Draft of Section 13.4: Compound Interest

Compound interest is without a doubt the most important human accomplishment of all time. In this section we see why.

The formula for compound interest is \( A = P(1+r)^n \) where \( P \) is the initial amount of money or principal, \( A \) is after \( n \) time periods, and \( r \) will be the interest rate per time period. For example, if the compounding period is one year and the money is deposited for four years and \( P = 516 \) and \( r = .06 \), then \( n = 4 \) and the formula gives \( A = 651.44 \). If instead you go ahead and then make the compounding period to be one month in this example, then \( r = .005 \), \( n = 48 \) and \( A = 655.57 \).

The formula works because, after the first time period the amount \( A \) becomes \( A = P + Pr = P(1 + r) \). At the end of the second time period the amount is \( A = P(1 + r) + P(1 + r)r = P(1 + r)^2 \). And so on. So \( A = P(1 + r)^n \) for all positive integers \( n \) is the general formula.
Example: $5000 is deposited in a savings account at a large bank in a small town near Chicago. The population of the town is 6500. The account pays 4% interest per year compounded monthly. The money is left undisturbed in this account for ten years. During this time the value of the U.S. dollar rises 8% against the Japanese yen. After ten years by how much, in dollars, has the account increased in value?

Solution: Clearly $P$ is 5000. It is absolutely trivial to see that $r = .04/12 = .0033$. So to find $n$ you just multiply 12 by 10 and I get 120 which we know to be $n$ and she can use this value of $n$ in the formula. So $A = 5000(1 + .00333)^{120} = 7451.19$. So the amount is $7451.19$. So the increase after 10 years is $2451.19$.

Example: If in 1790 Benjamin Franklin had invested $100 at 10% percent interest compounded annually and the amount accumulated undisturbed, then it would be worth an enormous amount of money today.

The Rule of 72: A useful for method for mental computations involving compound interest is called the Rule of 72. It states that the amount of time it takes for an investment to double in value at a compound interest rate of $R$ percent compounded annually is approximately $72/R$ years. For example, it will take approximately $72/5 = 14.4$ years for money invested at 5 percent to double. The rule can also be used to determine approximately what interest rate is needed to have an investment double in a given number of years. For example, if one wants to know what interest rate is needed to double the principal in 10 years, divide 72 by 10 to obtain the rate of 7.2 percent. Rule: $R = 72/T$.

The justification for the Rule of 72 is that if in the formula $A = P(1 + r)^n$ we let $A$ be 2 and $P$ be 1, then $n$ is the number of years for doubling at rate $r$ per year. $2 = (1 + r)^n$. We saw in Chapter 7, when we studied the number $e$, that as the positive real number $h$ gets close to 0, the value of $(1 + h)^{1/h}$ gets arbitrarily close to $e$ from below. From

$$2 = \left(1 + r^{1/r}\right)^{nr}$$

we get that 2 is approximately equal to $e^{nr}$, with $e^{nr}$ being an overestimate. Do ln both sides. Use $1 = \ln e$. Get $.6931 \approx nr, \ n \approx .6931/r, \ n \approx 69.31/R$ for $R$ interest rate $r$ in percent. But this is an underestimate for $n$. Replace 69.31 by 72 to get a better estimate and to use an integer with many integer divisors to facilitate mental arithmetic by the human brain working quickly on one’s feet. That’s all, folks.

Example: Estimate the amount of money that $1000 would grow to at 8 percent per year, compounded annually, for 90 years. By the rule of 72 we see that the money doubles every 9 years. So in 90 years $1000 grows to approximately $1,024,000.

Example: A hedge fund manager gets two competing proposals: The first is to invest one billion dollars in Cicada Futures and get back four billion dollars in 14 years. The second is to put one billion dollars in the Lilliputian Ponzi Scheme Fund that will pay 12 percent per year interest compounded continuously for the next 14 years. The manager must decide instantly which is the better investment for the 14 year period. She uses the rule of 72 to compute in her head that the Cicada Futures proposal has an approximate annual return of 11.2%.