A ROTATED NONCONFORMING RECTANGULAR MIXED ELEMENT FOR ELASTICITY

GERARD AWANOU

Abstract. We present in this paper a low order nonconforming mixed element for plane elasticity on rectangular meshes. The 3 dimensional space of rigid body motions is used to approximate the displacement and a 16 dimensional space is used to discretize the space of symmetric tensors. This element may be viewed as the rectangular analogue of the nonconforming Arnold-Winther element and is related to a discrete version of the elasticity differential complex with a nonconforming H^2 element related to the rotated Q_1 element.

1. Introduction

It is well known that it is extremely difficult to construct mixed finite elements for elasticity in stress displacement formulation. In a pioneering work, Arnold and Winther constructed the first mixed elements for plane elasticity with symmetric stress fields, using polynomial shape functions, [2].

Previous works circumvent the difficulty of the symmetry condition by using composite elements, weakening or abandoning the symmetry condition, cf. [2] and the references therein. As explained in [2], vertex degrees of freedom are unavoidable for a finite element space for the stress field with continuous symmetric matrix fields if one imposes interelement continuity only by means of quantities defined on the edges. Simpler nonconforming elements which avoid vertex degrees of freedom were constructed in [3].

In a very recent paper, [1], we introduced conforming mixed elements for elasticity on rectangular meshes which may be viewed as analogues of the triangular elements. It was reasonable to expect that nonconforming analogues could be constructed using polynomials of lower degree. Our work has created interest in the construction of low order nonconforming rectangular mixed elements for elasticity, c.f. [5], and [6]. The purpose of this paper is to present a nonconforming mixed element on rectangular meshes which is the analogue of the element of lowest order in [3]. Our element involves 16 degress of freedom for the stress and 3 degrees of freedom for the displacement with a convergence rate in L^2 of $\mathrm{O}(h)$ for the stress and $\mathrm{O}(h)$ for the displacement. The paper is organized as follows: In the second section, we introduce the notations, the stability conditions and the elasticity sequence. A "rotated" polynomial version of the sequence is then constructed. It involves a nonconforming rotated

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H^2 element which is described in section 3. Our mixed element and its analysis is presented in section 4. We finish with some concluding remarks.

2. Preliminaries

2.1. **Notations.** Let Ω be a simply connected polygonal domain of $\mathbb{R}^2$, occupied by a linearly elastic body which is clamped on $\partial\Omega$ and let $H(\text{div}, \Omega, \mathbb{S})$ be the space of square-integrable fields taking values in the space $\mathbb{S}$ of 2×2 symmetric matrices, and which have square integrable divergence. We denote as usual by $L^2(\Omega, \mathbb{R}^2)$ the space of square integrable vector fields with values in $\mathbb{R}^2$ and $H^k(K, X)$ the space of functions with domain $K \subset \mathbb{R}^2$, taking values in the finite dimensional space X, and with all derivatives of order at most k square integrable. For our purposes, X will be either $\mathbb{S}, \mathbb{R}^2,$ or $\mathbb{R}$, and in the latter case, we simply write $H^k(X)$. The norms in $H^k(K, X)$ and $H^k(K)$ are denoted respectively by $||\cdot||_{H^k}$ and $||\cdot||_k$. We use the usual notations of $\mathcal{P}_k(K, X)$ for the space of polynomials on K with values in X of total degree less than k and $\mathcal{P}_{k_1,k_2}(K, X)$ for the space of polynomials of degree at most k_1 in x and of degree at most k_2 in y. We write $\mathcal{P}_k$ and $\mathcal{P}_{k_1,k_2}$ respectively when $X = \mathbb{R}$. So if we define $M_{k_1,k_2} = \{\, x^i, y^j \,|\, 0 \leq i \leq k_1, 0 \leq j \leq k_2 \,\}$, then $\mathcal{P}_{k_1,k_2} = \text{span}(M_{k_1,k_2})$.

We will often consider spaces of polynomials on the reference element $\hat{K} = [0,1] \times [0,1]$. We denote its vertices by $v_1 = (0,0), v_2 = (1,0), v_3 = (1,1)$ and $v_4 = (0,1)$. Let us also denote by e_i the edge $\langle v_i, v_{i+1} \rangle, i = 1, \ldots, 4$ with $v_5 \equiv v_1$.

For a vector field $v : \Omega \to \mathbb{R}^2, \nabla v$ is the matrix field with rows the gradient of each component of v and $\epsilon(v) = \left[\nabla v + \nabla v^T\right]/2$. For a matrix field $\tau, \text{div}\, \tau$ is the vector obtained by applying the divergence operator row wise and $\sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$. We will denote by $\begin{pmatrix} \mathcal{P}_{k_1,k_2} & \mathcal{P}_{k_3,k_4} \\ \mathcal{P}_{k_3,k_4} & \mathcal{P}_{k_5,k_6} \end{pmatrix}_{\mathbb{S}}$ the space of symmetric matrix fields $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ such that $\sigma_{11} \in \mathcal{P}_{k_1,k_2}$, $\sigma_{12} = \sigma_{2,1} \in \mathcal{P}_{k_3,k_4}$ and $\sigma_{22} \in \mathcal{P}_{k_5,k_6}$. Similarly the notation $\begin{pmatrix} \mathcal{P}_{k_1,k_2} \\ \mathcal{P}_{k_3,k_4} \end{pmatrix}$ for $\mathcal{P}_{k_1,k_2} \times \mathcal{P}_{k_3,k_4}$ will also be used.

The solution $(\sigma, u) \in H(\text{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ of the elasticity problem can be characterized as the unique critical point of the Hellinger-Reissner functional

$$\mathcal{J}(\tau, v) = \int_\Omega \left(\frac{1}{2} A\tau : \tau + \text{div}\, \tau \cdot v - f \cdot v \right) dx.$$

The compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is given, bounded and symmetric positive definite uniformly with respect to $x \in \Omega$, and the body force $f \in L^2(\Omega, \mathbb{R}^2)$ is also given. The unknowns, σ and u, represent the stress and displacement fields respectively.

2.2. **Stability Conditions.** We discuss in this paper nonconforming mixed elements for the elasticity problem in the sense that the stress space Σ_h is not contained in

$H(\mathrm{div}, \Omega, \mathbb{S})$ because the normal component of the discrete stress field is not continuous across element edges. The displacement space V_h satisfies $V_h \subset L^2(\Omega, \mathbb{R}^2)$. We extend the functional $\mathcal{J}$ to $\Sigma_h \times V_h$ by replacing the div operator by the divergence operator div_h applied element by element. The mixed finite element solution (σ_h, u_h) is then determined as the unique critical point of $\mathcal{J}$ over $\Sigma_h \times V_h$.

It is known form Brezzi's theory that for stable approximations the following conditions are sufficient:

- There exists a positive constant c_1, such that $||\tau||_{H(\mathrm{div}_h)} \leq c_1 ||\tau||_{L^2}$, if $\tau \in \Sigma_h$ and $\int_\Omega \mathrm{div}_h \tau \cdot v \, dx = 0$ for all $v \in V_h$.
- There exists a positive constant c_2 such that $\forall v \in V_h, v \neq 0, \exists \tau \in \Sigma_h, \tau \neq 0$ with $\int_\Omega \mathrm{div}_h \tau \cdot v \, dx \geq c_2 ||\tau||_{H(\mathrm{div}_h)} ||v||_{L^2}$. Here $||\tau||_{H(\mathrm{div}_h)} = (||\tau||_{L^2} + ||\mathrm{div}_h \tau||_{L^2})^{1/2}$.

It can be shown that for a stable approximation, the following two conditions are also sufficient:

- $\mathrm{div}_h \Sigma_h \subset V_h$
- There exists a linear operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$, such that there is a constant c independent of h with $||\Pi_h \tau||_0 \leq c ||\tau||_1$ for all $\tau \in H^1(\Omega, \mathbb{S})$, and such that $\mathrm{div}_h \Pi_h \tau = P_h \mathrm{div}_h \tau$ for all $\tau \in H^1(\Omega, \mathbb{S})$ where $P_h : L^2(\Omega, \mathbb{R}^2) \to V_h$ denotes the L^2 projection.

The main tool used in the design of the mixed methods presented in this paper is the elasticity differential complex, [2]:

$$0 \longrightarrow \mathcal{P}_1(\Omega) \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{J} C^\infty(\Omega, \mathbb{S}) \xrightarrow{\mathrm{div}} C^\infty(\Omega, \mathbb{R}^2) \longrightarrow 0, \tag{1}$$

where J is the Airy stress operator defined by

$$Jq := \begin{pmatrix} \frac{\partial^2 q}{\partial y^2} & -\frac{\partial^2 q}{\partial x \partial y} \\ -\frac{\partial^2 q}{\partial x \partial y} & \frac{\partial^2 q}{\partial x^2} \end{pmatrix}.$$

Notice that on $\partial\Omega$, if t and n denote the unit tangent and normal vectors,

$$(Jq)n \cdot n = \frac{\partial^2 q}{\partial s^2} \text{ and } (Jq)n \cdot t = -\frac{\partial^2 q}{\partial s \partial n}.$$

2.3. **Rotated Polynomial Sequence.** Let $k \geq 0$ be an integer. We recall, [1], that the sequence

$$0 \longrightarrow \mathcal{P}_1(\Omega) \xrightarrow{\subset} \mathcal{P}_{k+2,k+2}(\Omega) \xrightarrow{J} \begin{pmatrix} \mathcal{P}_{k+2,k} & \mathcal{P}_{k+1,k+1} \\ \mathcal{P}_{k+1,k+1} & \mathcal{P}_{k,k+2} \end{pmatrix}_{\mathbb{S}} \xrightarrow{\mathrm{div}} \begin{pmatrix} \mathcal{P}_{k+1,k} \\ \mathcal{P}_{k,k+1} \end{pmatrix} \longrightarrow 0 \tag{2}$$

is exact. It is immediate that each operator maps a space in the sequence into the next and like for (1), the kernel of J is $\mathcal{P}_1(\Omega)$. To prove that the kernel of div is the image of J one uses again the exactness of (1). To verify the surjectivity of the last map, one can notice that the alternating sum of the dimensions of the spaces in the sequence is zero. Next, analogous to the definition of the rotated Q_1 space c.f. [4], denoted here

$$\tilde{\mathcal{P}}_{1,1} = \text{span}(\{1, x, y, x^2 - y^2\}),$$

we introduce the rotated space

$$\tilde{\mathcal{P}}_{3,3} = \mathcal{P}_3 \oplus \text{span}(\{x^3 y, xy^3\}) \oplus b_K \tilde{\mathcal{P}}_{1,1}. \tag{3}$$

Recall that the degrees of freedom of $\tilde{\mathcal{P}}_{1,1}$ are the averages on each edge or equivalently, $\int_e q b_e, q \in \tilde{\mathcal{P}}_{1,1}$, where b_e is a quadratic polynomial vanishing at the endpoints of e. We have

Lemma 2.1.

$$\tilde{\mathcal{P}}_{3,3} = \text{span}(M_{3,3} \setminus \{x^3 y^3\}) \oplus \text{span}(\{p_0(x, y)\}),$$

where $p_0(x, y) = x^4 y^2 - x^2 y^4 - x^4 y + x y^4$.

Proof. It is enough to prove the identity on the reference element $\hat{K}$. Clearly $\mathcal{P}_3 \oplus \text{span}(\{x^3 y, xy^3\}) \oplus b_{\hat{K}} \,\text{span}(\{1, x, y\}) \subset \text{span}(M_{3,3} \setminus \{x^3 y^3\})$. Next $x(x-1)y(y-1)(x^2 - y^2) = p_0(x, y) + x^2 y^3 - x^3 y^2 + x^3 y - xy^3$. Therefore $\tilde{\mathcal{P}}_{3,3} \subset \text{span}(M_{3,3} \setminus \{x^3 y^3\}) \oplus \text{span}(\{p_0(x, y)\})$. To prove the reverse inclusion, notice that $p_0(x, y) = b_{\hat{K}}(x - y + x^2 - y^2) + xy^2 - x^2 y$. Moreover, $\text{span}(M_{3,3} \setminus \{x^3 y^3\}) \subset \mathcal{P}_3 \oplus \text{span}(\{x^3 y, xy^3\}) \oplus b_{\hat{K}}$ since the spaces on both sides of the inequality have the same dimension. □

Let us introduce the polynomials

$$\begin{aligned}
p_{11}(x, y) &= \frac{\partial^2 p_0(x, y)}{\partial y^2}(x, y) = 2x^4 - 12x^2 y^2 + 12xy^2, \\
p_{22}(x, y) &= \frac{\partial^2 p_0(x, y)}{\partial x^2}(x, y) = -2y^4 + 12x^2 y^2 - 12yx^2, \\
p_{12}(x, y) &= \frac{\partial^2 p_0(x, y)}{\partial x \partial y}(x, y) = 8x^3 y - 8xy^3 - 4x^3 + 4y^3, \\
p_3(x, y) &= \frac{\partial p_{12}}{\partial y}(x, y) = \frac{\partial p_{11}}{\partial x}(x, y) = 8x^3 - 24xy^2 + 12y^2, \text{ and} \\
p_4(x, y) &= \frac{\partial p_{12}}{\partial x}(x, y) = \frac{\partial p_{22}}{\partial y}(x, y) = -8y^3 + 24yx^2 - 12x^2.
\end{aligned}$$

Then

$$Jp_0(x, y) = \begin{pmatrix} p_{11}(x, y) & -p_{12}(x, y) \\ -p_{12}(x, y) & p_{22}(x, y) \end{pmatrix}, \ \text{div} \begin{pmatrix} \text{span}(\{p_{11}\}) & \text{span}(\{p_{12}\}) \\ \text{span}(\{p_{12}\}) & \text{span}(\{p_{22}\}) \end{pmatrix} = \begin{pmatrix} \text{span}(\{p_3\}) \\ \text{span}(\{p_4\}) \end{pmatrix}.$$

We also introduce the rotated spaces

$$\begin{aligned}
\tilde{\mathcal{P}}_{3,1} &= \text{span}(M_{3,1} \setminus \{x^3 y\}) \oplus \text{span}(\{p_{11}(x, y)\}), \\
\tilde{\mathcal{P}}_{1,3} &= \text{span}(M_{1,3} \setminus \{xy^3\}) \oplus \text{span}(\{p_{22}(x, y)\}), \\
\tilde{\mathcal{P}}_{2,2} &= \text{span}(M_{2,2} \setminus \{x^2 y^2\}) \oplus \text{span}(\{p_{12}(x, y)\}), \\
\tilde{\mathcal{P}}_{2,1} &= \text{span}(M_{2,1} \setminus \{x^2 y\}) \oplus \text{span}(\{p_3(x, y)\}) \text{ and} \\
\tilde{\mathcal{P}}_{1,2} &= \text{span}(M_{1,2} \setminus \{xy^2\}) \oplus \text{span}(\{p_4(x, y)\}),
\end{aligned}$$

and claim that the rotated polynomial sequence

$$0 \longrightarrow \mathcal{P}_1(\Omega) \overset{\subset}{\longrightarrow} \tilde{\mathcal{P}}_{3,3}(\Omega) \overset{J}{\longrightarrow} \begin{pmatrix} \tilde{\mathcal{P}}_{3,1} & \tilde{\mathcal{P}}_{2,2} \\ \tilde{\mathcal{P}}_{2,2} & \tilde{\mathcal{P}}_{1,3} \end{pmatrix}_{\mathbb{S}} \overset{\mathrm{div}}{\longrightarrow} \begin{pmatrix} \tilde{\mathcal{P}}_{2,1} \\ \tilde{\mathcal{P}}_{1,2} \end{pmatrix} \longrightarrow 0 \tag{4}$$

is exact, the proof being analogue to the one for (2) with $k = 1$. We show that the kernel of div is the image of J. Let $\tau \in \begin{pmatrix} \tilde{\mathcal{P}}_{3,1} & \tilde{\mathcal{P}}_{2,2} \\ \tilde{\mathcal{P}}_{2,2} & \tilde{\mathcal{P}}_{1,3} \end{pmatrix}_{\mathbb{S}}$ such that $\operatorname{div} \tau = 0$. By the exactness of (1), $\tau = J(q), q \in C^\infty(\Omega)$. Expressing that the components of Jq are in the required spaces, one obtains immediately that $q \in \tilde{\mathcal{P}}_{3,3}$.

3. A Rotated H^2 nonconforming finite element

We present in this subsection a nonconforming H^2 finite element, Q_h which together with our mixed element described in the next section form a discrete version of the elasticity complex (4). On each rectangle K, we take $Q_K = \tilde{\mathcal{P}}_{3,3}(K)$ with dimension 16. The degrees of freedom are given in the next lemma.

Lemma 3.1. *An element q of Q_K is uniquely determined by the degrees of freedom*

- *derivatives up to order 1 at each vertex ($3 \times 4 = 12$ degrees of freedom),*
- *average of $\partial q/\partial n$ on each edge (4 degrees of freedom).*

Proof. It is enough to prove the lemma for the reference rectangle $\hat{K} = [0, 1] \times [0, 1]$. Let us assume that all degrees of freedom vanish. By definition of $\tilde{\mathcal{P}}_{3,3}$, on each edge, $q \in Q_{\hat{K}}$ is a polynomial of degree 3 with double roots at the vertices. Then q vanishes identically on $\partial\hat{K}$ and we have $q = \overline{q} b_{\hat{K}}, \overline{q} \in \tilde{\mathcal{P}}_{1,1}(\hat{K})$. On each edge $e, \partial q/\partial n = \overline{q}\partial b_{\hat{K}}/\partial n = \overline{q} b_e$ where b_e is a quadratic polynomial vanishing at the endpoints of the edge e. Then the average of $\overline{q} b_e$ vanishes on each edge, hence, $\overline{q} \equiv 0$. □

The finite element Q_h is assembled the usual way. Elements of Q_h are globally continuous and of class C^1 at the vertices. The element is nonconforming since $\partial q/\partial n \in \mathcal{P}_4(e, \mathbb{R})$ for each edge e, and only its values at the vertices and the first moment on each edge are determined. The interpolation operator $I_h : C^\infty(\Omega) \to Q_h$ is defined by requiring

$$I_h q(x) = q(x) \text{ for each vertex } x,$$
$$(\nabla I_h)q(x) = (\nabla q)(x) \text{ for each vertex } x, \text{ and}$$
$$\int_e \frac{\partial I_h q}{\partial n}(s)\, ds = \int_e \frac{\partial q}{\partial n}(s)\, ds \text{ for all edges } e.$$

4. Rotated Mixed Element for Elasticity

We now describe our mixed element. Let $V(K) = \{(a - cy, b + cx) \,|\, a, b, c \in \mathbb{R}\}$ be the space of rigid body motions and

$$\Sigma_K = \left\{ \tau \in \begin{pmatrix} \tilde{\mathcal{P}}_{3,1} & \tilde{\mathcal{P}}_{2,2} \\ \tilde{\mathcal{P}}_{2,2} & \tilde{\mathcal{P}}_{1,3} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau \in V_K \right\}.$$

The dimension of Σ_K is at least 16, since the dimension of $\begin{pmatrix} \tilde{\mathcal{P}}_{3,1} & \tilde{\mathcal{P}}_{2,2} \\ \tilde{\mathcal{P}}_{2,2} & \tilde{\mathcal{P}}_{1,3} \end{pmatrix}_{\mathbb{S}}$ is 25 and the condition $\operatorname{div} \tau \in V_K$ imposes 9 constraints. We choose as degrees of freedom the first two moments of $\tau n \cdot n$ and $\tau n \cdot t$ on each edge. Hence the dimension of Σ_K is 16 by the following lemma.

Lemma 4.1. *An element of Σ_K is uniquely determined by the first two moments of $\tau n \cdot n$ and $\tau n \cdot t$ on each edge.*

Proof. It is enough to prove the lemma on the reference rectangle $\hat{K}$. We assume that all degrees of freedom for $\tau \in \Sigma_{\hat{K}}$ vanish and we show that $\tau \equiv 0$. For each $v \in V_{\hat{K}}, \epsilon(v) = 0$ and each component of v is linear. It follows from

$$\int_{\hat{K}} (\operatorname{div} \tau) \cdot v \, dx = - \int_{\hat{K}} \tau : \epsilon(v) \, dx + \int_{\partial \hat{K}} \tau n \cdot v \, ds,$$

that $\operatorname{div} \tau = 0$. From the exactness of the sequence (4), $\operatorname{div} \tau = 0 \Rightarrow \tau = Jq, q \in \tilde{\mathcal{P}}_{3,3}(\hat{K})$. Now $\partial^2 q / \partial s^2 = \tau n \cdot n = 0$ on each edge. Hence q is linear on each edge. Adding a linear function, we may assume that q vanishes on the edges e_1 and e_4. We can therefore write $q = xy\overline{q}$ for some polynomial $\overline{q}(x, y)$.

Next, $q(1, y) = y\overline{q}(1, y)$ must be linear in y so $\overline{q}(1, y)$ is a constant. Trivial calculations yield $\partial^2 q / \partial s \partial n = -\overline{q} - x\partial\overline{q}/\partial x$ on e_1. Since the average of $\tau n \cdot t$ is zero on each edge and $\tau n \cdot t = -\partial^2 q / \partial s \partial n$, we have $\int_0^1 \left(\overline{q} + x \frac{\partial \overline{q}}{\partial x} \right)(x, 0) \, dx = 0$. On the other hand, by integration by parts, $\int_0^1 x \frac{\partial \overline{q}}{\partial x}(x, 0) \, dx = - \int_0^1 \overline{q}(x, 0) \, dx + \Big[x\overline{q}(x, 0) \Big]_{x=0}^{x=1}$. We conclude that $\overline{q}(1, 0) = 0$ and since $\overline{q}(1, y)$ is constant, $q \equiv 0$ on the edge $\{x = 1\}$ which is e_2. But q is linear on e_3 and vanishes at the vertices (since $q \equiv 0$ on e_4 and e_2), therefore $q \equiv 0$ on e_3 and we get $q \equiv 0$ on $\partial \hat{K}$. We have $q = b_{\hat{K}} \tilde{q}, \tilde{q} \in \tilde{\mathcal{P}}_{1,1}(\hat{K})$, using the definition of $\tilde{\mathcal{P}}_{3,3}$.

On each edge $e, \partial q / \partial n = \tilde{q} \partial b_{\hat{K}} / \partial n = \tilde{q} b_e$ where b_e is a quadratic polynomial vanishing at the endpoints of the edge e. Since the first moment of $\tau n \cdot t$ vanishes on each edge, it follows that the average of $\tilde{q} b_e$ vanishes on each edge, hence, $\tilde{q} \equiv 0$. This concludes the proof. □ □

We now describe the finite elements on the rectangular partition $\mathcal{T}_h$. We denote by V_h the space of vector fields which belongs to V_K for each rectangle $K \in \mathcal{T}_h$ and Σ_h denotes the space of matrix fields which belong piecewise to Σ_K subject to the continuity conditions that $\tau n \cdot n$ for $\tau \in \Sigma_h$ is continuous across edges (since $\tau n \cdot n$ is linear on each edge for $\tau \in \Sigma_K$) as well as the first two moments of $\tau n \cdot t$. The space Σ_h is a nonconforming subspace of $H(\mathrm{div}, \Omega, \mathbb{S})$ since $\tau n \cdot t$ is quadratic on each edge and only the first two moments are continuous.

We define our interpolation operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ by

$$\int_e (\Pi_h \tau - \tau) n \cdot n \; v \, ds = 0 \text{ for all edges } e \text{ and } v \in \mathcal{P}_1(e),$$

$$\int_e (\Pi_h \tau - \tau) n \cdot t \; v \, ds = 0 \text{ for all edges } e \text{ and } v \in \mathcal{P}_1(e).$$

Moreover, for $\tau \in H^1(\Omega, \mathbb{S}), K \in \mathcal{T}_h$ and $v \in V_K$, we have

$$\int_K (\mathrm{div}\, \Pi_h \tau - \mathrm{div}\, \tau) \cdot v \, dx = - \int_K (\Pi_h \tau - \tau) : \epsilon(v) \, dx + \int_{\partial K} (\Pi_h \tau - \tau) n \cdot v \, ds.$$

The argument being similar to the one in the proof of the previous lemma, we conclude that the commutativity property

$$\mathrm{div}_h \Pi_h = P_h \, \mathrm{div}_h$$

holds. It remains to show that there is a constant c independent of h with $||\Pi_h \tau||_0 \leq c||\tau||_1$ for all $\tau \in H^1(\Omega, \mathbb{S})$.

For this note that the interpolation operator Π_h is local to the partition and preserve piecewise constant matrix fields. Hence by scaling any element to a similar element with unit diameter using translation, rotation and dilatation, and using a compactness argument, we obtain

$$||\Pi_h \tau - \tau||_0 \leq ch||\tau||_1, \tag{5}$$

from which the boundedness of Π_h follows,

$$||\Pi_h \tau||_0 \leq c||\tau||_1. \tag{6}$$

We can now describe our discrete version of the elasticity sequence

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & C^\infty(\Omega) & \xrightarrow{J} & C^\infty(\Omega, \mathbb{S}) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega, \mathbb{R}^2) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle I_h} & & \big\downarrow{\scriptstyle \pi_h} & & \big\downarrow{\scriptstyle P_h} & & \\
0 & \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & Q_h & \xrightarrow{J_h} & \Sigma_h & \xrightarrow{\mathrm{div}} & V_h & \longrightarrow & 0,
\end{array}$$

where J_h denotes the operator J applied element by element. It is not difficult to verify the diagram commutes, see [1] for similar computations.

We now give the error estimates which proofs follow closely [3].

Theorem 4.2. *Let (σ, u) and (σ_h, u_h) be the unique critical points of the Hellinger-Reissner functional over $H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ and $\Sigma_h \times V_h$ respectively. Then*

$$||\,\mathrm{div}\,\sigma - \mathrm{div}_h\,\sigma_h||_0 \leq ch^m||\,\mathrm{div}\,\sigma||_m, \quad 0 \leq m \leq 1,$$
$$||\sigma - \sigma_h||_0 \leq ch||u||_2,$$
$$||u - u_h||_0 \leq ch||u||_2.$$

Proof. The mixed discretization is: Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$\int_\Omega (A\sigma_h : \tau + \mathrm{div}_h\,\tau \cdot u_h)\,dx = 0,\ \tau \in \Sigma_h \tag{7}$$

$$\int_\Omega \mathrm{div}_h\,\sigma_h \cdot v\,dx = \int_\Omega f \cdot v\,dx, v \in V_h. \tag{8}$$

We have

$$\int_\Omega (A\sigma : \tau + \mathrm{div}_h\,\tau \cdot u)dx = E_h(u, \tau),\ \tau \in \Sigma_h, \tag{9}$$

where

$$E_h(u,\tau) = \sum_{e \in \mathcal{E}_h} \int_e [\tau n \cdot t]\,u \cdot t\,ds$$

measures the consistency error for $u \in H^1(\Omega, \mathbb{R}^2)$ and $v \in \Sigma_h$. Here $\mathcal{E}_h$ denotes the set of all edges of the partition $\mathcal{T}_h$ and $[\tau n \cdot t]$ denotes the jump of $\tau n \cdot t$ across the associated edge. Since $[\tau n \cdot t]$ has mean value zero on each edge, it follows by shape regularity and a scaling argument [3] that

$$|E_h(u,\tau)| \leq c||\tau||_0||u||_1, \quad u \in H^1(\Omega, \mathbb{R}^2), \tau \in \Sigma_h.$$

Moreover since the first two moments of $[\tau n \cdot t]$ are continuous, $E_h(v, \tau) = 0$ for any continuous piecewise bilinear function v. Therefore

$$|E_h(u,\tau)| = |E_h(u - u_h^I, \tau)| \leq ch||\tau||_0||u||_2, \quad \tau \in \Sigma_h, u \in H^2(\Omega, \mathbb{R}^2), \tag{10}$$

where u_h^I denotes interpolation in the space of piecewise bilinear continuous functions.

Recall that $\mathrm{div}_h\,\Sigma_h \subset V_h$ and define $||\tau||_A^2 := \int_\Omega A\tau : \tau\,dx$. As in [3], it is not difficult to obtain

$$\mathrm{div}_h\,\sigma_h = P_h\,\mathrm{div}\,\sigma \tag{11}$$

$$\int_\Omega A(\sigma - \sigma_h) : (\Pi_h\sigma - \sigma_h)\,dx = E_h(u, \Pi_h\sigma - \sigma_h) \tag{12}$$

$$||\sigma - \sigma_h||_A^2 \leq ||\sigma - \Pi_h\sigma||_A^2 + 2E_h(u, \Pi_h\sigma - \sigma_h). \tag{13}$$

Next, $||\Pi_h\sigma - \sigma_h||_A^2 = \int_\Omega A(\sigma - \sigma_h) : (\Pi_h\sigma - \sigma_h)\,dx + \int_\Omega A(\Pi_h\sigma - \sigma) : (\Pi_h\sigma - \sigma_h)\,dx$

$$= E_h(u, \Pi_h\sigma - \sigma_h) + \int_\Omega A(\Pi_h\sigma - \sigma) : (\Pi_h\sigma - \sigma_h)\,dx,$$

using (12). By the equivalence of the norms, $||\cdot||_A$ and L^2,

$$||\Pi_h\sigma - \sigma_h||_0 \leq ch||u||_2 + c||\Pi_h\sigma - \sigma||_0, \quad u \in H^2(\Omega, \mathbb{R}^2). \tag{14}$$

Now, let $\tau \in H^1(\Omega, \mathbb{S})$ such that $\operatorname{div} \tau = P_h u - u_h$ with $||\tau||_1 \leq c||P_h u - u_h||_0$. Then using the commutativity property we have $\operatorname{div}_h \Pi_h \tau = P_h \operatorname{div} \tau = P_h u - u_h$ and using (6), $||\Pi_h \tau||_0 \leq c||\tau||_1 \leq c||P_h u - u_h||_0$. Moreover, since $\operatorname{div}_h \Sigma_h \subset V_h$, by (9), (7) and (10)

$$\begin{aligned}
||P_h u - u_h||_0^2 &= \int_\Omega \operatorname{div}_h \Pi_h \tau \cdot (P_h u - u_h)\, dx \\
&= \int_\Omega \operatorname{div}_h \Pi_h \tau \cdot (u - u_h)\, dx = \int_\Omega \operatorname{div}_h \Pi_h \tau \cdot u\, dx - \int_\Omega \operatorname{div}_h \Pi_h \tau \cdot u_h\, dx \\
&= -\int_\Omega A\sigma : \Pi_h \tau + E_h(u, \Pi_h \tau) + \int_\Omega A\sigma_h : \Pi_h \tau\, dx \\
&= -\int_\Omega A(\sigma - \sigma_h) : \Pi_h \tau\, dx + E_h(u, \Pi_h \tau) \\
&\leq c||\sigma - \sigma_h||_0 ||P_h u - u_h||_0 + ch||P_h u - u_h||_0 ||u||_2, \text{ for } u \in H^2(\Omega, \mathbb{R}^2).
\end{aligned}$$

That is,

$$||P_h u - u_h||_0 \leq c\left(||\sigma - \sigma_h||_0 + h||u||_2\right), \quad u \in H^2(\Omega, \mathbb{R}^2). \tag{15}$$

Moreover, by shape regularity, we have the estimate

$$||P_h v - v||_0 \leq ch^m ||v||_m, \text{ for all } v \in H^m(\Omega), \quad 0 \leq m \leq 1. \tag{16}$$

With these preliminaries, we can now give the error estimates. By (16) and (11), $||\operatorname{div} \sigma - \operatorname{div}_h \sigma_h||_0 = ||(I - P_h) \operatorname{div} \sigma||_0 \leq ch^m || \operatorname{div} \sigma||_m$, $0 \leq m \leq 1$.
Using (5) and a norm equivalence,

$$||\sigma - \Pi_h \sigma||_A \leq ch||\sigma||_1,$$

so that using (13), (10), (14) and (5),

$$\begin{aligned}
||\sigma - \sigma_h||_A^2 &\leq ch^2||\sigma||_1^2 + 2E_h(u, \Pi_h \sigma - \sigma_h) \\
&\leq ch^2||\sigma||_1^2 + ch||\Pi_h \sigma - \sigma_h||_0 ||u||_2 \\
&\leq ch^2||\sigma||_1^2 + ch^2||u||_2^2 + ch^2||u||_2||\sigma||_1.
\end{aligned}$$

Since $\sigma = \epsilon(u)$, we get

$$||\sigma - \sigma_h||_0 \leq ch||u||_2.$$

Finally for the displacement, we have by (16), (15) and the above result

$$\begin{aligned}
||u - u_h||_0 &\leq ||u - P_h u||_0 + ||P_h u - u_h||_0 \\
&\leq ch||u||_1 + c\left(||\sigma - \sigma_h||_0 + h||u||_2\right), \\
&\leq ch||u||_1 + ch||u||_2 + ch||u||_2 \leq ch||u||_2.
\end{aligned}$$

□

5. Concluding remarks

It is probably possible to extend the element described here to higher order nonconforming elements. One would need higher order versions of the rotated Q_1 element. We also want to point out that the divergence free elements in the local stress space of the element described in [6], namely $\mathcal{P}_3 \oplus \text{span}(\{x^3y, xy^3\})$ turns out to be a subspace of the local space of our nonconforming H^2 element. The rotated nonconforming H^2 element is a generalization of the rotated Q_1 element on quadrilaterals of Rannacher and Turek, [4]. This makes us believe that our rectangular elements can be extended to quadrilateral meshes. Finally, we note that extensions to mixed or pure traction boundary conditions are straightforward.

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Northern Illinois University, Department of Mathematical Sciences, Dekalb, IL, 60115

E-mail address: awanou@math.niu.edu

URL: http://www.math.niu.edu/~awanou