Math 181H: **Hour Exam 2** (Solutions)

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Problem 1: Find the length of the curve $9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3}, 3)$.

Solve $x = \frac{2}{3}y^{\frac{3}{2}}$ *so* $\frac{dx}{dy} = \sqrt{y}$ *and length is* $\int_0^3 \sqrt{1 + \sqrt{y}^2}\, dy$

$= \int_0^3 (1+y)^{\frac{1}{2}}\, dy = \frac{2}{3}[(1+y)^{\frac{3}{2}}]_0^3\, dy = \frac{2}{3}[4^{\frac{3}{2}} - 1^{\frac{3}{2}}] = \frac{14}{3}$.

Problem 2: Find the length of the parametric curve $x = \cos t, y = t + \sin t$ between $t = 0$ and $t = \pi$. (Hint: need half-angle to integrate)

We see $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = 1 + \cos t$, *so length is* $\int_0^\pi \sqrt{(-\sin t)^2 + (1+\cos t)^2}\, dt$

$= \int_0^\pi \sqrt{1 + 2\cos t + \sin^2 t + \cos^2 t}\, dt = \int_0^\pi \sqrt{2 + 2\cos t}\, dt = \int_0^\pi \sqrt{4\cos^2 \frac{t}{2}}\, dt$

$= 2\int_0^\pi \cos\frac{t}{2}\, dt = 2.2[\sin\frac{t}{2}]_0^\pi = 4[1-0] = 4$.

Problem 3: Compute the length of the polar curve $r = 2\sin\theta$ from $\theta = 0$ to $\theta = \pi$. (What *is* the curve?)

Have $\frac{dr}{d\theta} = 2\cos\theta$, *so length is* $\int_0^\pi \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$

$= \int_0^\pi \sqrt{4(\sin^2\theta + \cos^2\theta)}\, d\theta = 2\int_0^\pi 1 d\theta = 2[\theta]_0^\pi = 2\pi$.

The curve is the circle with radius 1 centered at $(0,1)$.

Problem 4: Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis, between $x = 0$ and $x = 2$.

Using $y' = \frac{1}{2\sqrt{x}}$ *area is* $2\pi\int_0^2 \sqrt{x}\sqrt{1 + (\frac{1}{2\sqrt{x}})^2}\, dx = 2\pi\int_0^2 \sqrt{x(1 + \frac{1}{4x})}\, dx$

$= 2\pi\int_0^2 \sqrt{x + \frac{1}{4}}\, dx = 2\pi\frac{2}{3}[(x+\frac{1}{4})^{\frac{3}{2}}]_0^2 = \frac{4\pi}{3}[(\frac{9}{4})^{\frac{3}{2}} - (\frac{1}{4})^{\frac{3}{2}}] = \frac{4\pi}{3}[\frac{27}{8} - \frac{1}{8}] = \frac{13\pi}{3}$.

Problem 5: Find the area of the surface generated by rotating the parametric curve $x = \cos t, y = 1 + \sin t$ about the x-axis between $t = 0$ and $t = 2\pi$.

Using $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$ *area is* $2\pi\int_0^{2\pi}(1+\sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$

$= 2\pi\int_0^{2\pi}(1+\sin t)(1)\, dt = 2\pi[t - \cos t]_0^{2\pi} = 2\pi[(2\pi - 1) - (0-1)] = 4\pi^2$.

Problem 6: The graph of a curve satisfying $\frac{dy}{dx} = x\sqrt{y}$ passes through $(0,1)$. Find $y = f(x)$.

Separate and integrate: $\int \frac{1}{\sqrt{y}}\, dy = \int x\, dx$, *so* $2\sqrt{y} = \frac{x^2}{2} + C$.

From $2\sqrt{1} = \frac{0^2}{2} + C$ *we get* $C = 2$, *so* $2\sqrt{y} = \frac{x^2}{2} + 2$ *yields* $y = (\frac{x^2+4}{4})^2$.

Problem 7: Find the center of mass of the plane lamina bounded by $y = x^3$ and the x-axis between $x = 0$ and $x = 2$.

First, area is $\int_0^2 x^3\, dx = [\frac{x^4}{4}]_0^2 = \frac{16}{4} = 4$.

So $x_{cm} = \frac{1}{4}\int_0^2 x.x^3\, dx = \frac{1}{4}\int_0^2 x^4\, dx = \frac{1}{4}[\frac{x^5}{5}]_0^2 = \frac{8}{5}$.

And $y_{cm} = \frac{1}{4}\frac{1}{2}\int_0^2 (x^3)^2\, dx = \frac{1}{8}\int_0^2 x^6\, dx = \frac{1}{8}[\frac{x^7}{7}]_0^2 = \frac{16}{7}$. *So cm* $= (\frac{8}{5}, \frac{16}{7})$.

Problem 8: Find the area inside the cardioid $r = 2(1 + \cos\theta)$.
(That is, θ runs from 0 to 2π). Sketch the graph.

$\frac{1}{2}\int_0^{2\pi} 2^2(1+\cos\theta)^2\, d\theta = 2\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta\, d\theta = 2\int_0^{2\pi} 1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\, d\theta$
$= 2[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta]_0^{2\pi} = 2[\frac{3}{2}2\pi + 2.0 + \frac{1}{4}0] = 6\pi.$

Problem 9: Find an equation for the parabola with vertex $(2, 0)$ and directrix the y-axis. Sketch, indicating focus.

Directrix is vertical, with vertex to right, so parabola "opens to the right".
Vertex shifted by 2 in x-direction, by 0 in y; so form is $(y-0)^2 = c(x-2)$.
Also focus is same distance 2 from vertex but away from directrix, that is at $(4, 0)$.
Line $x = 4$ at focus is distance 4 from directrix, so y-value is 4 at $x = 4$.
See then that $c = 8$, so $y^2 = 8(x-2)$.