Math 181H: **PRACTICE Exam 2** Prof. S. Smith, March 1992
(based on Exam 1 of old Math 132, April 1990)

Problem 1: Find the length of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$.
Note $y' = \frac{1}{3}\frac{3}{2}(x^2+2)^{\frac{1}{2}}2x = x\sqrt{x^2+2}$, *so length is*
$\int_0^3 \sqrt{1+(x\sqrt{x^2+2})^2}\,dx = \int_0^3 \sqrt{1+x^2(x^2+2)}\,dx = \int_0^3 \sqrt{x^4+2x^2+1}\,dx$
$= \int_0^3 x^2+1\,dx = [\frac{x^3}{3}+x]_0^3 = 12.$

Problem 2: Find the length of the parametric curve $x = t^3$, $y = t^2$, $0 \le t \le 4$.
Have $\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t$, *so length is* $\int_0^4 \sqrt{(3t^2)^2+(2t)^2}\,dt = \int_0^4 \sqrt{9t^4+4t^2}\,dt.$
$= \int_0^4 t\sqrt{9t^2+4}\,dt$ *(Subst:* $u = 9t^2+4$*)* $= \frac{1}{27}[(9t^2+4)^{\frac{3}{2}}]_0^4 = \frac{1}{27}[(9.4^2+4)^{\frac{3}{2}} - 4^{\frac{3}{2}}] = \frac{8}{27}[37^{\frac{3}{2}} - 1].$

Problem 3: Find the length of the polar curve $r = e^{-\theta}$, $0 \le \theta \le 3\pi$.
Have $\frac{dr}{d\theta} = -e^{-\theta}$, *so length is* $\int_0^{3\pi} \sqrt{(e^{-\theta})^2+(-e^{-\theta})^2}\,d\theta = \int_0^{3\pi} \sqrt{2e^{-2\theta}}\,d\theta$
$= \int_0^{3\pi} \sqrt{2}e^{-\theta}\,d\theta = \sqrt{2}[-e^{-\theta}]_0^{3\pi} = \sqrt{2}[-e^{-3\pi} - (-1)] = \sqrt{2}[1 - e^{-3\pi}].$

Problem 4: Find the area of the surface generated by rotating about the x-axis the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$.
Have $\frac{dy}{dx} = 3x^2$, *so area is* $2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx.$
(Subst: $u = 1+9x^4$*)* $= \frac{2\pi}{54}[(1+9x^4)^{\frac{3}{2}}]_0^1 = \frac{\pi}{27}[10^{\frac{3}{2}} - 1].$

Problem 5: Find the area of the surface generated by rotating about the x-axis the parametric curve $x = \frac{t^2}{2}, y = t$ between $t = 1$ and $t = 3$.
Have $\frac{dx}{dt} = t, \frac{dy}{dt} = 1$, *so area is* $2\pi\int_1^3 t\sqrt{t^2+1^2}\,dt.$
(Subst: $u = t^2+1$*)* $= \frac{2\pi}{3}[(t^2+1)^{\frac{3}{2}}]_1^3 = \frac{2\pi}{3}[10^{\frac{3}{2}} - 1].$

Problem 6: Find the area of the region enclosed by one loop of the polar curve $r = 3\sin 2\theta$.
Draw to see 4 loops, one in $[0, \frac{\pi}{2}]$. *So area is* $\frac{1}{2}\int_0^{\frac{\pi}{2}} (3\sin 2\theta)^2\,d\theta$
$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2 2\theta\,d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{9}{4}[\theta - \frac{\sin 4\theta}{4}]_0^{\frac{\pi}{2}} = \frac{9}{4}[(\frac{\pi}{2} - 0) - 0] = \frac{9\pi}{8}.$

Problem 7: Sketch the polar curve $r = 2(1 - \sin\theta)$.

Problem 8: Determine the function y of x satisfying the differential equation $\frac{dy}{dx} = 2xy^2$, subject to the condition $y(1) = 1$.

Separate and integrate: $\int \frac{1}{y^2}\, dy = \int 2x\, dx$, *so* $-\frac{1}{y} = x^2 + C$.

Then $1 = y(1)$ *gives* $-\frac{1}{1} = 1^2 + C$ *so* $C = -2$. *Hence* $-\frac{1}{y} = x^2 - 2$ *yields* $y = -\frac{1}{x^2-2}$.

Problem 9: Find the center of mass of the triangle with vertices (0,0) (1,0) (1,2).

So right triangle, bounded by $y = 0, x = 1, y = 2x$; *area* $\frac{1}{2}1.2 = 1$.

Then $x_{cm} = \frac{1}{1}\int_0^1 x(2x)\, dx = 2\int_0^1 x^2\, dx = 2[\frac{x^3}{3}]_0^1 = \frac{2}{3}$.

And $y_{cm} = \frac{1}{2}\int_0^1 (2x)^2\, dx = 2\int_0^1 x^2\, dx = 2[\frac{x^3}{3}]_0^1 = \frac{2}{3}$. *So cm at* $(\frac{2}{3}, \frac{2}{3})$.

Problem 10: Find an equation for the ellipse with foci $(\pm 1, 0)$ and vertices $(\pm 2, 0)$.

As foci and vertices have $y = 0$, *axis is on* x*-axis.*

From vertices with $x = \pm 2$ *get major axis* $a = 2$.

Also 4 is sum of distances from foci to (2,0), so 2 is distance from each focus to (0,b).

This gives hypotenuse length 2 for right triangle (0,0) (1,0) (0,b) so $b = \sqrt{3}$.

So equation is $\frac{x^2}{4} + \frac{y^2}{3} = 1$.