - c\|_{H^1(\Omega)^3}
\]

for a constant \( \gamma_1 < 1 \). As a consequence \( c^{(n+1)} \) converges to \( c \).
**Newton’s iterations** We are interested in the sequence $c^{(n+1)}$ defined by

$$
\nu \hat{K}c^{(n+1)} + \bar{B}(c^{(n)})c^{(n+1)} + \tilde{B}(c^{(n)})c^{(n+1)} + L^T \lambda^{(n+1)} =
\begin{align*}
\bar{M}F + \bar{B}(c^{(n)})c^{(n)}
\end{align*}
Lc^{(n+1)} = \overline{G}.
$$

$\tilde{B}$ is defined such that $\tilde{B}(c)d = \bar{B}(d)c$. We have the following convergence result. There exists $r > 0$ such that if $||c - c^{(0)}||_{H^1(\Omega)^3} < r$, there is a unique $c^{(n+1)}$ solution of the system and $||c - c^{(n)}||_{H^1(\Omega)^3} < r$ for all $n$ with $||c - c^{(n+1)}||_{H^1(\Omega)^3} \leq \frac{1}{r} ||c - c^{(n)}||_{H^1(\Omega)^3}$. Moreover, if there’s $\eta < 1$ such that $||c - c^{(0)}||_{H^1(\Omega)^3} = r\eta$, then $c^{(n)}$ converges to $c$ as

$$
||c - c^{(n)}||_{H^1(\Omega)^3} \leq \frac{1}{r^{2^n - 1}} ||c - c^{(0)}||_{H^1(\Omega)^3}^{2^n}, \quad n = 1, 2, \ldots
$$
Practical computation of $c$  The previous methods all involve to find $c$ solution of a singular system of type

$$
\begin{pmatrix}
A & L^T \\
L & 0
\end{pmatrix}
\begin{pmatrix}
c \\
\lambda
\end{pmatrix} =
\begin{bmatrix}
F \\
G
\end{bmatrix},
$$

with $A$ non symmetric.

Under the hypothesis that $\nu$ is sufficiently large or $\|F\|_{L^2(\Omega)^3}$ is sufficiently small, the symmetric part $(A)_s$ of $A$ is positive definite with respect to $L$ in the sense that $x^T(A)_s x \geq 0$ and $x^T(A)_s x = 0$ with $Lx = 0$ implies $x = 0$. 
We show that the later condition is sufficient for the solution $\mathbf{c}$ to be unique. Indeed if $(\mathbf{d}, \beta)$ is another solution we have
\[ A(\mathbf{c} - \mathbf{d}) + L^T(\lambda - \beta) = 0. \]

So, with $\mathbf{e} = \mathbf{c} - \mathbf{d}$,
\[ \mathbf{e}^T((A)_s\mathbf{e} + (A)_{as}\mathbf{e} + L^T(\lambda - \beta)) = 0 \]
\[ Le = 0. \]

Here $(A)_{as}$ denotes the antisymmetric part of $A$. We have $\mathbf{e}^T((A)_{as}\mathbf{e} = 0$ and $\mathbf{e}^T L = 0$. Therefore $\mathbf{e}^T(A)_s\mathbf{e} = 0$ with $Le = 0$. Thus $\mathbf{c} = \mathbf{d}$.

This suggests that we can retrieve the solution $\mathbf{c}$ by computing any least squares solution of the system.
We consider for \( l= 0, 1, 2, \ldots \), the sequence of problems

\[
\begin{pmatrix}
A & L^T \\
L & -\epsilon I
\end{pmatrix}
\begin{bmatrix}
c^{(l+1)} \\
\lambda^{(l+1)}
\end{bmatrix}
= 
\begin{bmatrix}
F \\
G - \epsilon \lambda^{(l)}
\end{bmatrix},
\]

where \( \lambda^{(0)} \) is a suitable initial guess for example \( \lambda^{(0)} = 0 \), and \( I \) is the identity matrix. Let also assume that \( A \) is a matrix of size \( n \times n \); \( c, F \in \mathbb{R}^n \); \( L \) is a matrix of size \( m \times n \) and \( \lambda, G \in \mathbb{R}^m \).
Theorem

Suppose that the linear system (of the discrete problem) has a unique solution \( c \). Assume that \( A_s = \frac{1}{2}(A + A^T) \) the symmetric part of \( A \) is positive definite with respect to \( L \), i.e., \( x^T A_s x \geq 0 \) and \( x^T A_s x = 0 \) with \( Lx = 0 \) implies \( x = 0 \). Then, the sequence \( (c^{(l+1)}) \) defined by the iterative method converges to the solution \( c \) for any \( \epsilon > 0 \). Furthermore,

\[
\|c - c^{(l+1)}\| \leq C \epsilon \|c - c^{(l)}\|
\]

for some constant \( C \) independent of \( \epsilon \) and \( l \).
Theorem

Suppose that the linear system (of the discrete problem) has a unique solution $c$. Assume that $A_s = \frac{1}{2}(A + A^T)$ the symmetric part of $A$ is positive definite with respect to $L$, i.e., $x^T A_s x \geq 0$ and $x^T A_s x = 0$ with $Lx = 0$ implies $x = 0$. Then, the sequence $(c^{(l+1)})$ defined by the iterative method converges to the solution $c$ for any $\epsilon > 0$. Furthermore,

$$\|c - c^{(l+1)}\| \leq C\epsilon\|c - c^{(l)}\|$$

for some constant $C$ independent of $\epsilon$ and $l$.

Proof

We first show that $c^{(l+1)}$ and $\lambda^{(l+1)}$ are well-defined. Let us first rewrite the iterative method system as follows.
\[ Ac^{(l+1)} + L^T \lambda^{(l+1)} = F \quad \text{and (1)} \]
\[ Lc^{(l+1)} - \varepsilon \lambda^{(l+1)} = G - \varepsilon \lambda^{(l)} \quad (2). \]

Multiplying (2) on the left by \( L^T \) and substituting \( L^T \lambda^{(l+1)} \) into (1) and rewriting (2), we get

\[
\left( A + \frac{1}{\varepsilon} L^T L \right) c^{(l+1)} = -L^T \lambda^{(l)} + F + \frac{1}{\varepsilon} L^T G \quad (3)
\]
\[
\lambda^{(l+1)} + \frac{1}{\varepsilon} Lc^{(l+1)} = \lambda^{(l)} + \frac{1}{\varepsilon} G.
\]

To show that the iterative method system is solvable under the hypotheses of the theorem, we need only to show that \( A + \frac{1}{\varepsilon} L^T L \) is invertible.
Since $A$ is a square matrix, it is enough to show that

$$(A + \frac{1}{\epsilon}L^T L)x = 0 \Rightarrow x = 0.$$  

That is,

$$0 = x^T(A + \frac{1}{\epsilon}L^T L)x = x^T(A_s + \frac{1}{\epsilon}L^T L)x = x^T A_s x + \frac{1}{\epsilon}(Lx)^T(Lx)$$

since $x^T A_s x = 0$. It follows that

$$x^T A_s x = 0 \text{ and } (Lx)^T(Lx) = 0.$$

By the assumptions on $A$, i.e., $A_s$ is assumed to be symmetric positive definite with respect to $L$, we get $x = 0$. Hence, the new iterative linear system is invertible and $c^{(l+1)}$ and $\lambda^{(l+1)}$ are well-defined.
We now show that $c^{(l+1)}$ converges to $c$. Let also $u^{(l+1)} = c^{(l+1)} - c$ and $p^{(l+1)} = \lambda^{(l+1)} - \lambda$. We have

$$\begin{cases}
(A + \frac{1}{\epsilon} L^T L)u^{(l+1)} + L^T p^{(l)} = 0 \\
p^{(l+1)} = p^{(l)} + \frac{1}{\epsilon} Lu^{(l+1)}.
\end{cases}$$

$$\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 = \frac{2}{\epsilon} (A_s u^{(l+1)}, u^{(l+1)}) + \frac{1}{\epsilon^2} \|Lu^{(l+1)}\|^2.$$ 

We conclude that since $A_s$ is nonnegative,

$$\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 \geq 0,$$

and the sequence $\{\|p^{(l)}\|\}$ is seen to be decreasing.
Being bounded below by 0, it converges; hence \( \|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 \) converges to 0 which implies that \((A_s u^{(l+1)}, u^{(l+1)})\) and \(\|Lu^{(l+1)}\|^2\) converge to 0. Since \(A_s + \frac{1}{\epsilon} L^T L\) is positive definite, it follows that \(u^{(l+1)}\) converges to 0 and finally \(c^{(l+1)}\) converges to \(c\).
Sketch of proof of convergence rate

We prove that
\[ \|c - c^{(l+1)}\| \leq C\epsilon\|c - c^{(l)}\|, \]

Recall that \( u^{(l+1)} = c^{(l+1)} - c \) and \( p^{(l+1)} = \lambda^{(l+1)} - \lambda \). We showed that
\[ \|p^{(l+1)}\| \leq \|p^{(l)}\|, \quad \text{for all } l \]
i.e. that \( (p^{(l)}) \) is a decreasing sequence. We also have
\[
\begin{cases}
(A + \frac{1}{\epsilon}LT L)u^{(l+1)} + LTp^{(l)} = 0 \\
p^{(l+1)} = p^{(l)} + \frac{1}{\epsilon}Lu^{(l+1)},
\end{cases}
\]
from which it follows that
\[ Au^{(l+1)} + LTp^{(l+1)} = 0 \]
We write \( u^{(l+1)} = \hat{u}^{(l+1)} + \overline{u}^{(l+1)} \) with \( \hat{u}^{(l+1)} \in \text{Ker}(L) \) and \( \overline{u}^{(l+1)} \in \text{Im}(L^T) \). Note that \( L : \text{Im}(L^T) \to \text{Im}(L) \) has a bounded inverse, so there exists \( k_0 > 0 \) such that

\[
\| \overline{u}^{(l+1)} \| \leq \frac{1}{k_0} \| Lu^{(l+1)} \|
\]

from which it follows that

\[
\| \overline{u}^{(l+1)} \| \leq \frac{2\epsilon}{k_0} \| p^{(l)} \|
\]

To get a bound on \( \| \hat{u}^{(l+1)} \| \), we notice that \( A \) is invertible on \( \text{Ker}(L) \) since \( A + \frac{1}{\epsilon} L^T L \) is invertible. This gives for some \( \alpha_0 > 0 \),

\[
\| \hat{u}^{(l+1)} \| \leq \frac{1}{\alpha_0} \sup_{v_0 \in \text{Ker}(L)} \frac{(v_0, A\hat{u}^{(l+1)})}{\| v_0 \|} = \sup_{v_0 \in \text{Ker}(L)} \frac{-v_0^T A\overline{u}^{(l+1)}}{\| v_0 \|} \leq \| A \| \| \overline{u}^{(l+1)} \|
\]
Putting together, we obtain

\[ \| u^{(l+1)} \| \leq C \epsilon \| p^{(l)} \|, \quad \text{for some constant } C > 0 \]

To finish, we need a bound on \( \| p^{(l)} \| \) in terms of \( \| u^{(l)} \| \). It can be shown that one can choose \( \lambda_0 \) such that \( p^{(l)} \in \text{Im}(L) \) and since \( L^T : \text{Im}(L) \rightarrow \text{Im}(L^T) \) has a bounded inverse,

\[ \| p^{(l)} \| \leq \frac{1}{k_0} \| L^T p^{(l)} \|. \]

This completes the proof since \( L^T p^{(l)} = -A u^{(l)} \).
Computational Experiments on the 3D Stokes Equations

Let $\Omega \subset \mathbb{R}^3$ be a cube with sides of length 1. We consider the vector field $\mathbf{u} = (u_1, u_2, u_3)$ with a pressure $p$.

\[
\begin{align*}
  u_1 &= -\exp(x + 2y + 3z) \\
  u_2 &= 2 \exp(x + 2y + 3z) \\
  u_3 &= -\exp(x + 2y + 3z) \\
  p &= x(1 - x)z(1 - z).
\end{align*}
\]
Table 1 Approximation Errors from Trivariate Spline Spaces on $\mathcal{T}_1$

<table>
<thead>
<tr>
<th>degrees</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$3.3633\times 10$</td>
<td>$5.9431\times 10$</td>
<td>$4.0397\times 10$</td>
<td>$1.3466 \times 10^3$</td>
</tr>
<tr>
<td>4</td>
<td>$1.7010\times 10$</td>
<td>$4.4374\times 10$</td>
<td>$3.5368\times 10$</td>
<td>$3.8562 \times 10^2$</td>
</tr>
<tr>
<td>5</td>
<td>$2.3804$</td>
<td>$7.3711$</td>
<td>$5.9629$</td>
<td>$9.8470 \times 10^1$</td>
</tr>
<tr>
<td>6</td>
<td>$3.9620\times 10^{-1}$</td>
<td>$1.2238$</td>
<td>$1.0311$</td>
<td>$2.7404 \times 10^1$</td>
</tr>
<tr>
<td>7</td>
<td>$6.7456\times 10^{-2}$</td>
<td>$1.9789 \times 10^{-1}$</td>
<td>$1.6260 \times 10^{-1}$</td>
<td>$6.8411$</td>
</tr>
<tr>
<td>Rate</td>
<td>$1.56\times 10^7 d^{-9.8294}$</td>
<td>$3.22\times 10^7 d^{-9.6203}$</td>
<td>$2.32\times 10^7 d^{-9.5463}$</td>
<td>$8.50 \times 10^6 d^{-7.13}$</td>
</tr>
</tbody>
</table>

Table 2 Approximation Errors from Trivariate Spline Spaces on $\mathcal{T}_2$

<table>
<thead>
<tr>
<th>degrees</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.5083\times 10$</td>
<td>$1.8709\times 10$</td>
<td>$1.5222\times 10$</td>
<td>$4.4382 \times 10^2$</td>
</tr>
<tr>
<td>4</td>
<td>$9.4142\times 10^{-1}$</td>
<td>$2.2094$</td>
<td>$1.8373$</td>
<td>$3.5278 \times 10^1$</td>
</tr>
<tr>
<td>5</td>
<td>$9.1619\times 10^{-2}$</td>
<td>$2.2322 \times 10^{-1}$</td>
<td>$2.0176 \times 10^{-1}$</td>
<td>$5.8199$</td>
</tr>
<tr>
<td>6</td>
<td>$8.5128\times 10^{-3}$</td>
<td>$2.3520 \times 10^{-2}$</td>
<td>$1.9276 \times 10^{-2}$</td>
<td>$7.1884 \times 10^{-1}$</td>
</tr>
<tr>
<td>Rate</td>
<td>$9.31\times 10^6 d^{-11.5631}$</td>
<td>$1.24\times 10^7 d^{-11.1692}$</td>
<td>$1.09\times 10^7 d^{-11.1901}$</td>
<td>$1.05 \times 10^7 d^{-9}$</td>
</tr>
</tbody>
</table>
$L^2$ norm of the error versus degree on $\mathcal{T}_1$ (rate $1.6777 \times 10^7 d^{-9.8962}$) and $\mathcal{T}_2$ (rate $7.7013 \times 10^6 d^{-11.8503}$)
$H^1$ norm of the error versus degree on $\mathcal{T}_1$ (rate $1.6777 \times 10^7 d^{-9.8962}$) and $\mathcal{T}_2$ (rate $7.7013 \times 10^6 d^{-11.8503}$)
Lid Driven Cavity Flow Problem

3D fluid profile in the $x - y$ plane

Presentation at the 10th Annual Conference for African American Researchers in the Mathematical Sciences, June 24, 2004 – p.44/47
3D fluid profile in the $y - z$ plane
3D fluid profile in the $x - z$ plane

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Work in Progress

- Time dependent Navier-Stokes
Work in Progress

- Time dependent Navier-Stokes
- Extension to variational inequalities
Work in Progress

- Time dependent Navier-Stokes
- Extension to variational inequalities
- Thank You!