Math 181H: **PRACTICE Hour Exam 3** Spring 1992, S. Smith
(based on Hour Exam 2 of quarter course Math 132, May 1990)

Problem 1: Find the sum (if convergent) of $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \ldots$.
(geometric series) $\frac{9}{100}\sum_{n=0}^{\infty}(\frac{1}{100})^n = \frac{9}{100}\frac{1}{1-\frac{1}{100}} = \frac{9}{100}\frac{100}{99} = \frac{1}{11}$.

Problem 2: Test convergence of $\sum_{k=0}^{\infty}\frac{k^2}{k^3+3}$.
Integral test: $\int_1^{\infty}\frac{x^2}{x^3+3} = \frac{1}{3}\ln(x^3+3)|_1^{\infty} = \infty$, *so diverges.*

Problem 3: Test convergence of $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}$.
Comparison test: below $5\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$, *so converges.*

Problem 4: Test convergence of $\sum_{n=0}^{\infty}\frac{n^2}{2^n}$.
Root test: $\lim_n(\frac{n^2}{2^n})^{\frac{1}{n}} = \frac{1^2}{2} < 1$, *so converges. (Also ratio test).*

Problem 5: Find the interval of convergence of $\sum_{n=1}^{\infty}(-1)^n\frac{(x+1)^n}{n(n+1)}$.
(ratio:) $\lim_n\frac{|x+1|^{n+1}}{(n+1)(n+2)}\frac{n(n+1)}{|x+1|^n} = \lim_n\frac{1}{1+\frac{1}{n}}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}|x+1| = |x+1|$; *so radius 1 about* $x=-1$.
At $x=-2$, *get* $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n(n+1)}$, *converges by comparison* $< \sum\frac{1}{n^2}$.
At $x=0$, *get* $\sum_{n=1}^{\infty}\frac{(-1)^n 1^n}{n(n+1)}$, *coverges by alternating test (decr,* $\to 0$*).*
So the interval of convergence is $[-2,0]$.

Problem 6: Find the n-th Taylor polynomial of $f(x)=\cos x$ at $a=\frac{\pi}{4}$ for $n=3$.
Compute $f(x)=\cos x, f'(x)=-\sin x, f''(x)=-\cos x, f'''(x)=\sin x$;
at $x=a=\frac{\pi}{4}$, *get values* $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$.
So $\frac{\sqrt{2}}{2}[1-(x-\frac{\pi}{4})-\frac{1}{2!}(x-\frac{\pi}{4})^2+\frac{1}{3!}(x-\frac{\pi}{4})^3]$.

Problem 7: Express $\int_0^1\frac{\sin x}{\sqrt{x}}dx$ as the sum of an infinite series, and compute the integral to three decimal places.
Usual $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$, *so* $\frac{\sin x}{\sqrt{x}} = x^{\frac{1}{2}}-\frac{x^{\frac{5}{2}}}{3!}+\frac{x^{\frac{9}{2}}}{5!}-\ldots$.
Then $\int_0^1\frac{\sin x}{\sqrt{x}}dx = [\frac{2x^{\frac{3}{2}}}{3}-\frac{2x^{\frac{7}{2}}}{7.3!}+\frac{2x^{\frac{11}{2}}}{11.5!}-\ldots]_0^1$.
So series is $2\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)(2n+1)!}$, *approx* $2(\frac{1}{3}-\frac{1}{7.3!}+\frac{1}{11.5!})$, *since* $|\frac{-2}{15.7!}| < \frac{1}{2000}$.

Problem 8: Compute a power series at $a=0$ for $\frac{1}{(1-x^2)}$.
Shortcut: Use geometric series $f(x)=\frac{1}{1-x}=1+x+x^2+\ldots$
so $\frac{1}{1-x^2}=f(x^2)=1+x^2+x^4+\ldots=\sum_{n=0}^{\infty}x^{2n}$.

Problem 9: Find the Taylor Series of $f(x)=\cos 2x$ about $x=\frac{\pi}{4}$.
Sequence of derivs: $\cos 2x, -2\sin 2x, -4\cos 2x, 8\sin 2x, 16\cos 2x, -32\sin 2x, \ldots$;
values at $x=a=\frac{\pi}{4}$ *are:* $0,-2,0,8,0,-32,\ldots$.
So series is $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^{2n+1}}{(2n+1)!}(x-\frac{\pi}{4})^{2n+1}$.