The second boundary value problem for a discrete Monge-Ampère equation

Gerard Awanou

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

Outline

Non-imaging optics refers to the optimal transfer of light radiation between a source and a target. This requires an accurate control of light for the design of projection displays, laser weapons, medical illuminators . . .

Parallel far field reflector problem

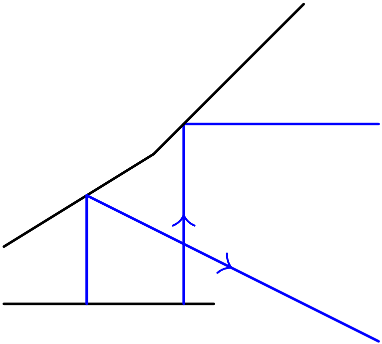


FIGURE – Incoming light distribution is reflected onto a set of directions

The position and shape $u(x)$ of the (convex) reflector solves the second boundary value problem for the Monge-Ampère equation

$$g(Du(x)) \det D^2 u(x) = f(x), x \in X$$
$$Du(X) = Y,$$

where $f \in L^1(X), f \geq 0$ is the incoming light distribution density and $g \in L^1(Y), g > 0$ is the prescribed irradiance density.

Conservation of energy : $\int_X f(x)dx = \int_Y g(y)dy$.

Solutions are unique up to a constant.

Review : Computational nonimaging geometric optics, Notices of the American Mathematical Society 2021.

Differential equation

$$g(Du(x)) \det D^2 u(x) = f(x), x \in X$$

Boundary condition

$$Du(X) = Y$$

For a smooth strictly convex function u and smooth boundaries ∂X and ∂Y

$$Du(\partial X) = \partial Y.$$

Prins, 2014.

Numerical approaches to the 2nd boundary condition

Iterated projection algorithm Froese 2012.

Defining function of Y Benamou-Froese-Oberman 14

In a (mixed) least squares sense Prins et al 2014

Enforced throughout the source domain $Du(X) = Y$ Lindsey and Rubinstein 17, Froese 2019

Other approaches include work by Kawecki, Lakkis, and Pryer 2018, Bonnet and Mirebeau 2022, Brusca and Hamfeldt 2023.

An approach based on convex extension. Equation is written as an equation on $\mathbb{R}^2$. Benamou and Duval 2018. For X convex

$$
\begin{aligned}
g(Du(x)) \det D^2 u(x) &= f(x), x \in X \\
\det D^2 u(x) &= 0, x \in \mathbb{R}^2 \setminus X \\
\min_{e \in \mathcal{S}^1} e^T D^2 u(x) e &= 0, x \in \mathbb{R}^2 \setminus X.
\end{aligned}
$$

Finally, with $\sigma_{\overline{Y}}$ denoting the support function of $\overline{Y}$, i.e. $p \in Y$ iff $p \cdot e \leq \sigma_{\overline{Y}}(e), \forall e \in \mathbb{R}^d$, the condition $Du(X) \subset \overline{Y}$ is rewritten as

$$
\sup_{e \in \mathcal{S}^1} e^T Du(x) - \sigma_{\overline{Y}}(e) = 0, x \in \mathbb{R}^2 \setminus \overline{X}.
$$

Discretization is done on a bounded domain X' such that $X \subsetneq U \subset X'$ for an open set U.

Method of supporting paraboloids

The target density g is approximated by a sum of Dirac masses $\sum_{i=1}^{M} r_i \delta_{P_i}$ for $P_i \in Y$ and $r_i > 0$ for all i.

Energy conservation reads $\sum_{i=1}^{M} r_i = \int_X f(x)\, dx$.

The solution is given by the graph of the convex function

$$u_M(x) = \max_{i=1,\ldots,M} x \cdot P_i - b_i,$$

with rays in the region

$$W_i(b) = \{\, x \in X, x \cdot P_i - b_i \geq x \cdot P_j - b_j \text{ for all } j = 1, \ldots, M \,\},$$

reflected in the direction P_i. We thus need

$$\int_{W_i(b)} f(x)\, dx = r_i, i = 1, \ldots, M.$$

Note that $Du_M \subset \{P_1, \ldots, P_M\} \subset Y$.

Semi-discrete optimal transport

The solution $u_M(x) = \max_{i=1,\ldots,M} x \cdot P_i - b_i$ from the method of supporting paraboloids induces a map $T : X \to Y$ defined by

$$T(x) = P_i \text{ for } x \in W_i(b).$$

This map is optimal in the sense that it minimizes the total cost

$$\int_X c(x, T(x))\, dx,$$

where $c(x, y) = |x - y|^2$ is the cost of moving " mass at x to y", among all "measure preserving maps" from X to Y.

The solution $u_M(x) = \max_{i=1,\dots,M} x \cdot P_i - b_i$ also induces a mapping $\psi : Y \to \mathbb{R}$, the Legendre transform of u_M, such that $\psi(P_i) = b_i$.

The convex envelope $\hat{\psi}$ of ψ, i.e. the largest convex function below ψ, solves in a weak sense (an equation similar to) a Monge-Ampère equation $f(D\hat{\psi}) \det D^2\hat{\psi} = g$.

For a direct approximation of a solution of $g(Du) \det D^2 u = f$, one first solves $f(D\psi) \det D^2\psi = g$ for the discrete mapping $\psi : \{P_1, \dots, P_N\} \to \mathbb{R}$ by rewriting the subdifferential in terms of the so-called Laguerre cells of ψ.

Note the change of point of view : u_M is obtained not by seeking $u_M(x), x \in X$ but by the values b_i at P_i of its Legendre transform.

Related to the proposed approach for Dirichlet b.c.

Oliker-Prussner method (1988)
Implementation uses implicitly the computation of Laguerre cells or power diagrams

Mirebeau 2015 : medius between wide stencils and power diagrams

Neilan and Zhang 2018 : 2D implementation based on edge flips

Possible advantages of the method

Semi-discrete optimal transport has a complexity $O(N^2)$. The complexity of a version of this method is $O(N)$.

Convex extensions were used by Benamou and Duval in 2018. Here existence of a solution, uniqueness results and convergence of u_h and not just for its convex envelope.

Step 1 : Oliker-Prussner discretization for the second boundary condition
Step 2 : method medius between wide stencils and power diagrams

Find a convex function $u \in C^2([a,b])$ such that

$$g(u'(x))u''(x) = f(x), x \in (a,b) \quad (1)$$

$$u'((a,b)) = (\alpha, \beta). \quad (2)$$

$x \mapsto u'(x)$ is a surjective mapping from (a,b) onto (α, β). Since u is convex, u' is increasing and hence (2) is equivalent to

$$u'(a) = \alpha \text{ and } u'(b) = \beta.$$

Similar to $Du(X) = Y$ discretized by $Du(\partial X) = \partial Y$.
Compatibility : change of variable $x \to \gamma(x) = u'(x) = p$ (gradient mapping), provided that u' is one-to-one, i.e. u is strictly convex

$$\int_a^b f(x)dx = \int_a^b g(u'(x))u''(x)dx = \int_\alpha^\beta g(p)dp.$$

Aleksandrov solutions

Monge-Ampère measure $M[u](B) = \int_{\gamma(B)} g(p)\, dp$
Replace $\gamma(x)$ by subgradient mapping for non smooth solutions

$$\partial u(x_0) = \{\, p \in \mathbb{R} : u(x) \geq u(x_0) + p(x - x_0), \text{ for all } x \in X \,\}.$$

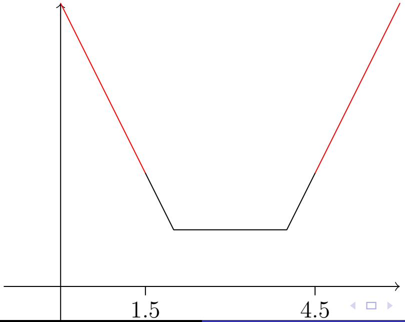
Find u convex such that

$$\int_{\partial u(B)} g(p)\, dp = \int_B f(x)\, dx, \text{ for all Borel sets } B \subset (a, b)$$

Second boundary condition in terms of asymptotic cone.
Given $x_0 \in (a,b)$ and $p \in \partial u(x_0)$ the affine function $L(x) = u(x_0) + p(x - x_0)$ is said to be subtangent to u at x_0.
Define

$$\overline{u}(x) = \sup\{\, u(y) + p(x-y), y \in (a,b) \text{ and } p \in \partial u(y)\,\}.$$

It can be shown that $\overline{u}$ is a convex extension of u.



To get the extension $\overline{u}$, one needs to know $\partial u(y)$ for all $y \in (a, b)$. The extension can also be obtained with just the knowledge of $\partial u(a, b) = (\alpha, \beta)$.

Epigraph of $\overline{u}$ is given by

$$\operatorname{epi} \overline{u} = \{ (x, y) \in \mathbb{R}^2, y \geq \overline{u}(x) \}.$$

K denotes the epigraph of $k(x) = \max\{ \alpha x, \beta x \}$. For $(r, s) \in \mathbb{R}^2$, $(r, s) + K$ is the epigraph of the function

$$k_{(r,s)}(x) = \max\{ \alpha(x - r) + s, \beta(x - r) + s \}.$$

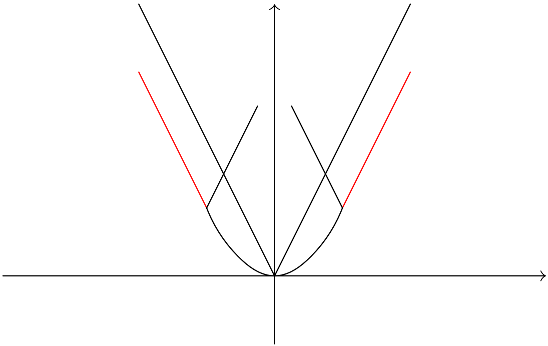


FIGURE – The function $u(x) = x^2$ solves $u'' = 2$ on (-1,1) with $\partial u(-1,1) = (-2,2)$. Its extension to $\mathbb{R}$ is also shown. Here $\alpha = -2$ and $\beta = 2$. The graphs of $k_{0,0}$, $k_{(-1,1)}$ and $k_{(1,1)}$ are shown. Their epigraphs are completely contained in the epigraph of the extension $\overline{u}$ of u.

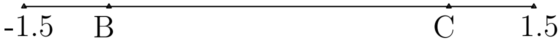
For $x \notin (a,b)$

$$\overline{u}(x) = \inf_{y \in (a,b)} k_{(y,u(y))}(x)$$
$$= \inf_{y \in (a,b)} \max\{\, \alpha(x-y)+u(y), \beta(x-y)+u(y) \,\}$$
$$\overline{u}(x) = \min\{\, k_{(a,\overline{u}(a))}(x), k_{(b,\overline{u}(b))}(x) \,\}.$$

$u \in C(a,b)$ is extended by continuity to $[a,b]$.

Example in 1D of piecewise linear convex function

Assume that $\Omega = (-1.5, 1.5)$ and $\Omega^* = (-2, 2)$.



Consider the piecewise linear convex function on Ω with vertices at $B(-1, 0)$ and $C(1, 0)$.

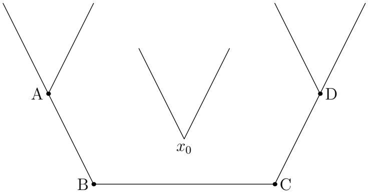
Recall that

$$\partial u(x) = \{\, p \in \mathbb{R}, u(y) \geq u(x) + p(y - x), \forall y \in (a, b) \,\}.$$

For $x \notin \{B, C\}$, $u'(x)$ is constant. At those points $\partial u(x) \in \{\, -2, 0, 2 \,\}$ with $|\partial u(x)| = 0$. We have $\partial u(B) = [-2, 0]$ and $\partial u(C) = [0, 2]$. Thus $|\partial u(B)| = 2$ and $|\partial u(C)| = 2$. Also, $\partial u(\Omega) = \overline{\Omega^*}$.

The epigraph of u is a bounded convex set.

The epigraph of the extension of u is an unbounded convex set.



We have $\partial u(\Omega) = \partial u(\mathbb{R}) = \overline{\Omega^*}$.

Monge-Ampère equation reformulated in terms of asymptotic cone.
Find u convex on Ω which solves equation in Ω and given outside of Ω by

$$u(x) = \min_{s \in \partial X} u(s) + k_Y(x - s).$$

g-curvature of convex functions

Let v be a convex function on $\mathbb{R}^d$.

$$\chi_v(E) = \cup_{x \in E} \chi_v(x).$$

g-curvature as the set function

$$\omega(g, v, E) = \int_{\chi_v(E)} g(p) dp.$$

Extend f and g by 0 to $\mathbb{R}^d$ with equation in measures

$$\omega(g, u, E) = \int_E f(x) dx \text{ for all Borel sets } E \subset \overline{X}$$
$$\chi_u(\overline{X}) = \overline{Y}.$$

Approximation by piecewise linear convex functions which have subdifferential a polygon

Sequences of polygons $K^* \subset Y$ $K^* \to Y$. To K^* one associates a cone K which is the epigraph of

$$\max_{j=1,\ldots,N} x \cdot a_j^*,$$

where a_j^* is a vertex of K^*.

Find a piecewise linear convex function u_h with asymptotic cone K such that

$$\omega(g, u_h, x) = \sum_{x \in X_h} c_x \delta_x,$$

where $\sum_{x \in X_h} c_x \delta_x \to \mu_f$.

Subdifferential of the extension

$$\chi_u(y) = \{\, q \in \mathbb{R}^d : \tilde{u}(z) \geq \tilde{u}(x) + q \cdot (z - x), \text{ for all } z \in \mathbb{R}^d \,\}.$$

For X, Y bounded convex $\partial u(X) = Y$ implies $\chi_u(\overline{X}) = \chi_u(\mathbb{R}^d) = \overline{Y}$.

Assume first that Y is polygonal with vertices $a_i^*, i = 1, \ldots, N$. Recall the support function of Y : $p \in \mathbb{R}^d, k_Y(p) = \sup_{y \in Y} p \cdot y$.

Choose a set of vectors V_0 of normals to facets of Y. Then $p \in Y$ iff $p \cdot e \leq k_Y(e)$ for all $e \in V_0$.

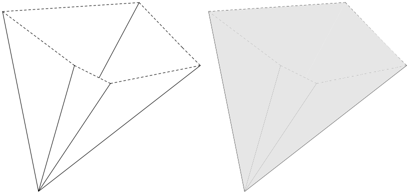


FIGURE – Polyhedral angle in $\mathbb{R}^3$

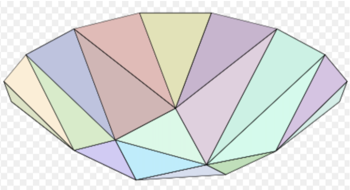


FIGURE – Piecewise linear convex function. Reference : wikipedia.

Observation 1 If $\chi_u(\partial\overline{X}) \subset \overline{Y}$ then $\chi_u(\overline{X}) \subset \overline{Y}$.

Observation 2 By a theorem of Aleksandrov, the set of normals to the facets of a polygon cannot all lie in a half space.

For a point $x \in \partial X$, points outside of X are needed to evaluate $\chi_u(x)$.

Orthogonal lattice with mesh length h : $\mathbb{Z}^d_h = \{ mh, m \in \mathbb{Z}^d \}$.
Put $X_h = X \cap \mathbb{Z}^d_h$ and

$$\partial X_h = \{ x \in X_h \text{ such that for some } i = 1, \ldots, d, x + hr_i \notin X_h \text{ or } x - hr_i \notin X_h \}.$$

Put $\mathcal{N}_h = X_h \cup \{ x + he, e \in V_0, x \in \partial X_h \}$.

Assume now that u is piecewise linear convex with vertices at $x \in X_h$, i.e. the epigraph of u is an unbounded polygon with vertices at $x \in X_h$.

For x in the interior of X, $\chi_u(x)$ depends only on $u|_X$.

However, for $x \in \partial X$, we need all the points $x + he$, $e \in V_0$.

Let $z \in \partial X_h$ and put $r = z + he$ for some $e \in V_0$, we must have for $q \in \chi_u(z)$

$$q \cdot (r - z) \leq u(r) - u(z).$$

Sufficient condition to ensure that $\chi_u(\partial \overline{X}_h) \subset \overline{Y}$:
$u(r) - u(z) \leq k_Y(r - z)$ for all $z \in \partial X_h$.

Extension formula : $u(r) = \min_{z \in \partial X_h} u(z) + k_Y(r - z)$.

The unknown are the (finite set of) mesh values $\{ u_h(x), x \in X_h \}$ and the second boundary condition is enforced implicitly using the discrete extension formula

$$u_h(x) = \min_{y \in \partial X_h} \max_{1 \leq j \leq N} (x - y) \cdot a_j^* + u_h(y).$$

(for X polygonal)

The min and the max are over a finite number of points.

This is sometimes called Oliker-Prussner method. Here for the second boundary value problem.

Discrete convexity For a function v_h on $\mathbb{Z}_h^d$, $e \in \mathbb{Z}^d$ and $x \in X_h$

$$\Delta_{he} v_h(x) = v_h(x + he) - 2v_h(x) + v_h(x - he).$$

The unknown in the numerical scheme is a mesh function v_h on X_h which is extended to $\mathbb{Z}^d$ using the extension formula, and which is discrete convex ($\Delta_{he} v_h(x) \geq 0$).

Stencil

Definition

A stencil V is a set valued mapping from X_h to the set of finite subsets of $\mathbb{Z}^d \setminus \{0\}$.

Minimal stencil V_{min} is symmetric with respect to the origin, contains the elements of the canonical basis of $\mathbb{R}^d$ and the set of normals V_0.

Extended mesh $\mathcal{N}_h^2 = X_h \cup \{ x + he, x \in X_h, e \in V_{min} \}$.

Maximal stencil V_{max} such that $e \in V_{max}(x)$ iff $x + he \in \mathcal{N}_h^2$.

$$V_{min} \subset V(x) \subset V_{max}(x).$$

For convergence, and $f \in C(\overline{X})$, choose $V(x)$ to be in addition symmetric with respect to the origin and with vectors with co-prime coordinates.

$$\partial_V v_h(x) = \{ p \in \mathbb{R}^d, p \cdot (he) \geq v_h(y) - v_h(y - he) \forall e \in V(x) \}.$$

$$\omega_V(g, v_h, x) = \int_{\partial_V v_h(x)} g(p) dp, x \in X_h.$$

find $u_h \in \mathcal{C}_h$ with asymptotic cone K such that

$$\omega_V(g, u_h, \{ x \}) = \int_{C_x} \tilde{f}(t) dt, x \in X_h,$$

where C_x with $C_x \cap X_h = \{ x \}$ form a partition of X.

Recall the extension formula

$$u_h(r) = \min_{z \in \partial X_h} u_h(z) + k_Y(r - z).$$

In the case $V = V_{max}$, u_h coincides with its convex envelope and was essentially studied by Bakelman. Existence, uniqueness and convergence of the discretization follows for $f \in L^1(X)$.

$V_{min} \subset V(x) \subsetneq V_{max}(x)$

Existence of solutions follows from the convergence of a damped Newton's method.

Discrete convex mesh functions with asymptotic cone K are Lipschitz continuous with a uniform Lipschitz bound, i.e.

$$|v_h(x) - v_h(y)| \leq C||x - y||_1.$$

If X is a rectangle, which does not require a loss of generality, a subsequence converges to a convex function which is shown to be a viscosity solution for $f \in C(\overline{X})$.

Uniqueness holds under various assumptions for V not necessarily equal to V_{max}

Possibility of having $X_h = X_h^1 \cup X_h^2$ with discrete Monge-Ampère equations on either which do not "interact".

Convergence of the discretization follows from weak convergence of Monge-Ampère measures

Theorem

Let K_m^* be bounded convex polygonal domains increasing to $\overline{Y}$. Then the convex solution u_m of

$$\omega(g, u, E) = \int_E f_{K_m^*}(x) dx \text{ for all Borel sets } E \subset \overline{X}$$
$$\chi_u(\overline{X}) = K_m^*$$
$$u(x^0) = \alpha,$$

for $x^0 \in X$ and $\alpha \in \mathbb{R}$ converges uniformly on compact subsets of X to the solution u with $u(x^0) = \alpha$.

A numerical experiment

Exact solution $u(x,y) = x^2/2 + xy + y^2$ with $X = (0,1)^2$ and Y is the polygon of area 1 with vertices $(0,0),(1,1),(1,2)$ and $(1,3)$.

Take $g(x,y) = x + y$ with corresponding right hand side $f(x,y)$.

Initial guess $\partial v_h(X) \subset Y$

Stencil V was taken as $V = -V_1 \cup V_1$ where V_1 consists of the vectors $(1,0),(0,1),(1,1),(1,-1),(2,1),(-1,2),(1,2)$ and $(-2,1)$.

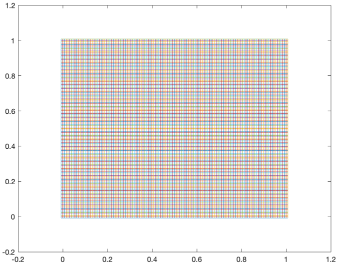
$$g(Du)\,\det D^2 u = f + u(h,h)$$

Quadrature rules and a damped Newton's method

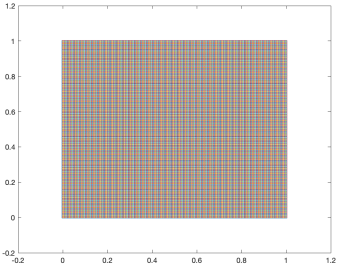
h	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
Error for u	$2.72\ 10^{-4}$	$8.01\ 10^{-5}$	$2.31\ 10^{-5}$	$6.52\ 10^{-6}$	$1.82\ 10^{-6}$
Rate		1.76	1.79	1.82	1.84
Error for Du	$6.27\ 10^{-3}$	$3.30\ 10^{-3}$	$1.56\ 10^{-3}$	$8.23\ 10^{-4}$	$3.92\ 10^{-4}$
Rate		0.93	1.07	0.93	1.07

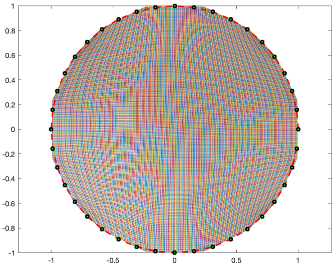
TABLE – Maximum errors for a smooth solution.

Constant density on a square mapped to constant density on the unit disc $h = 1/2^7$



Constant density on a square mapped to Gaussian $e^{-0.5(x^2+y^2)}$ on the unit disc $h = 1/2^8$





Some references

Awanou, Gerard, On the weak convergence of Monge-Ampère measures for discrete convex mesh functions. Acta Appl. Math. 172 (2021), Paper No. 6, 31 pp.

Awanou, Gerard, The second boundary value problem for a discrete Monge-Ampère equation. Journal of Scientific Computing, volume 97, article 22. (2023)