Math 220: Solutions to EXAM II, 11/19/99

Be sure to show all work in the answer book. Indicate where you have used a TI-89 to get an answer. Please put a BOX around all final answers. The more work you show, the more partial credit you may get. Indicate your TA Section time as 10:00 or 12:00.

1. (a) Show that the solution $h(t)$ to
   
   $$ay'' + by' + cy = \delta(t), \quad y(0) = y'(0) = 0$$
   
   is the same as
   
   $$ay'' + by' + cy = 0, \quad y(0) = 0, \quad y'(0) = 1/a$$

   (b) Write a convolution integral using $h(t)$ which solves
   
   $$ay'' + by' + cy = \sin(t), \quad y(0) = y'(0) = 0$$

2. Solution: Taking Laplace transforms we have $a(s^2 Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 1$. Using the IC and solving for $Y(s)$ gives $Y(s) = 1/(as^2 + bs + c)$. For the second equation we have $a(s^2 Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0$. Using the IC we have $as^2 Y(s) - a(1/a) + bsY(s) + cY(s) = 0$. Solving for $Y(s)$ gives $Y(s) = 1/(as^2 + bs + c)$. Since the Laplace transforms are equal, the $h(t)$, which are the inverse transforms are equal. Using convolutions the solution the third part is $y(t) = h(t) * \sin(t)$ or $\int_0^t h(t - \tau) \sin(\tau) \, d\tau$.

3. Find $y(t)$ for the initial value problem: ($u(t)$ is the Heaviside function.)

   $$y' + 6y = [1 - u(t - 2)]e^{-6t}, \quad y(0) = 1$$

4. Solution: Using Laplace transforms and writing $-u(t-2)e^{-6t} = -e^{-12}u(t-2)e^{-6(t-2)}$ gives $sY(s) - 1 + 6Y(s) = 1/(s + 6) - e^{-12}e^{-2s}/(s + 6)$. Solving for $Y(s)$ gives $Y(s) = 1/(s + 6) + 1/(s + 6)^2 - e^{-12}e^{-2s}/(s + 6)^2$. Inverting gives $y(t) = e^{-6t} + te^{-6t} - e^{-12}(t - 2)u(t-2)e^{-6(t-2)} = e^{-6t} + te^{-6t} - (t-2)u(t-2)e^{-6t}$.

   Or, in terms of a convolution integral we have $y(t) = e^{-6t} + \int_0^t (1-u(\tau-2))(e^{-6\tau})(e^{-6(t-\tau)}) \, d\tau$.

5. Using the power series method, find the first four non zero terms in the series solution about $x = 0$ of

   $$y' + 2xy = 0, \quad y(0) = 1$$

6. Solution: We plug in the series $y(x) = \sum a_n x^n$ to get the recursion relation $\sum_{n=0}^\infty na_n x^{n-1} + 2 \sum_{n=0}^\infty a_n x^{n+1} = 0$. Replacing $n - 1 = p$ in the first sum and $n + 1 = p$ in the second we have $\sum_{p=0}^{p} (p + 1)a_{p+1}x^p + 2 \sum_{p=1}^\infty a_{p-1}x^p = 0$. This gives $(p + 1)a_{p+1} = -2a_{p-1}$ for $p = 1, 2, 3$ etc. The first sum gives $a_1 = 0$. From the IC we know that $a_0 = 1$. So we have $a_0 = 1, a_1 = 0, a_2 = -a_0, a_3 = 0, a_4 = -a_2/2 = a_0/2$ etc. The final answer is $y(x) = 1 - x^2 + x^4/2 - x^6/6 + ...$
7. Find \( x(t) \):

\[
x' - y' = (\sin t)u(t - \pi), \quad x(0) = 1 \\
x + y' = 0, \quad y(0) = 1
\]

8. Solution: Taking Laplace transforms and solving for \( X(s) \) we have for \( F(s) = \mathcal{L}\{\sin(t)u(t - \pi)\}(s) \), \( sX(s) - 1 - (sY(s) - 1) = F(s) \), \( X(s) + sY(s) - 1 = 0 \); \( sX(s) - X(s) = 1 + F(s) \), now inverting we have \( x(t) = e^{-t} + \int_0^t e^{-(t-\tau)} \sin \tau \cdot u(\tau - \pi) \, d\tau \) or \( x(t) = e^{-t} + e^{-t} \ast (\sin t)u(t - \pi) \).

9. Consider the following equation for the damped oscillator with displacement \( y(t) \):

\[
y'' + 10y' + 25y = u(t - 4\pi)\cos(2(t - 4\pi)), \quad y(0) = 1, \quad y'(0) = 0.
\]

(a) Is this SHM underdamped, overdamped, or critically damped?
(b) Solution: It is critically damped since the roots are repeated \( r_{1,2} = -5 \)
(c) Make a rough sketch of the solution for \( 0 \leq t \leq 4\pi \).
(d) Solution: The graph should be a very quick exponential decay. If you look close to the origin the slope of the curve will be zero because the IC \( y'(0) = 0 \) In fact \( y(t) = e^{-5t} + 5te^{-5t} \).
(e) Make a rough sketch of the solution for \( 10\pi \leq t \leq 12\pi \)
(f) Solution: The answer here is a sine curve with constant amplitude since by the time \( t \) has reached \( 10\pi \) the part of the solution due to the IC has died out. The forcing function will keep the motion going. Using the method of undetermined coefficients you expect a linear combination of \( \sin \) and \( \cos \) with no exponential decay present.