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Cardinalities of finite distributive lattices

by

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Dedicated to Professor Hermann Boerner to the occasion of his 7oth birthday

Despite the wealth of information available on the class of distributive lattices, general results on the cardinalities of the finite members of this class are rather scarce. This paper presents an algorithm which computes from a finite partially ordered set X the cardinality of the distributive lattice S(X), whose partially ordered set of join irreducibles is isomorphic to X.

The algorithm is described in Section 2. This section also includes a discussion on how the algorithm can readily be used for rapid paper and pencil calculations. A computer implementation is also presented.

Section 3 contains some general results on the cardinality of the lattice S(X), when the poset X is given. These results are obtained with the help of the notion of the incidence ring of a partially ordered set.

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Finally Section 4 consists of numerical results obtained using the algorithm and the techniques from Section 3. Included are some results on the cardinalities of free algebraic structures having distributive lattice reducts. In particular the cardinality of the free distributive lattice on seven generators is determined.

Section 1. Background

In what follows L always denotes a finite distributive lattice and X is a finite partially ordered set (poset). An element $a \in L$ is <u>join irreducible</u> if $a \neq 0$ and if $a = b + c$ then $a = b$ or $a = c$. The set of all join irreducibles in L forms a poset J(L).
A subset Y of X is a <u>semi-ideal</u> if $x \leq y$ and $y \in Y$ imply $x \in Y$. The collection of all semi-ideals of X, ordered by set theoretic inclusion, forms a distributive lattice S(X). A very useful result of lattice theory states that $J(S(X)) \simeq X$ and $S(J(L)) \simeq L$. In particular, $a \in L$ corresponds to the semi-ideal $\{x \mid x \in J(L), x \leq a\}$ of the poset J(L).

An <u>anti-chain</u> is a totally unordered set. A semi-ideal $Y \subseteq X$ is uniquely determined by the anti-chain of its maximal elements. Similarly any anti-chain $A \subseteq X$ determines a semi-ideal of X in a natural way, i.e. $\{x \mid x \in X, \exists a \in A\ x \leq a\}$. Thus, the cardinality of S(X) is equal to the mumber of anti-chains in X.

For elements $x,y \in X$ $[x,y]$ is the (possibly empty) interval $\{a \mid a \in L,\ x \leq a \leq y\}$, and $(x]$ is the semi-ideal $\{a \mid a \in L,\ a \leq x\}$.

A <u>chain</u> is a totally ordered set, in particular

$\underline{n} = \{0,\dots,n-1\}$ endowed with the natural order is the n-element chain. If Y and Z are posets, then Hom(Y,Z) denotes the set of all order homomorphisms from Y to Z. If, moreover, Y and Z are bounded then $Hom_{0,1}(Y,Z)$ is the set of all order homomorphisms which respect the bounds. The dual of X will be denoted by $\check{X}$.

Details on this an other matters of distributive lattices can be found in [1], [5], [11].

A few finite posets Y have lattices S(Y) which are easily described. Thus $S(\emptyset)$ is the one-element lattice and more generally $S(\underline{n}) \simeq \underline{n+1}$. At the other extreme the lattice of semi-ideals of the n-element anti-chain is simply the Boolean lattice $\underline{2}^n$. Other examples are given by the following.

<u>Proposition 1.1</u>

Let Y,Z be finite posets, and n a natural number. Then:

(1) $S(\check{Y}) \simeq \check{S(Y)}$

(2) $S(Y + Z) \simeq S(Y) \times S(Z)$

(3) $S(Y \times Z) \simeq Hom(\check{Y}, S(Z))$

(4) $S(\underline{2}^n) \simeq FD_{0,1}(n)$, the free bounded distributive lattice on n generators .

Proof:

(1),(2) and (3) are well known, see e.g. [5], p.57, while (4) is a consequence of the more general observation that $S(Y \times Z)$ is isomorphic to the free product of S(Y) and S(Z) in the category of bounded distributive lattices, see [11], p.129.

Combining (3) and (4) we get the next results which have served as a basis for several approaches to compute the cardinality of $FD_{o,1}(n)$, see e.g. [9],[1o] and also Section 4 of this paper.

Corollary 1.2

(1) $FD_{o,1}(n+1) \simeq \{<a,b> \mid a,b \in FD_{o,1}(n),\ a \leq b\}$

(2) $FD_{o,1}(n+2) \simeq \{<a,b,c,d> \mid a,b,c,d \in FD_{o,1}(n), a \leq b \leq d,\ a \leq c \leq d\}$.

In spite of Proposition 1.1 (4) and Corollary 1.2 the problem of finding a general formula for the cardinality of $FD_{o,1}(n)$ is probably the oldest unsolved problem in lattice theory. The lack of success on this particular problem emphasizes the general difficulties of these enumeration problems.

Section 2. An algorithm

The algorithm for finding $|S(X)|$ from the poset X is simply an iteration of Theorem 2.1 below. We need two bits of notation. If $Y \subseteq X$ then $X \setminus Y$ is the poset obtained by deleting from X all elements of Y; and for $x \in X$, $\text{cone}(x) = \{y \mid y \in X,\ y \leq x \text{ or } y \geq x\}$.

Theorem 2.1

For any $x \in X$:

$$|S(X)| = |S(X \setminus \{x\})| + |S(X \setminus \text{cone}(x))|.$$

Proof:

As already noted $|S(X)|$ is the number of anti-chains of X. If an anti-chain A of X contains x then A cannot contain any other element in cone(x). Hence, the number of anti-chains in X which contain x is $|S(X \setminus \text{cone}(x))|$. Moreover, the set of anti-chains which do not contain x has cardinality $|S(X \setminus \{x\})|$.

As an example of the use of Theorem 2.1, let W_n be the poset having n elements as drawn in Figure 1.

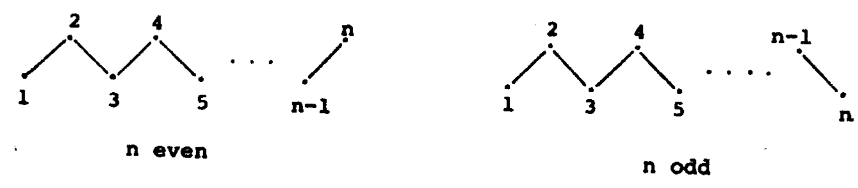

Figure 1

So $W_o = \emptyset$ and W_1 is the one element poset. Letting x = n in Theorem 2.1, it follows that $|S(W_n)| = |S(W_{n-1})| + |S(W_{n-2})|$. Since $|S(W_o)| = 1$ and $|S(W_1)| = 2$, we obtain the following:

Example 2.2

The cardinality of $S(W_{n-1})$ is the n^{th} Fibonacci number.

More importantly, Theorem 2.1 may be used to find $|S(X)|$ for any finite poset X. Let $X = \{x_1,\dots,x_n\}$ be any enumeration of X. Form a binary tree T(X) with nodes labeled by subposets and edges labeled by members of X in the following way: The root node of T(X) is labeled by X. Coming out of the root are edges labeled x_1 going to the nodes labeled $X \setminus \{x_1\}$ and $X \setminus \text{cone}(x_1)$. In general, if a node is labeled by the poset Y and if $i = \min\{j \mid x_j \in Y\}$ then the branches emanating from Y are labeled x_i and go to nodes $Y \setminus \{x_i\}$ and $Y \setminus \text{cone}(x_i)$. If $Y = \emptyset$ then this is a terminal node.

It follows from Theorem 2.1 that $|S(X)|$ is equal to the number of terminal nodes in $T(X)$.

As an example, Figure 2 is a poset X corresponding to $J(L)$ for the free distributive lattice on 3 generators.

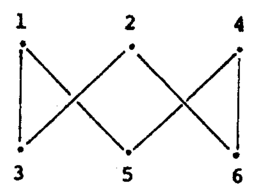

Figure 2

Figure 3 is $T(X)$. The tree $T(X)$ is drawn with the node $Y \setminus \{x_i\}$ on the lower left of node Y.

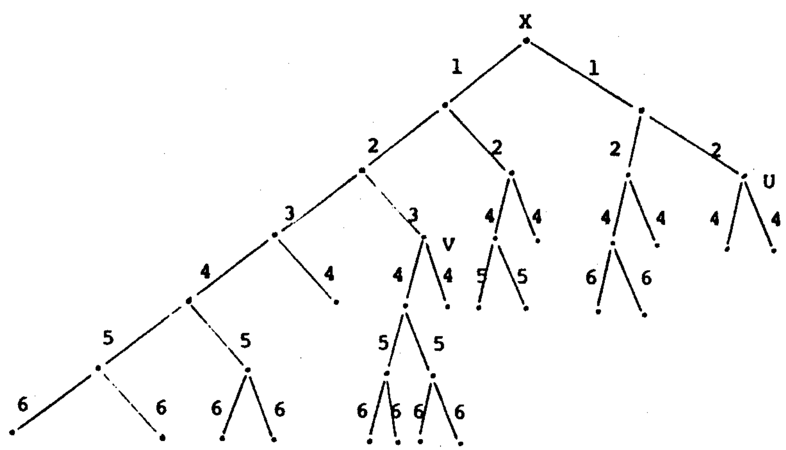

Figure 3

For example, the node labeled U is the subposet {4} and the node labeled V is the subposet {4,5,6}. Each terminal node N corresponds to the anti-chain in X whose elements consist of those x_i for which the branch $Y \setminus \mathrm{cone}(x_i)$ was taken in reaching N.

In practice, for paper and pencil calculations, the entire tree $T(X)$ need not be drawn since Proposition 1.1 (1),(2) and the remarks preceding Proposition 1.1 may be used to determine the contribution of entire branches of $T(X)$. Also, the elements of X may be initially labeled in a judicious fashion.

Also, for some families of posets, Theorem 2.1 can be applied to determine recursion formulas for the cardinalities of the corresponding lattices of semi-ideals. Example 2.2 is one such, another is given by:

<u>Example 2.3</u>

Let n,m be natural numbers, and let $W_{n,m} = W_n \times \underline{m}$, where W_n is the poset of Example 2.2 . Then

$$|S(W_{o,m})| = |S(W_{n,o})| = 1$$

and for $m,n \geq 1$ the recursion

$$|S(W_{n,m})| = |S(W_{n,m-1})| + \sum_{\substack{i+j=n-1 \\ i \text{ even}}} |S(W_{i,m-1})| \cdot |S(W_{j,m})|$$

holds.

Proof:

Since $W_{n,o} = W_{o,m} = \emptyset$ it is clear that the first statement holds. So let $m,n \geq 1$. Set $X_o = W_{n,m}$, and for $i=1,\dots,n$ define X_i recursively by $X_i = X_{i-1} \setminus \{x_i\}$, where the elements x_i are as indicated in Figure 4.

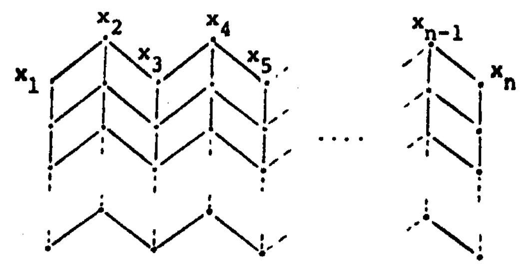

Figure 4

e claim that for even $i < n-2$ the following holds:

1) $|S(X_i)| = |S(X_{i+2})| + |S(W_{n-i-1,m})| \cdot |S(W_{i,m-1})|$.

o prove this claim observe that a two-fold application of

heorem 2.1 yields

2) $|S(X_i)| = |S(X_{i+2})| + |S(X_i \setminus \mathrm{cone}(x_{i+1}))| + |S(X_{i+1} \setminus \mathrm{cone}(x_{i+2}))|$

ow $X_i \setminus \mathrm{cone}(x_{i+1}) \cong W_{i,m-1} + Y_i$, where again by Theorem 2.1

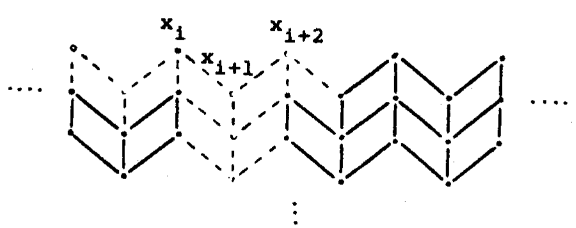

Figure 5

$(Y_i)| = |S(\breve{W}_{n-i-1,m})| - |S(\breve{W}_{n-i-3,m})|$. Thus by Proposition 1.1

),(2) we get

) $|S(X_i \setminus \mathrm{cone}(x_{i+1}))| = |S(W_{i,m-1})| \cdot (|S(W_{n-i-1,m})| - |S(W_{n-i-3,m})|)$

reover, $X_{i+1} \setminus \mathrm{cone}(x_{i+2}) \cong W_{i,m-1} + \breve{W}_{n-i-3,m}$, hence again by

oposition 1.1 (1),(2)

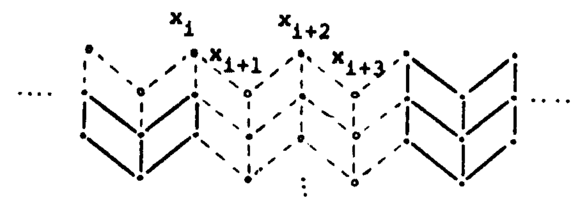

Figure 6

(4) $|S(X_{i+1} \setminus \mathrm{cone}(x_{i+2}))| = |S(W_{i,m-1})| \cdot |S(W_{n-i-3,m})|$.

Thus (2), (3) and (4) together imply the claim.

A similar argument applies to verify (1) in the case $i = n-2$ and n even. Moreover, if n is uneven one easily sees that $|S(X_{n-1})| = |S(X_n)| + |S(W_{n-2,m-1})|$. In any case $|S(X_n)| = |S(W_{n,m-1})|$. Together with (1) this completes the proof.

The numerical values for $|S(W_{n,m})|$ will be listed in Section 4.

In general, however, such a tractable decomposition of a poset will not be obtainable, and a crude paper and pencil calculation of $T(X)$ may become unwieldy. So we have written a computer program which counts the number of terminal nodes of $T(X)$.

It is a backtrack program which when given an anti-chain $A = \{x_{i_1}, \dots, x_{i_k}\} \subseteq X$, finds another $x_i \in X$ for which $A \cup \{x_i\}$ is also an anti-chain and then continues with $A \cup \{x_i\}$ until no more elements can be adjoined. The program then "backtracks" and removes the previously adjoined element and finds another, if possible. Consult [15] and the references therein for a general discussion of backtracking.

The details are as follows. Let $X = \{x_1,\dots,x_n\}$ be an enumeration of X in such a way that $x_i \le x_j$ implies $i \ge j$. Form the n×n-incidence matrix B of X, i.e.

$$B(i,j) = \begin{cases} 1 & \text{if } x_i \le x_j \\ 0 & \text{if } x_i \not\le x_j \end{cases} .$$

Thus B is lower triangular. For i fixed let $C(i)$ be the number of anti-chains containing x_i and not containing any x_j for $j > i$. We obtain $C(i)$ as follows. Let j_1 be the first index $\le i$ such that $B(i,j_1) = 0$. If no such index exists then $C(i) = 1$ (i.e. only the anti-chain $\{x_i\}$ is counted). Form $A(1,k) = \max\{B(i,k),B(j_1,k)\}$ for $1 \le k \le n$. Thus $A(1,k) = 0$ for $k \le j_1$ if and only if $\{x_i,x_{j_1},x_k\}$ is an anti-chain in X. Again let j_2 be the first index $\le j_1$ such that $A(1,j_2) = 0$ and form $A(2,k) = \max\{A(1,k),B(j_2,k)\}$ for $1 \le k \le n$. This procedure is repeated until $A(m,k) = 1$ for all $k \le j_m$. This yields a maximal anti-chain $\{x_i,x_{j_1},\dots,x_{j_m}\}$. Adding 1 to the current value of $C(i)$, which is 0 at the beginning, we backtrack by investigating the remaining values of $A(m-1,j_m+1),\dots,A(m-1,n)$ and repeat the above process. Thus we exhibit all anti-chains of the given kind. This includes $\{x_i\}$ which is obtained in the last step.

Clearly, $|S(X)| = 1 + \sum_{i=1}^{n} C(i)$, where 1 is the contribution of the empty set.

Moreover, storing the row vector $A(m,k)_{k=1}^{n}$ whenever an anti-chain is counted and adding the zero row we represent $S(X)$ as a sublattice of $\underline{2}^n$, which can be used for further computations on $S(X)$.

Section 3. General results

The notion of the incidence algebra has been extensively studied in [19]. We use it as a framework for our investigations.

Let $A(X)$ be the ring defined by

$$A(X) = \{\alpha \mid \alpha : X^2 \longrightarrow \mathbb{Z},\ \alpha(x,y) = 0 \text{ if } x \not\le y\}$$

with pointwise addition and the multiplication given by

$$\alpha\beta(x,y) = \sum_{z \in X} \alpha(x,z)\beta(z,y) .$$

This ring is easily seen to be isomorphic with a subring of the ring of upper triangular $|X|$ by $|X|$ matrices over $\mathbb{Z}$. Let $\pi \in A(X)$ be the incidence function, i.e.

$$\pi(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x \not\le y \end{cases} .$$

π is invertible and its inverse is the so-called Möbius function of the poset X.

The following lemma - already stated in [19] - shows how combinatorial properties of X are related to arithmetical properties of π.

<u>Lemma 3.1</u>

If $x,y \in X$, $x \le y$ then for any natural number $n \ge 1$:
$\pi^n(x,y) = |\mathrm{Hom}(\underline{n-1},[x,y])| = |\mathrm{Hom}_{0,1}(\underline{n+1},[x,y])|$.

This lemma has an immediate corollary, which however could have been proven directly.

Corollary 3.2

For any natural number $n \geq 1$:

$$|S(X \times \underline{n})| = \sum_{I \in \check{S}(X)} |S(I \times \underline{n-1})| \ .$$

Proof:

Consider $A(S(X))$, then by Lemma 3.1 and Proposition 1.1 (3) we have

$$|S(X \times \underline{n})| = \pi^{n+1}(\emptyset, X) = \sum_{I \in \check{S}(X)} \pi^{n}(\emptyset, I)\pi(I,X) =$$

$$= \sum_{I \in \check{S}(X)} |\mathrm{Hom}(\underline{n-1}, S(I))| \cdot 1 = \sum_{I \in \check{S}(X)} |S(I \times \underline{n-1})| \ .$$

We are going to apply Corollary 3.2 in the special case where X is a product of two finite chains, in order to obtain a closed formula for $|S(\underline{k} \times \underline{m} \times \underline{n})|$. This result has already been stated in [2o] .

Theorem 3.3

For natural numbers k,m,n :

$$|S(\underline{k} \times \underline{m} \times \underline{n})| = \prod_{j=o}^{k-1} \frac{\binom{n+m+j}{m}}{\binom{m+j}{m}} \ .$$

Proof:

By Proposition 1.1 (3) $|S(\underline{k} \times \underline{m})| \cong \mathrm{Hom}(\underline{k}, \underline{m+1})$, hence any semi-ideal I of $\underline{k} \times \underline{m}$ corresponds uniquely to a k-tuple $\langle a_1, \dots, a_k \rangle$ with $m \geq a_1 \geq \dots \geq a_k \geq o$. For abbreviation let $B_{k,m}$ be the set of all these k-tuples endowed with the inherited partial order. Moreover, for any k-tuple $\langle a_1, \dots, a_k \rangle \in B_{k,m}$ let $I_{a_1,\dots,a_k}$ be the corresponding ideal. We define a function $f: B_{k,m} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $f(a_1,\dots,a_k,n) = |S(I_{a_1,\dots,a_k} \times \underline{n})|$.

Then $f(a_1,\dots,a_k,0) = 1$ for all $\langle a_1,\dots,a_k \rangle \in B_{k,m}$ and by Corollary 3.2 f satisfies the following recursion (for $n \geq 1$)

$$(1) \qquad f(a_1,\dots,a_k,n) = \sum_{i_k=o}^{a_k} \sum_{i_{k-1}=i_k}^{a_{k-1}} \cdots \sum_{i_1=i_2}^{a_1} f(i_1,\dots,i_k,n-1) \ .$$

We claim:

$$(2) \qquad f(a_1,\dots,a_k,n) = \begin{vmatrix} \binom{a_1+n}{n} & \binom{a_1+n}{n-1} & & \binom{a_1+n}{n-k+1} \\ \binom{a_2+n}{n+1} & \binom{a_2+n}{n} & & \binom{a_2+n}{n-k+2} \\ & & & \\ \binom{a_k+n}{n+k-1} & \binom{a_k+n}{n+k-2} & & \binom{a_k+n}{n} \end{vmatrix} \ .$$

The proof is a straightforward induction using the recursion (1), the alternating multilinearity of the determinant and the following identity for binomial coefficients:

$$\sum_{j=j_1}^{j_2} \binom{b+j}{c} = \binom{b+j_2+1}{c+1} - \binom{b+j_1}{c+1} \ .$$

Since $\underline{k} \times \underline{m} = I_{m,\dots,m}$ we infer that $S(\underline{k} \times \underline{m} \times \underline{n}) = f(m,m,\dots,m,n)$. The theorem is now proven by induction on k.

Obviously $f(m,n) = \binom{m+n}{n} = \prod_{j=o}^{o} \frac{\binom{n+m+j}{m}}{\binom{m+j}{m}}$, which settles the case $k = 1$. Now suppose $k > 1$ and the theorem is proven for k-1. By (2) $f(m,m,\dots,m,n) = \det D$, where $D(i,j) = \binom{m+n}{n+i-j}$.

We form a new matrix D' with

$$D'(i,j) = \begin{cases} D(i,j) & \text{if } j = 1 \\ D(i,j) - D(i,j-1)\frac{D(1,j)}{D(1,j-1)} & \text{if } j \neq 1 \end{cases}$$

Clearly $\det D = \det D'$, and a straightforward verification yields

$$D'(i,j) = \begin{cases} D(i,j) & \text{if } j = 1 \\ \binom{m+n+1}{n+1+i-j} \cdot \frac{i-1}{m-1+i} & \text{if } j \neq 1 \end{cases} .$$

Thus

$$D' = \begin{pmatrix} \binom{n+m}{n} & 0 & 0 & \cdots \\ \binom{n+m}{n+1} & \frac{1}{m+1} \cdot \binom{n+1+m}{n+1} & \frac{1}{m+2} \cdot \binom{n+1+m}{n} & \cdots \\ \binom{n+m}{n+2} & \frac{2}{m+1} \cdot \binom{n+1+m}{n+2} & \frac{2}{m+2} \cdot \binom{n+1+m}{n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

This shows:

$$f(\underbrace{m,m,\ldots,m}_{k\text{-times}},n) = \binom{n+m}{n} \frac{1}{\binom{m+k-1}{k-1}} f(\underbrace{m,m,\ldots,m}_{k-1\text{-times}},n+1),$$

and by the induction hypothesis

$$f(m,m,\ldots,m,n) = \frac{\binom{n+m}{m}}{\binom{m+k-1}{k-1}} \prod_{j=o}^{k-2} \frac{\binom{n+1+m+j}{m}}{\binom{m+j}{m}} = \prod_{j=o}^{k-1} \frac{\binom{n+m+j}{m}}{\binom{m+j}{m}}$$

which was to be proven.

We have not been successful in finding a closed formula for $|S(\underline{j} \times \underline{k} \times \underline{m} \times \underline{n})|$. For $|S(\underline{2} \times \underline{2} \times \underline{2} \times \underline{n})|$ a step by step application of Corollary 3.2 yielded:

$$|S(\underline{2} \times \underline{2} \times \underline{2} \times \underline{n})| = 48\binom{n+8}{8} - 96\binom{n+7}{7} + 63\binom{n+6}{6} - 15\binom{n+5}{5} + \binom{n+4}{4}.$$

This may serve as an indication of the difficulties which arise when trying to generalize Theorem 3.3 to the product of four chains.

Nevertheless, Lemma 3.1 together with Proposition 1.1 (3) may be used to give some numerical results. These will be presented in Section 4. The following observations are also more or less devoted to numerical applications.

If Y is a finite bounded poset let $\alpha_Y \in A(X)$ be defined by

$$\alpha_Y(a,b) = \begin{cases} |\mathrm{Hom}_{o,1}(\check{Y},[a,b])| & \text{if } a \leq b \\ 0 & \text{if } a \nleq b \end{cases} .$$

Note that $\alpha_{\underline{n}} = \pi^{n-1}$, in particular $\alpha_{\underline{2}} = \pi$. As an immediate consequence we have:

<u>Lemma 3.4</u>

$$|S(X \times Y)| = \sum_{I,J \in S(X)} \alpha_Y(I,J) .$$

Proof:

Follows directly from Proposition 1.1 (3) .

If Y and Z are finite bounded posets, then $Y \dotplus Z$ denotes their order sum with 1_Y and 0_Z identified. Moreover, let $Y * Z$ be their coproduct in the category of bounded posets, i.e. $Y * Z$ is the cardinal sum of Y and Z with $0_Y, 0_Z$ resp. $1_Y, 1_Z$ identified.

<u>Proposition 3.5</u>

In the ring A(X), $\alpha_{Y \dotplus Z} = \alpha_Z \cdot \alpha_Y$.

Proof:

Since $(Y \dotplus Z)\check{} \simeq \check{Z} \dotplus \check{Y}$ we have that for each pair $a,b \in X$ with $a \leq b$:

$$\mathrm{Hom}_{o,1}((Y \dotplus Z)\check{},[a,b]) = \bigcup_{a \leq c \leq b} (\mathrm{Hom}_{o,1}(\check{Z},[a,c]) \times \mathrm{Hom}_{o,1}(\check{Y},[c,b])).$$

This union is clearly disjoint, hence

$$\alpha_{Y \dotplus Z}(a,b) = \sum_{a \leq c \leq b} \alpha_Z(a,c) \cdot \alpha_Y(c,b) = \alpha_Z \cdot \alpha_Y(a,b) .$$

This Proposition admits a Corollary which generalizes Corollary 3.2 .

Corollary 3.6

$$|S(X \times (\underline{2} + Z))| = \sum_{I \in S(X)} |S(I \times Z)| .$$

Proof:

By Lemma 3.4 and Proposition 3.5 we have

$$|S(X \times (\underline{2} + Z))| = \sum_{I,J \in S(X)} \alpha_{\underline{2}+Z}(I,J) = \sum_{I,J \in S(X)} \alpha_Z \cdot \alpha_{\underline{2}}(I,J) =$$

$$= \sum_{I,J \in S(X)} \sum_{K \in S(X)} \alpha_Z(I,K) \cdot \pi(K,J) = \sum_{I,J \in S(X)} \sum_{K \leq J} \alpha_Z(I,K) =$$

$$= \sum_{J \in S(X)} \sum_{I,K \leq J} \alpha_Z(I,K) = \sum_{J \in S(X)} |S(J \times Z)|.$$

Concerning $Y * Z$ we get:

Proposition 3.7

For all $a,b \in X$ $\quad \alpha_{Y * Z}(a,b) = \alpha_Y(a,b) \cdot \alpha_Z(a,b)$.

Proof:

It is clear that $\check{Y * Z} \simeq \check{Y} * \check{Z}$, and thus for $a \leq b$
$Hom_{o,1}(\check{Y * Z},[a,b]) \simeq Hom_{o,1}(\check{Y},[a,b]) \times Hom_{o,1}(\check{Z},[a,b])$.
If $a \nleq b$ then the statement is trivial.

As a typical application of Proposition 3.7 let us consider M_n, the finite modular lattice of length 2 with n atoms. Obviously M_n is the coproduct of n copies of $\underline{3}$ in the category of bounded posets. Thus we have:

Corollary 3.8

$$|S(X \times M_n)| = \sum_{I,J \in S(X)} (\pi^2(I,J))^n .$$

Section 4. Numerical results

4.1 A natural candidate for application of the backtrack program of Section 2 is the free bounded distributive lattice on n generators, $FD_{o,1}(n)$. By Proposition 1.1 (4) $FD_{o,1}(n) \simeq S(\underline{2}^n)$. The program was run for each poset $\underline{2}^n$, $n \leq 6$, and the results are listed below. The n=6 case took 5 minutes CPU time on an IBM 37o. These agree with other published results, e.g. [6],[22]. Another approach to enumerating $|FD_{o,1}(n)|$ is found in [18], while asymptotic results are given in [14] and [23].

n	1	2	3	4	5	6
$\lvert FD_{o,1}(n)\rvert$	3	6	2o	168	7581	7828354

For $|FD_{o,1}(7)|$ there are conflicting results in the literature; [7] and [17]. Moreover, the magnitude is too great to allow for a direct count of the elements. On the basis of Corollary 1.2 (2) and the complete information on $FD_{o,1}(5)$ obtained by the algorithm we found $|FD_{o,1}(7)|$ = 2 414 682 o4o 998 . This confirms the result of [7]. The computation was based on the following consequence of Corollary 1.2 (2):

$$|FD_{o,1}(n+2)| = \sum_{x,y \in FD_{o,1}(n)} ~~|(x]| \cdot |[y)|~~ \quad |(x \wedge y]| \cdot |[x \vee y)|$$

This computation was done in collaboration with Alan J. Burger [2].

4.2 Let B_ω be the variety of pseudocomplemented distributive lattices. For a general discussion of B_ω consult [1] or [11]. The free algebra $FB_\omega(n)$ is finite, for n finite. The poset $J(FB_\omega(n))$ has been characterized in [3], [16], [21] .

Using the algorithm on these posets gives the following:

n	o	1	2
$\lvert J(FB_\omega(n))\rvert$	1	4	22
$\lvert FB_\omega(n)\rvert$	2	7	626

It can be argued that $\lvert FB_\omega(3)\rvert \geq \lvert FD_{o,1}(8)\rvert$, and therefore is beyond calculation at this moment, since it is known that $\lvert FD_{o,1}(8)\rvert \geq 2^{7o}$.

4.3 The class of de Morgan algebras is studied, for example, in [4], [8] and [12]. There are three nontrivial varieties of de Morgan algebras: Boolean algebras, Kleene algebras and the entire variety of de Morgan algebras. The free Boolean algebra on n generators has cardinality 2^{2^n}. The free de Morgan algebra on n generators is lattice-isomorphic to $FD_{o,1}(2n)$. The free Kleene algebra on n generators, $FM_1(n)$, is described in [4], where it is shown that

$$J(FM_1(n)) \cong \{\langle X,Y\rangle \mid X,Y \subseteq \underline{n},\ X\cap Y = \emptyset \ \text{ or } \ X\cup Y = \underline{n}\}$$

whence it follows that $\lvert J(FM_1(n))\rvert = 2\cdot 3^n - 2^n$. Using this characterization of $J(FM_1(n))$ the following was obtained:

n	o	1	2	3
$\lvert FM_1(n)\rvert$	2	6	84	43918

4.4 In Example 2.3 a recursion formula for $\lvert S(W_{m,n})\rvert$ was found. The numerical values for $m \leq 1o$, $n \leq 7$ are as follows:

$\lvert S(W_{m,n})\rvert$:

m \ n	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8
2	3	6	1o	15	21	28	36
3	5	14	3o	55	91	14o	2o4
4	8	31	85	19o	371	658	1o86
5	13	7o	246	671	1547	3164	5916
6	21	157	7o7	2353	64o5	151o6	31998
7	34	353	2o37	8272	26585	723o2	1735o2
8	55	793	5864	29o56	11o254	345775	94ooo5
9	89	1782	16886	1o2o91	457379	1654o92	5o9422o
1o	144	4oo4	4862o	358671	1897214	791197o	276o4798

4.5 With the help of the results of Section 3 we obtained the following:

n	$\lvert S(\underline{2}\times\underline{2}\times\underline{3}\times\underline{n})\rvert$	$\lvert S(\underline{2}\times\underline{2}\times\underline{4}\times\underline{n})\rvert$	$\lvert S(\underline{2}\times\underline{2}\times\underline{5}\times\underline{n})\rvert$	$\lvert S(\underline{2}\times\underline{2}\times\underline{2}\times\underline{n})\rvert$
1	5o	1o5	196	168
2	887	349o	11196	7581
3	879o	59542	3o796o	16o948
4	59542	65o644	5157o98	2o68224
5	3o796o	5157o98	6o112692	18561984
6	13o161o	32o46856	53o962446	127234oo8
7	47o1698	164489o84	376472734o	7o6987164
8	14975675	7235o9159	22326282261	332o153661
9	43o25762	28o1747767	11415849o576	13583619496
1o	113414717	9748942554	515o632388lo	4953oo7o161

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