Math 181: **FINAL EXAM** NAME: ______________________

(Combined Sections): May 5, 1992 SS# ______________________

For safety, please write your name on each page.
If additional work space is needed, use the back of the preceding page.

		(Scoring:)
1,2,3.	________	(30: 10 each)
4,5,6.	________	(30: 10 each)
7,8,9.	________	(30: 10 each)
10,11,12.	________	(30: 10 each)
13.	________	(20 pts)
14.	________	(20 pts)
15.	________	(20 pts)
16.	________	(20 pts)
TOTAL.	________	(200 pts)

Circle your TA and discussion time:

(Prof. Fong)
- Boris: 8 9 10
- Rasoulian: 8 9 11

(Prof. Radford)
- Shi: 11 12 1
- Xiong: 10 11 12

(Prof. Friedlander)
- Chai: 12 1 2
- Yang: 12 1 2

(Prof. Smith)
- Tan: 11

Multiple-choice Problems

In Problems 1 through 12, **CIRCLE** the correct answer.
(There is *no* partial credit, but you must show work to receive credit).

Problem 1 A curve has parametric equations $x = t + t^2$, $y = t + e^t$. Find the slope of the tangent line at the point $(0, 1)$.

a) -2

b) -1

c) 0

d) 1

e) 2

Answer e): $\frac{dx}{dt} = 1 + 2t$, $\frac{dy}{dt} = 1 + e^t$, $\frac{dy}{dx} = \frac{1+e^t}{1+2t}$. *At* $(0, 1)$ *we have* $t = 0$, *so the slope is 2.*

Problem 2 Find the length of the polar curve $r = e^{-\theta}$ from $\theta = 0$ to $\theta = \ln 2$.

a) $2\sqrt{2}$

b) $\sqrt{2}$

c) $2\pi\sqrt{2}$

d) $\pi(2^{\frac{3}{2}} - 1)$

e) $\dfrac{\sqrt{2}}{2}$

Answer e): $\frac{dr}{d\theta} = -e^{-\theta}$, *so length is* $\int_0^{\ln 2} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta = \int_0^{\ln 2} \sqrt{2e^{-2\theta}}\, d\theta$
$= \sqrt{2} \int_0^{\ln 2} e^{-\theta}\, d\theta = \sqrt{2} \cdot -e^{-\theta}|_0^{\ln 2} = \sqrt{2}(-\frac{1}{2} + 1) = \frac{\sqrt{2}}{2}$.

Problem 3 Find the area of the surface of revolution obtained by rotating $y = x^2/2$ for $0 \le x \le 1$ about the y-axis.

a) $(4\sqrt{2} - 2)\pi/3$

b) $(2\sqrt{2} - 2)\pi/3$

c) $(4\sqrt{2} + 1)\pi/3$

d) $(4\sqrt{2} + 2)\pi/3$

e) $(2\sqrt{2} + 2)\pi/3$

f) $(4\sqrt{2} - 1)\pi/3$

Answer a): $y = x^2/2$, $y' = x$, *so* $ds = \sqrt{1 + x^2}\, dx$. *The area of the surface is*
$S = \int_0^1 2\pi x\, ds = 2\pi \int_0^1 x\sqrt{1 + x^2}\, dx = \frac{2\pi}{3}(1 + x^2)^{3/2}]_0^1 = \frac{\pi}{3}(4\sqrt{2} - 2)$.

Problem 4 Evaluate $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$.

a) $1/4$

b) $1/3$

c) $1/2$

e) $\ln 2$

f) $\ln 3$

g) $\ln 4$

h) divergent

Answer a): (by parts) $u = \ln x, dv = \frac{dx}{x^3}$ *gives* $\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} + \int \frac{dx}{2x^3} = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$.
Then $\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left[(-\frac{\ln t}{2t^2} - 0) + (-\frac{1}{4t^2} + \frac{1}{4})\right] = \frac{1}{4}$.

Problem 5 Find $\displaystyle\int \sin^3 x\,dx$.

a) $\cos^3 x/3$

b) $\sin^3 x/3$

c) $-\cos x + \cos^3 x/3$

d) $\sin x - \sin^3 x/3$

e) $\cos x - \sin^3 x/3$

f) $\sin x - \cos^3 x/3$

Answer c): $\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx = -\cos x + \frac{1}{3}\cos^3 x$.

Problem 6 Find $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

a) $\sqrt{x^2+4}$

b) $x\sqrt{x^2+4}$

c) $-\sqrt{x^2+4}/4x$

d) $-\sqrt{x^2+4}/x^2$

e) $x/\sqrt{x^2+4}$

f) $1/\sqrt{x^2+4}$

Answer c): $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ *converts* $\int \frac{dx}{x^2\sqrt{x^2+4}}$
into $\int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \int \frac{\sec\theta\,d\theta}{4\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{4}\frac{1}{\sin\theta} = -\frac{\sqrt{x^2+4}}{4x}$.

Problem 7 Find $\sum_{n=0}^{\infty} 2^{2n} 5^{1-n}$.

a) 10

b) 25

c) 2

d) 5

e) divergent

Answer b): $\sum_{n=0}^{\infty} 2^{2n} 5^{1-n} = 5 \sum_{n=0}^{\infty} (\frac{4}{5})^n = 5[\frac{1}{1-4/5}] = 25.$

Problem 8 Which of the following series is convergent?

a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

b) $\sum_{n=1}^{\infty} \frac{3n}{4n^2 - 1}$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1}$

d) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

e) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Answer c): converges since it is an alternating series with $a_{n+1} \leq a_n$ *and* $a_n \longrightarrow 0$.
[a), b), d) diverge by limit comparison with harmonic; e) since $n^n/n! \longrightarrow \infty$ *].*

Problem 9 Which of the following series is divergent?

a) $\sum_{n=1}^{\infty} \left(\frac{n+4}{2n-1}\right)^n$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

d) $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+1}\right)^n$

e) $\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

Answer d): diverges by the root test. [The others converge: a) by the root test, b) by the ratio test, c) by the alternating test, e) is a geometric series with $r = 8/9$ *].*

Problem 10 Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n(x-1)^n}{4^{n+1}}$.

a) 1

b) 2

c) 3

d) 4

e) 5

Answer d): The ratio $\frac{(n+1)|x-1|^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{n|x-1|^n} \longrightarrow \frac{|x-1|}{4}$.
The series converges when $|x-1| < 4$, *diverges when* $|x-1| > 4$, *so* $R = 4$.

Problem 11 Find $\lim_{x\to\infty} x^{3/x^2}$.

a) $3/2$

b) e

c) $e^{3/2}$

d) $\ln(3/2)$

e) ∞

f) 1

Answer f): $x^{3/x^2} = e^{\frac{3\ln x}{x^2}} \sim e^{3\frac{1}{x}/2x} = e^{\frac{3}{2x^2}} \longrightarrow 1$.

Problem 12 A bacteria colony in a culture has been growing exponentially. If there were 80 bacteria at the end of 3 minutes and 160 at the end of 5 minutes, how many were present initially?

a) 40

b) $160/2^{3/2}$

c) $80/2^{3/2}$

d) $40/2^{3/2}$

e) $80/\ln 2$

f) $160/\ln 2$

Answer c): $P(t) = P(0)e^{kt}$, $80 = P(0)e^{3k}$, $160 = P(0)e^{5k}$.
Thus $2 = e^{2k}$, $k = \ln\sqrt{2}$, *and* $P(0) = 80/e^{3\ln\sqrt{2}} = 80/2^{3/2}$.

In the remaining problems 13-16, **show all your work**. (There *is* partial credit).

Problem 13 Evaluate $\displaystyle\int \frac{3x^2+x+4}{x^3+x}\,dx$.

Answer: $\frac{3x^2+x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ *gives* $3x^2+x+4 = A(x^2+1)+(Bx+C)x$.
So (const) $A = 4$; *(x)* $C = 1$; *and then* (x^2) $3 = A + B = 4 + B$ *gives* $B = -1$.
So integrate: $\int \frac{4}{x}\,dx - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C$.

Problem 14 Find the area inside the polar graph of $r = 4\sin\theta$ but outside the polar graph of $r = 2\sin\theta$.

Answer: The circles touch only at the origin. So the desired area is
$A = 2\cdot\frac{1}{2}\int_0^{\pi/2}(16\sin^2\theta - 4\sin^2\theta)\,d\theta = \int_0^{\pi/2} 6(1-\cos 2\theta)\,d\theta = (6\theta - 3\sin 2\theta)]_0^{\pi/2} = 3\pi$.

Problem 15

a) Solve the differential equation $\dfrac{dQ}{dt} = 2Q(5-Q)$ for $Q(t)$, with initial condition $Q(0) = 1$.
b) Find $\lim\limits_{t\to\infty} Q(t)$.

Answer: Separating variables gives $\frac{dQ}{Q(5-Q)} = dt$.
Now $\frac{1}{Q(5-Q)} = \frac{1}{5}(\frac{1}{Q} + \frac{1}{5-Q})$, *so* $\ln\left(\frac{Q}{5-Q}\right) = 10t + C$
(note $Q(0) = 1 > 0$, $5 - Q(0) = 4 > 0$, *so no absolute values are needed).*
The initial condition gives $C = \ln\frac{1}{4}$. *For a):* $\ln(\frac{4Q}{5-Q}) = 10t$, $\frac{4Q}{5-Q} = e^{10t}$, $Q(t) = \frac{5e^{10t}}{4+e^{10t}}$.
Finally for (b), $\lim_{t\to\infty} Q(t) = 5$.

Problem 16

a) Find the Taylor series for $f(x) = (1+x)^{-1/2}$ expanded about $x = 0$.
b) Let $R_3(x)$ be the third remainder term (that is, beginning with the x^4-term) of the series in a). Find $M > 0$ such that $|R_3(x)| \le M$ for $0 \le x \le 1$.

Answer: $f^{(0)}(x) = (1+x)^{-1/2}$, $f^{(1)}(x) = (-\frac{1}{2})(1+x)^{-3/2}$, $f^{(2)}(x) = (-\frac{1}{2})(-\frac{3}{2})(1+x)^{-5/2}$,
$f^{(3)}(x) = (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(1+x)^{-7/2}$, $f^{(4)}(x) = (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(1+x)^{-9/2}$.
So $f^{(n)}(x) = (-1)^n \frac{1\cdot 3 \cdots (2n-1)}{2^n}(1+x)^{\frac{2n+1}{2}}$, *and the Taylor series is* $\sum_0^\infty (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n n!} x^n$.
The remainder term $|R_3(x)| = \frac{1\cdot 3\cdot 5\cdot 7}{16\cdot 24}|1+z|^{-9/2} \le \frac{105}{384}$ *for* $0 \le x \le 1$.