Maximal n-generated subdirect products

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Abstract

For n a positive integer and K a finite set of finite algebras, let $\mathbf{L}(n, K)$ denote the largest n-generated subdirect product whose subdirect factors are algebras in K. When K is the set of all n-generated subdirectly irreducible algebras in a locally finite variety $\mathcal{V}$, then $\mathbf{L}(n, K)$ is the free algebra $\mathbf{F}_{\mathcal{V}}(n)$ on n free generators for $\mathcal{V}$. For a finite algebra $\mathbf{A}$ the algebra $\mathbf{L}(n, \{\mathbf{A}\})$ is the largest n-generated subdirect power of $\mathbf{A}$.

For every n and finite $\mathbf{A}$ we provide an upper bound on the cardinality of $\mathbf{L}(n, \{\mathbf{A}\})$. This upper bound depends only on n and these basic parameters: the cardinality of the automorphism group of $\mathbf{A}$, the cardinalities of the subalgebras of $\mathbf{A}$, and the cardinalities of the equivalence classes of certain equivalence relations arising from congruence relations of $\mathbf{A}$. Using this upper bound on n-generated subdirect powers of $\mathbf{A}$, as $\mathbf{A}$ ranges over the n-generated subdirectly irreducible algebras in $\mathcal{V}$, we obtain an upper bound on $|\mathbf{F}_{\mathcal{V}}(n)|$. And if all the n-generated subdirectly irreducible algebras in $\mathcal{V}$ have congruence lattices that are chains, then we characterize in several ways those $\mathcal{V}$ for which this upper bound is obtained.

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1 Introduction

An algebra $\mathbf{B}$ that is a subalgebra of a direct product $\prod_{i\in I}\mathbf{A}_i$ is a *subdirect product* of the $\mathbf{A}_i$ if each coordinate projection pr_i maps $\mathbf{B}$ onto $\mathbf{A}_i$. Each $\mathbf{A}_i$ is called a *subdirect factor* of $\mathbf{B}$. If $\mathbf{B}$ is n-generated, then so is each of its subdirect factors. For a given set K of algebras of the same similarity type what is the largest size an n-generated algebra can be if it is a subdirect product whose subdirect factors are algebras taken from the set K? This paper investigates this question when K is a finite set of finite algebras.

Definition 1.1. *For n a positive integer and K a finite set of finite algebras of the same similarity type, $\mathbf{L}(n, K)$ denotes the largest n-generated subdirect product whose subdirect factors are algebras in K.*

If no algebra in K is n-generated or if K is empty, then $\mathbf{L}(n, K)$ is the 1-element algebra having the same similarity type as the algebras in K. For $\mathbf{A}$ a finite algebra, $\mathbf{L}(n, \mathbf{A})$ denotes $\mathbf{L}(n, \{\mathbf{A}\})$. The algebra $\mathbf{L}(n, \mathbf{A})$ is the largest n-generated subdirect power of $\mathbf{A}$.

An easy argument shows that in Definition 1.1, up to isomorphism, the algebra $\mathbf{L}(n, K)$ is unique.

If $\mathcal{V}$ is the variety generated by a finite set K of finite algebras of the same similarity type, then $|\mathbf{L}(n, K)| \leq |\mathbf{F}_{\mathcal{V}}(n)|$ holds for all integers n. Here $\mathbf{F}_{\mathcal{V}}(n)$ denotes the free algebra for $\mathcal{V}$ on n-free generators. However, when K is the set of all n-generated subdirectly irreducible algebras in a locally finite variety $\mathcal{W}$, then $|\mathbf{L}(n, K)| = |\mathbf{F}_{\mathcal{W}}(n)|$.

There are several classic results that have been the impetus for this paper. I group them together in the following omnibus theorem:

Primal Cluster Theorem. *Let $K = \{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ be a finite set of finite, nontrivial, pairwise nonisomorphic algebras of the same similarity type and let $\mathcal{V}$ be the variety of algebras generated by K. Then for all $n \geq 0$:*

$$|\mathbf{L}(n, K)| \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq \prod_{i=1}^{k} |A_i|^{|A_i|^n} \quad \text{[Birkhoff]}. \tag{1.1}$$

Moreover, the following are equivalent.

1. **(bound obtained):** $|\mathbf{L}(n, K)| = \prod_{i=1}^{k} |A_i|^{|A_i|^n}$ *for all $n \geq 0$.*

2. **(interpolation condition):** *For every $n \geq 0$ and for every $1 \leq i \leq k$, if given an n-ary operation $f_i\colon A_i^n \to A_i$, then there exists a term $t(x_1, \ldots, x_n)$ in the language of K for which $t^{\mathbf{A}_i} = f_i$ for all i.* [Foster]

3. **(algebraic condition):** *Each* $\mathbf{A}_i$ *is simple, has no proper subalgebras, has no proper automorphisms, and the variety generated by* K *is congruence permutable and congruence distributive.* [Pixley]

4. **(computational condition):** $|\mathbf{L}(m, K)| = \prod_{i=1}^{k} |A_i|^{|A_i|^m}$ *for* $m = \max\{2, k\}$*.* [Sioson, Sierpinski]

The inequalities displayed in (1.1) are from G. Birkhoff's classic theorem of 1935 that bounds the size of a finitely generated algebra. The notion of a set K of algebras being a primal cluster of algebras was introduced by A. F. Foster in a 1955 paper. The interpolation condition (2) is essentially his definition of a primal cluster when K consists of a finite set of finite algebras. That the interpolation condition implies the bound obtained condition is due to Foster.

In this Primal Cluster Theorem if $k = 1$ and $K = \{\mathbf{A}\}$, then the upper bound on $|\mathbf{L}(n, \mathbf{A})|$ is $|A|^{|A|^n}$ and if this bound is obtained the algebra is known as a *primal algebra*. For a primal algebra the interpolation condition is the statement that every n-ary operation on A is a term operation of $\mathbf{A}$. The algebraic condition may be phrased in a different terminology as stating that the algebra $\mathbf{A}$ has no proper subalgebras, is simple, rigid, and the variety generated by $\mathbf{A}$ is arithmetical. Note that a variety is $\mathcal{V}$ is arithmetical if and only if it has a Pixley term [14], i.e., a ternary term $t(x, y, z)$ such that $\mathcal{V} \models t(x, y, x) \approx t(x, y, y) \approx t(y, y, x) \approx x$. The computational condition for a primal algebra is essentially Sierpinski's theorem [16] that for every n every n-ary operation on A can be obtained as a composition of binary operations. We use the word *computational* here since by computing $|\mathbf{L}(2, \mathbf{A})|$ one can determine whether or not $\mathbf{A}$ is a primal algebra.

The initial motivation for this paper was to investigate how in Pixley's characterization of a primal algebra $\mathbf{A}$, the severe restrictions on the subalgebras, congruences, and automorphisms might be relaxed with a corresponding modification of the upper bound on the cardinality of $\mathbf{L}(n, \mathbf{A})$. This led to an extension of the Primal Cluster Theorem that takes into account basic numerical parameters based on the structure of the subalgebras, congruence relations, and automorphisms of the algebras in the class K. The parameters that are used in this upper bound place no restriction on the subalgebras or automorphisms of the algebras in K but do require that the congruence lattice $\operatorname{Con} \mathbf{A}$ of every $\mathbf{A} \in K$ is linearly ordered. Corollaries 3.10 and 3.11 present this upper bound. Theorems 7.1 and 9.2 are two extensions of the Primal Cluster Theorem given in this paper.

A central notion used throughout is a *valuation*, which is a function v from a finite set X of variables into an algebra $\mathbf{A}$ such that $v(X)$ generates all of $\mathbf{A}$. Section 2 contains notation and basic facts about valuations and presents the correspondence between a set U of valuations on $\mathbf{A}$ and an $|X|$-generated subdirect power of $\mathbf{A}$ embedded in $\mathbf{A}^U$, which is denoted $\mathbf{Ge}(X, U)$.

If U is a set of valuations from X to a finite algebra $\mathbf{A}$, and if C is a linearly ordered subset of the congruence lattice of $\mathbf{A}$, then in section 3 we define a subuniverse of $\mathbf{A}^U$, denoted $\Omega(\mathbf{A}, C, U)$, the cardinality of which is an upper bound for the size of $\mathbf{Ge}(X, U)$ for any algebra $\mathbf{A}'$ that has universe A, a congruence lattice that contains C, and the set U is contained in the set of valuations from X to $\mathbf{A}'$.

Sections 4, 5, and 6 present the interpolation, algebraic, and computational conditions that appear in our version of the Primal Cluster Theorem. Section 7 combines these results to state and prove this theorem. In section 8 we illustrate with a concrete example how our main theorem may be used in various ways and how a software package such as the Universal Algebra Calculator [10] can be utilized interactively with this theorem. The final section contains several applications of the main theorem.

The notation and terminology follows that of [3] and [12]. I am indebted to Alden Pixley for much helpful correspondence and conversation concerning this paper. I acknowledge with thanks the many useful comments and suggestions of an anonymous referee.

2 Valuations and subdirect representations

Given an algebra $\mathbf{S}$ generated by a set X and a subdirect product representation of $\mathbf{S}$ by algebras $(\mathbf{A}_i \mid i \in I)$ we view $\mathbf{S}$ as a subalgebra of $\prod_{i\in I} \mathbf{A}_i$. If $pr_i\colon \mathbf{S} \to \mathbf{A}_i$ is the projection map on coordinate i, then the function $v_i\colon X \to A_i$ defined by $v_i(x) = pr_i(x)$ for all $x \in X$ has the property that the set $v_i(X)$ generates all of $\mathbf{A}_i$. Thus every subdirect representation of $\mathbf{S}$ has a set of functions $v_i\colon X \to A_i$ associated with it, each with the property that $v_i(X)$ generates the algebra $\mathbf{A}_i$. This set of functions codes the subdirect representation. Functions of this kind play a central role in this paper.

Definition 2.1. *Let X be a set, $\mathbf{A}$ an algebra with universe A, and $u \in \mathbf{A}^X$.*

1. *The subalgebra of $\mathbf{A}$ generated by $u(X)$ is denoted $\mathbf{Alg}(u)$.*

2. *If $u(X)$ generates $\mathbf{A}$, then u is called a* valuation of X into $\mathbf{A}$. *The set of all valuations of X into $\mathbf{A}$ is denoted* $\mathrm{val}(X, \mathbf{A})$. *For K a class of algebras,* $\mathrm{val}(X, K)$ *denotes the collection of all $u \in \mathrm{val}(X, \mathbf{A})$ for all $\mathbf{A} \in K$.*

3. *If no set of size $|X|$ generates $\mathbf{A}$, then $\mathrm{val}(X, \mathbf{A}) = \emptyset$. If $\mathbf{A}$ and $\mathbf{B}$ are distinct algebras, then $\mathrm{val}(X, \mathbf{A}) \cap \mathrm{val}(X, \mathbf{B}) = \emptyset$.*

4. *For $u, v \in \mathrm{val}(X, \mathbf{A})$, $\overline{uv}$ denotes $\{(u(x), v(x)) \mid x \in X\}$.*
   We write $\mathbf{B} \lhd \mathbf{A}$ to denote that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$.

This definition of a valuation is adapted from the Foreward on Algebra in the 1948 edition of G. Birkhoff's *Lattice Theory* in which there is a *constructive* definition of the free algebra generated by a set X associated with an algebra $\mathbf{A}$. Here an arbitrary function $u \colon X \to A$ is called a "valuation". In our definition of a valuation we require that the set $u(X)$ also generates the algebra $\mathbf{A}$.

Conversely, if we start with a set U of valuations, then we may associate with U an algebra and a particular subdirect representation of this algebra as in Definition 2.2 below. We view a set U of valuations as a particularly efficient way to represent and to analyze a subdirect representation of an algebra. In what follows we present a calculus for working with valuations.

Definition 2.2. *Let K be a set of algebras of the same similarity type, X a nonempty set, and U a nonempty subset of* $\mathrm{val}(X, K)$. *For $x \in X$, let $x^* \in \prod_{v \in U} \mathbf{Alg}(v)$ denote the element given by $pr_v(x^*) = v(x)$ for all $v \in U$. We let $X^* = \{x^* \mid x \in X\}$.*

$\mathbf{Ge}(X, U)$ *denotes the subalgebra of $\prod_{v \in U} \mathbf{Alg}(v)$ generated by X^* and* $\mathrm{Ge}(X, U)$ *denotes the universe of $\mathbf{Ge}(X, U)$.*

If $U = \emptyset$, then $\mathbf{Ge}(X, U)$ is the 1-element algebra in the similarity type of K.

Since $v \in U$ in this definition is a valuation, it follows that $\mathbf{Ge}(X, U)$ is subdirectly embedded in $\prod_{v \in U} \mathbf{Alg}(v)$. Thus, for n a positive integer and K a set of algebras containing at least one algebra that is n-generated, the algebra $\mathbf{L}(n, K)$ may be viewed as subdirectly embedded in $\prod_{v \in U} \mathbf{Alg}(v)$ for $U = \mathrm{val}(X, K)$ with $|X| = n$. That is, $\mathbf{L}(n, K) \cong \mathbf{Ge}(X, \mathrm{val}(X, K))$. If no algebra in K is n-generated, then $\mathbf{L}(n, K)$ and $\mathbf{Ge}(X, U)$ are both the 1-element algebra in the similarity type of K.

In what follows we investigate $\mathbf{L}(n, K)$ by considering sets $U \subseteq \mathrm{val}(X, K)$ for which $\mathbf{L}(n, K)$ is isomorphic to $\mathbf{Ge}(X, U)$. To this end we present some definitions and notation.

Definition 2.3. *Let Z be a nonvoid set.*

1. *A* transversal *of an equivalence relation $\equiv$ on Z is any subset of Z consisting of precisely one element from every equivalence class of $\equiv$.*

2. *A* quasi-order *(also called a* preorder*) on Z is any binary relation $\preceq$ on Z that is reflexive and transitive.*

3. *Given a quasi-order $\preceq$ on Z, an equivalence relation $\sim$ is defined on Z by $y \sim z$ if and only if $y \preceq z$ and $z \preceq y$.*
   A partial order $\leq$ associated with $\preceq$ is defined on the set of equivalence classes of $\sim$ by $y/\sim\ \leq\ z/\sim$ if and only if $y \preceq z$.

4. *A* minimal transversal *of a quasi-order $(Z, \preceq)$ is any set $T \subseteq Z$ such that*
   (i) for every $z \in Z$ there exists $t \in T$ such that $t \preceq z$ and
   (ii) for every $t, t' \in T$, if $t \preceq t'$, then $t = t'$.

Thus, a minimal transversal of $(Z, \preceq)$ is obtained by choosing one element from each equivalence class of $\sim$ that is a minimal element in the partial order $\leq$ on $\sim$ equivalence classes.

Definition 2.4. *Let K be a set of algebras in a variety $\mathcal{V}$ and let $U \subseteq \mathrm{val}(X, K)$.*

1. *The quasi-order $\preceq$ is defined on U by $v \preceq w$ if and only if there exists a homomorphism $h \colon \mathbf{Alg}(v) \to \mathbf{Alg}(w)$ such that $h(v(x)) = w(x)$ for all $x \in X$.*

2. *Two valuations v and w in U are called* equivalent*, denoted $v \sim w$, if there exists an isomorphism h of $\mathbf{Alg}(v)$ onto $\mathbf{Alg}(w)$ such that $h(v(x)) = w(x)$ for all $x \in X$.*

3. *The set U is called* $\sim$-independent *if whenever $v, w \in U$ and $v \sim w$, then $v = w$. A set $V \subseteq \mathrm{val}(X, K)$ is* $\sim$-equivalent to *U, denoted $U \sim V$, if there exists a bijection b between U and V such that $b(u) \sim u$ for all $u \in U$.*

We note that if a homomorphism h from $\mathbf{Alg}(u)$ to $\mathbf{Alg}(w)$ is such that $h(u(x)) = w(x)$ for all $x \in X$, then h is necessarily onto since u and w are valuations. Thus, if u and w are valuations with $\mathbf{Alg}(u)$ finite and if $u \preceq w$, and $w \preceq u$, then $u \sim w$. Likewise, if $\mathbf{A}$ is finite and $u, w \in \mathrm{val}(X, \mathbf{A})$ with $u \preceq w$, then $u \sim w$. That is, for a finite algebra $\mathbf{A}$ the quasi-order $\preceq$ and the equivalence relation $\sim$ on $\mathrm{val}(X, \mathbf{A})$ are the same.

The next lemma follows easily from Definition 2.4

Lemma 2.5. *Suppose that K is a set of finite algebras in a variety $\mathcal{V}$ and that $U \subseteq \mathrm{val}(X, K)$. Let $V \subseteq U$ be a minimal transversal of the quasi-order $(U, \preceq)$. Then for every $W \subseteq U$ we have that W is $\sim$-equivalent to V if and only if W is a minimal transversal of $(U, \preceq)$.*

Lemma 2.6. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras and $X = \{x_1, \dots, x_n\}$.*

1. *Suppose $u \colon X \to A$ is a function. Then $|\mathrm{Sg}^{\mathbf{A}}(u(X))| = |A|$ if and only if $u \in \mathrm{val}(X, \mathbf{A})$.*
2. *Suppose $v \in \mathrm{val}(X, \mathbf{A})$ and $w \in \mathrm{val}(X, \mathbf{B})$. Then $|\mathrm{Sg}^{\mathbf{A}\times\mathbf{B}}(\overline{vw})| = |\mathbf{A}|$ if and only if there exists a homomorphism h from $\mathbf{A}$ to $\mathbf{B}$ such that $h(v) = w$ if and only if $v \preceq w$.*

Proof. The first item is immediate. For the second, if $|\mathrm{Sg}^{\mathbf{A}\times\mathbf{B}}(\overline{vw})| = |\mathbf{A}|$, then $|A| \geq |B|$ and the relation $\mathrm{Sg}^{\mathbf{A}\times\mathbf{B}}(\overline{vw})$ is functional. Since this relation is a subalgebra of $\mathbf{A} \times \mathbf{B}$, it is also the graph of a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. $\square$

Definition 2.7. *Let S, Y, and Z be sets with $S \subseteq Y$. Suppose $f \in Z^Y$.*

1. *We denote the projection of f on S by $f|_S$. When $f|_S$ viewed as a function we have $f|_S \colon S \to Z$.*
   If $P \subseteq Z^Y$, then $P|_S$ denotes $\{f|_S \mid f \in P\}$.
2. *For $y \in Y$ and depending on the context we write $f|_y$ or $f(y)$ for $f|_{\{y\}}$.*
3. *If θ is an equivalence relation on Z, then f is called θ-*constant on *S if $f(S)$ is a subset of a θ-class.*

Lemma 2.8. *Let K be a finite set of finite algebras with $U \subseteq U' \subseteq \mathrm{val}(X, K)$. If for each $w \in U'$ there is a $\gamma(w) \in U$ such that $\gamma(w) \preceq w$, then $\mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U')$.*

Proof. Let W denote $U' \setminus U$. We may assume W is nonvoid. By hypothesis there is a function $\gamma\colon W \to U$ such that $\gamma(w) \preceq w$ in the quasi-order of Definition 2.4. Let $h_w\colon \mathbf{Alg}(\gamma(w)) \to \mathbf{Alg}(w)$ be a homomorphism for which $h_w\,(\gamma(w)(x)) = w(x)$ for all $x \in X$. Define $h\colon \mathbf{Ge}(X, U) \to \mathbf{Ge}(X, U')$ for $f \in \mathbf{Ge}(X, U)$ by

$$h(f)(v) = \begin{cases} f(v) & \text{if } v \in U; \\ h_v(f(\gamma(v))) & \text{if } v \in W. \end{cases}$$

Then h is clearly a homomorphism since each h_w is. The map h is onto because it maps the generating set X^* of $\mathbf{Ge}(X, U)$ onto the corresponding generating set of $\mathbf{Ge}(X, U')$. Also, h is one-to-one since $\mathbf{Ge}(X, U)$ is isomorphic to $G(X, U')|_U$. □

Corollary 2.9. *Let K be a finite set of finite algebras with $U' \subseteq \mathrm{val}(X, K)$. If U is a minimal transversal of $(U', \preceq)$, then $\mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U')$.*

Proof. Since $U = \emptyset$ if and only if $U' = \emptyset$ we may assume both U and U' are nonempty. By hypothesis, for every $u \in U'$ there is a $w \in U$ such that $w \preceq u$. Hence the claim follows from Lemma 2.8. □

We note that the reduction of the set U' to the smaller set U in Corollary 2.9 is implemented in the current version of the Universal Algebra Calculator (UACalc) software developed by Ralph Freese, Emil Kiss, and Matt Valeriote [10]. This version contains an efficient algorithm that may be used for finding a subset U of U' such that for all $u' \in U'$ there is a $u \in U$ such that $u \preceq u'$. The set U is called a *thin set* of coordinates in this software. When the program computes the subalgebra of an m-th direct power generated by a set of m-tuples, there is an option to reduce m coordinates to as many as there are in a thin set, and thereby speed up the computation.

The paper [4] includes descriptions of various methods for computing the exact value of $|\mathrm{val}(X, \mathbf{B})|$ for $\mathbf{B}$ a finite algebra. These methods only involve numerical parameters determined by the subalgebras of $\mathbf{B}$.

We now have our first upper bound on the cardinality of an n-generated subdirect product.

Corollary 2.10. *Suppose $\mathcal{V}$ is a locally finite variety and $X = \{x_1, \ldots, x_n\}$ with $W \subseteq \mathrm{val}(X, \mathcal{V})$. Let $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$ be a transversal with respect to isomorphism of $\{\mathbf{Alg}(u) \mid u \in W\}$. Let U be a minimal transversal of $(W, \preceq)$.*

For each $1 \leq i \leq k$, let $U_i = \{u \in U \mid \mathbf{Alg}(u) = \mathbf{B}_i\}$. Then

$$\mathbf{Ge}(X, W) \cong \mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U_1 \cup \cdots \cup U_k) \lhd \prod_{i=1}^{k} \mathbf{Ge}(X, U_i)$$

and thus

$$|\mathbf{Ge}(X, W)| = |\mathbf{Ge}(X, U)| \leq \prod_{i=1}^{k} |B_i|^{|U_i|}. \tag{2.1}$$

Proof. Any transversal with respect to $\cong$ of the n-generated algebras in $\mathcal{V}$ is finite because the variety $\mathcal{V}$ is locally finite. Since $W = \emptyset$ if and only if $U = \emptyset$ if and only if all the $U_i = \emptyset$ we may assume that W, U and at least one U_i are nonempty. The $\mathbf{B}_i$ are pairwise distinct by so the U_i are pairwise disjoint by Definition 2.1(3). The isomorphism between $\mathbf{Ge}(X, W)$ and $\mathbf{Ge}(X, U)$ is Lemma 2.8 and the other displayed claims are immediate. □

If in Corollary 2.10 there are no onto homomorphisms from $\mathbf{B}_i$ to $\mathbf{B}_j$, for all $1 \leq i < j \leq k$, then the quasi-order $\preceq$ on W reduces to the equivalence relation $\sim$. In this situation, each U_i is a transversal with respect to $\sim$ of $W \cap \mathrm{val}(X, \mathbf{B}_i)$.

Remark 2.11. Lemma 2.8 and Corollary 2.10 provide an upper bound for $|\mathbf{L}(n, K)|$ when K is a finite set of finite algebras. Let $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$ be a transversal with respect to isomorphism of K, and let U be a minimal transversal of the quasi-order $\preceq$ on $\mathrm{val}(X, \{\mathbf{B}_1, \ldots, \mathbf{B}_k\})$. Then as in Corollary 2.10 we have

$$\mathbf{L}(n, K) \cong \mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, \bigcup_{i=1}^{k} U_i)$$

and

$$|\mathbf{L}(n, K)| \leq \prod_{i=1}^{k} |\mathbf{Ge}(X, U_i)| \leq \prod_{i=1}^{k} |B_i|^{|U_i|},$$

where $U_i = \{v \in U \mid \mathbf{Alg}(v) = \mathbf{B}_i\}$.

If the inequality of (2.1) in Corollary 2.10 is an equality, then for every $v \neq w \in U$, the projection of $\mathbf{Ge}(X, U)$ on $\{v, w\}$ is $\mathbf{Alg}(v) \times \mathbf{Alg}(w)$. This projection is also the subalgebra of $\mathbf{Alg}(v) \times \mathbf{Alg}(w)$ generated by $\{(v(x), w(x)) \mid x \in X\}$, i.e., $\mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw})$. Hence, if equality were to hold and if $v \neq w$ with $\mathbf{Alg}(v) = \mathbf{Alg}(w)$, then $\mathrm{Sg}^{\mathbf{Alg}(v)^2}(\overline{vw}) = \mathrm{Cg}^{\mathbf{Alg}(v)}(\overline{vw}) = \mathbf{Alg}(v)^2$.

3 Bounds incorporating congruence relations

We consider a finite algebra $\mathbf{A}$ and U an arbitrary nonvoid subset of $\mathrm{val}(X, \mathbf{A})$ with $X = \{x_1, \ldots, x_n\}$. Suppose $m \geq 0$ and $\mathrm{Con}\, \mathbf{A}$ contains the congruence relations

$$1_A = \theta_0 > \theta_1 > \cdots > \theta_m > \theta_{m+1} = 0_A.$$

Let C denote this chain of $m+2$ congruence relations $\{\theta_0, \ldots, \theta_\ell, \ldots, \theta_{m+1}\}$ in $\mathrm{Con}\, \mathbf{A}$.

We label the congruence classes of the θ_ℓ by means of strings of positive integers. For $\alpha = i_1 i_2 \ldots i_\ell$ a string of ℓ integers, we write $\lambda(\alpha)$ to denote the length of α, i.e., $\lambda(\alpha) = \ell$. The empty string is denoted ϵ. Each congruence class of θ_ℓ is denoted $A(i_1 i_2 \ldots i_\ell)$ where the $A(i_1 i_2 \ldots i_\ell)$ are defined inductively as follows. For $\ell = 0$, $A(\epsilon) = A$, which is the only congruence class of θ_0. For $\ell = 1$, the congruence classes of θ_1 are $A(1), A(2), \ldots, A(r(\epsilon))$ where $r(\epsilon)$ is the number of congruence classes of θ_1 in $A(\epsilon)$. In general, for $\ell > 0$, if $A(i_1 \ldots i_{\ell-1})$ is a congruence class of $\theta_{\ell-1}$, then the congruence classes of θ_ℓ contained in $A(i_1 \ldots i_{\ell-1})$ are

$$A(i_1 \ldots i_{\ell-1} 1),\ A(i_1 \ldots i_{\ell-1} 2),\ \ldots,\ A(i_1 \ldots i_{\ell-1} r(i_1 \ldots i_{\ell-1})),$$

where $r(i_1 \ldots i_{\ell-1})$ denotes the number of θ_ℓ congruence classes contained in $A(i_1 \ldots i_{\ell-1})$. That is, $r(i_1 \ldots i_{\ell-1})$ is the index of θ_ℓ restricted to $A(i_1 \ldots i_{\ell-1})$. Note that under this notation, if a is an arbitrary element of A, then there is a unique string α, with $\lambda(\alpha) = m+1$, such that $\{a\} = A(\alpha)$. Moreover, if $\alpha = i_1 \ldots i_{m+1}$, then for every $1 \leq \ell \leq m+1$ we have $1 \leq i_\ell \leq r(i_1 \ldots i_{\ell-1})$. Also, for every $0 \leq \ell \leq m$ and each θ_ℓ congruence class $A(i_1 \ldots i_\ell)$

$$A(i_1 \ldots i_\ell) = \bigcup_{i_{\ell+1}=1}^{r(i_1 \ldots i_\ell)} A(i_1 \ldots i_\ell\, i_{\ell+1}).$$

For each congruence relation θ_ℓ we define an equivalence relation $\equiv_\ell$ on the nonvoid set U of valuations by

$$v \equiv_\ell w \text{ if and only if } (v(x), w(x)) \in \theta_\ell \text{ for all } x \in X.$$

Thus,

$$1_U = \equiv_0\ \geq\ \equiv_1\ \geq\ \cdots \geq\ \equiv_m\ \geq\ \equiv_{m+1}\ =\ 0_U.$$

Note that $v \equiv_\ell w$ if an only if $\mathrm{Cg}^{\mathbf{A}}(\overline{vw}) \leq \theta_\ell$. The equivalence classes of the $\equiv_\ell$ are labelled analogously to the labelling for the congruence classes of the

θ_ℓ: $U = U(\epsilon)$ and $s(\epsilon)$ is the number of equivalence classes of $\equiv_1$ in $U(\epsilon)$. For $\ell > 0$ the equivalence classes of $\equiv_\ell$ are $U(\beta)$ where β is the string $j_1 \dots j_\ell$ with $1 \le j_\ell \le s(j_1 \dots j_{l-1})$ and where $s(j_1 \dots j_{l-1})$ denotes the number of $\equiv_\ell$ classes contained in $U(j_1 \dots j_{l-1})$.

With this notation, for every $u \in U$ there is a unique string $j_1 \dots j_{m+1}$ for which $\{u\} = U(j_1 \dots j_{m+1})$ and $1 \le j_\ell \le s(j_1 \dots j_{\ell-1})$ for every $1 \le \ell \le m+1$. Also, for every $\equiv_\ell$-class $U(j_1 \dots j_\ell)$,

$$U(j_1 \dots j_\ell) = \bigcup_{j_{\ell+1}=1}^{s(j_1 \dots j_\ell)} U(j_1 \dots j_\ell \, j_{\ell+1}),$$

and this union is a disjoint union of nonvoid sets.

Definition 3.1. *For a finite algebra* $\mathbf{A}$*, for a chain*

$$C = \{1_A = \theta_0 > \cdots > \theta_\ell > \cdots > \theta_{m+1} = 0_A\}$$

of congruence relations in Con $\mathbf{A}$ *having* $A(i_1 \dots i_\ell)$ *for* $1 \le i_\ell \le r(i_1 \dots i_{\ell-1})$ *the* θ_ℓ *congruence classes contained in* $A(i_1 \dots i_{\ell-1})$*, and for a nonvoid set* $U \subseteq \mathrm{val}(X, \mathbf{A})$ *with* $U(j_1 \dots j_\ell)$ *for* $1 \le j_\ell \le s(j_1 \dots j_{\ell-1})$ *the* $\equiv_\ell$ *equivalence classes contained in* $U(j_1 \dots j_{\ell-1})$*, let* $\Omega(\mathbf{A}, C, U) \subseteq A^U$ *denote*

$$\prod_{j_1=1}^{s(\epsilon)} \bigcup_{i_1=1}^{r(\epsilon)} \cdots \prod_{j_\ell=1}^{s(j_1 \dots j_{\ell-1})} \bigcup_{i_\ell=1}^{r(i_1 \dots i_{\ell-1})} \cdots \prod_{j_m=1}^{s(j_1 \dots j_{m-1})} \bigcup_{i_m=1}^{r(i_1 \dots i_{m-1})} A(i_1 \dots i_m)^{U(j_1 \dots j_m)},$$

with $\Omega(\mathbf{A}, C, U) = \{\emptyset\}$ *if* $U = \emptyset$*.*

Remark 3.2. Since for every ℓ the classes of θ_ℓ and of $\equiv_\ell$ are pairwise disjoint and finite, it follows that the cardinality of $\Omega(\mathbf{A}, C, U)$ is

$$\prod_{j_1=1}^{s(\epsilon)} \sum_{i_1=1}^{r(\epsilon)} \cdots \prod_{j_\ell=1}^{s(j_1 \dots j_{\ell-1})} \sum_{i_\ell=1}^{r(i_1 \dots i_{\ell-1})} \cdots \prod_{j_m=1}^{s(j_1 \dots j_{m-1})} \sum_{i_m=1}^{r(i_1 \dots i_{m-1})} |A(i_1 \dots i_m)|^{|U(j_1 \dots j_m)|}. \tag{3.1}$$

In particular, if $U = \emptyset$, then $|\Omega(\mathbf{A}, C, U)| = 1$ and if $C = \{0_A, 1_A\}$, then $|\Omega(\mathbf{A}, C, U)| = |A(\epsilon)|^{|U(\epsilon)|} = |A|^{|U|}$.

From the definition it follows that if $U(j_1 \dots j_m)$ is an arbitrary $\equiv_m$-class, then

$$\Omega(\mathbf{A}, C, U)|_{U(j_1 \dots j_m)} = \bigcup_{\substack{\text{all } \theta_m\text{-classes} \\ A(i_1 \dots i_m)}} A(i_1 \dots i_m)^{U(j_1 \dots j_m)} \tag{3.2}$$

and for $0 \le \ell < m$ and an arbitrary $\equiv_\ell$-class $U(j_1 \dots j_\ell)$

$$\Omega(\mathbf{A}, C, U)|_{U(j_1 \dots j_l)} = \bigcup_{\substack{\text{all } \theta_\ell\text{-classes} \\ A(i_1 \dots i_\ell)}} \prod_{j_{\ell+1}=1}^{s(j_1 \dots j_\ell)} \bigcup_{i_{\ell+1}=1}^{r(i_1 \dots i_\ell)} \cdots \prod_{j_m=1}^{s(j_1 \dots j_{m-1})} \bigcup_{i_m=1}^{r(i_1 \dots i_{m-1})} A(i_1 \dots i_m)^{U(j_1 \dots j_m)}. \quad (3.3)$$

Definition 3.3. *Let A be a set and f an n-ary partial operation on A.*

1. *The* support *of f, denoted* $\operatorname{supp}(f)$, *is the subset of A^n on which f is defined.*

2. *If θ is an equivalence relation on A, then f is said to* preserve *θ if whenever $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are in* $\operatorname{supp}(f)$ *with $(a_i, b_i) \in \theta$ for $1 \le i \le n$, then $(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta$.*

3. *f is said to be θ-*constant *on a subset S of* $\operatorname{supp}(f)$ *if there is an equivalence class C of θ such that $f(S) \subseteq C$.*

Remark 3.4. Suppose $\mathbf{A}$ is an algebra and $u \in U \subseteq \operatorname{val}(X, \mathbf{A})$. We may consider u as an n-tuple $(u(x_1), \dots, u(x_n))$. Any $f \in A^U$ may be viewed as an n-ary partial operation on A with support U. That is, for $u \in U$ we have $f(u) = f(u(x_1), \dots, u(x_n))$. Likewise $\Omega(\mathbf{A}, C, U)$ may be viewed as a collection of n-ary partial operations on A, each with support U.

We will have occasion to use the following observations involving automorphisms.

Lemma 3.5. *Suppose $\mathbf{A}$ is an arbitrary algebra with $\theta \in \operatorname{Con} \mathbf{A}$, $u \in \operatorname{val}(X, \mathbf{A})$, and $\alpha \in \operatorname{Aut} \mathbf{A}$.*

1. *The set $\alpha(\theta) = \{(\alpha(a), \alpha(b)) \mid (a, b) \in \theta\}$ is also in $\operatorname{Con} \mathbf{A}$.*

2. *The function $\alpha(u)\colon X \to A$ defined by $\alpha(u)(x_i) = \alpha(u(x_i))$ is a valuation.*

3. *If $\mathbf{A}$ is finite, $\operatorname{Con} \mathbf{A}$ is a chain $1_A = \theta_0 \succ \theta_1 \succ \cdots \succ \theta_m \succ \theta_{m+1} = 0_A$, and $v, w \in \operatorname{val}(X, \mathbf{A})$ with $v \equiv_\ell w$, then $\alpha(v) \equiv_\ell \alpha(w)$ and $\alpha(\theta_\ell) = \theta_\ell$.*

4. *If u is such that $\alpha(u) = u$, then α is the identity permutation ι.*

5. *If* $\mathbf{A}$ *is finite and* U *is a transversal with respect to* $\sim$ *of* $\mathrm{val}(X, \mathbf{A})$*, then* $\mathrm{val}(X, \mathbf{A})$ *is the disjoint union of the sets* $\alpha(U)$ *where* α *ranges over* $\mathrm{Aut}\,\mathbf{A}$*. Thus* $|U| = \frac{|\mathrm{val}(X,\mathbf{A})|}{|\mathrm{Aut}\,\mathbf{A}|}$.

Proof. The first four items are easily proved. The fifth follows from (4) and Burnside's Lemma. Section 2 of [1] also contains a proof of item (5). $\square$

Lemma 3.6. *Suppose that an algebra* $\mathbf{A}$*, a chain* C *of congruence relations, a set* U *of valuations, and* $\Omega = \Omega(\mathbf{A}, C, U)$ *are as in Definition 3.1. For* $1 \leq \ell \leq m$ *let* $U(j_1 \dots j_\ell)$ *be an* $\equiv_\ell$ *equivalence class and let* $h \in A^{U(j_1 \dots j_\ell)}$.

1. *For* $\ell = m$*:* h *is in* $\Omega|_{U(j_1 \dots j_m)}$ *if and only if there exists a* θ_m*-class* $A(i_1 \dots i_m)$ *such that* $h \in A(i_1 \dots i_m)^{U(j_1 \dots j_m)}$.

2. *For* $0 \leq \ell < m$*:* h *is in* $\Omega|_{U(j_1 \dots j_\ell)}$ *if and only if there exists a* θ_ℓ*-class* $A(i_1 \dots i_\ell)$ *such that for every* q*,* $1 \leq q \leq s(j_1 \dots j_\ell)$*, there exists* $h_q \in \Omega|_{U(j_1 \dots j_\ell q)}$ *and there exists* p_q *with* $1 \leq p_q \leq r(i_1 \dots i_\ell)$ *such that* $h_q \in A(i_1 \dots i_\ell p_q)^{U(j_1 \dots j_\ell q)}$ *and* $h = (h_1, \dots, h_q, \dots, h_{s(j_1 \dots j_\ell)})$.

Proof. The $\ell = m$ claim is immediate from (3.2). Suppose $0 \leq \ell < m$. It follows from (3.3) that $h \in \Omega|_{U(j_1 \dots j_\ell)}$ if and only if there exists a θ_ℓ-class $A(i_1 \dots i_\ell)$ such that

$$h \in \prod_{j_{\ell+1}=1}^{s(j_1 \dots j_\ell)} \bigcup_{i_{\ell+1}=1}^{r(i_1 \dots i_\ell)} \cdots \prod_{j_m=1}^{s(j_1 \dots j_{m-1})} \bigcup_{i_m=1}^{r(i_1 \dots i_{m-1})} A(i_1 \dots i_\ell \dots i_m)^{U(j_1 \dots j_\ell \dots j_m)},$$

for the given strings $j_1 \dots j_\ell$ and $i_1 \dots i_\ell$. This in turn will hold if and only if for all q, with $1 \leq q \leq s(j_1 \dots j_\ell)$ there exists h_q such that

$$h_q \in \bigcup_{i_{\ell+1}=1}^{r(i_1 \dots i_\ell)} \prod_{j_{\ell+2}=1}^{s(j_1 \dots j_\ell q)} \cdots \prod_{j_m=1}^{s(j_1 \dots j_\ell q j_{\ell+2} \dots j_{m-1})} \bigcup_{i_m=1}^{r(i_1 \dots i_{m-1})} A(i_1 \dots i_m)^{U(j_1 \dots j_\ell q j_{\ell+2} \dots j_m)},$$

for which $h = (h_1, \dots, h_q, \dots, h_{s(j_1 \dots j_\ell)})$. This is equivalent to there being for each q a p_q with $1 \leq p_q \leq r(i_1 \dots i_\ell)$ such that h_q is in

$$\prod_{j_{\ell+2}=1}^{s(j_1 \dots j_\ell q)} \bigcup_{i_{\ell+2}=1}^{r(i_1 \dots i_\ell p_q)} \cdots \prod_{j_m=1}^{s(j_1 \dots j_\ell q j_{\ell+2} \dots j_{m-1})} \bigcup_{i_m=1}^{r(i_1 \dots i_\ell p_q i_{\ell+2} \dots i_{m-1})} A(i_1 \dots i_\ell p_q i_{\ell+2} \dots i_m)^{U(j_1 \dots j_l q j_{\ell+2} \dots j_m)},$$

and $h = (h_1, , \ldots, h_q, \ldots, h_{s(j_1 \ldots j_\ell)})$. Note that for each q we have h_q in $\Omega(\mathbf{A}, C, U)|_{U(j_1 \cdots j_\ell q)}$ by (3.3). Also, each $A(i_1 \ldots i_\ell p_q)$ is a subset of $A(i_1 \ldots i_\ell)$. Thus, the claim holds for $\ell < m$. $\square$

The next lemma characterizes the elements of $\Omega(\mathbf{A}, C, U)$.

Lemma 3.7. *Suppose $\Omega(\mathbf{A}, C, U)$ is as in Definition 3.1. Let $f \in A^U$. Then $f \in \Omega(\mathbf{A}, C, U)$ if and only if f is θ_ℓ-constant on every $\equiv_\ell$-class $U(j_1 \ldots j_\ell)$ for all $0 \le \ell \le m$.*

Proof. For $f \in \Omega$, let h denote $f|_{U(j_1 \cdots j_\ell)}$. Since h is in $\Omega|_{U(j_1 \cdots j_\ell)}$, Lemma 3.6 applies. If $\ell = m$, then h is in $A(i_1 \ldots i_m)^{U(j_1 \cdots j_m)}$ for some θ_m-class $A(i_1 \ldots i_m)$. So h is θ_m-constant on $U(j_1 \ldots j_m)$. If $\ell < m$, then there exists a θ_ℓ-class $A(i_1 \ldots i_\ell)$ and there exist $h_q \in A(i_1 \ldots i_\ell p_q)^{U(j_1 \cdots j_\ell p)}$ for $1 \le q \le s(j_1 \ldots j_\ell)$ and $1 \le p_q \le r(i_1 \ldots i_\ell)$ such that $h = (h_1, \ldots, h_{s(j_1 \cdots j_\ell)})$. Therefore h is in $A(i_1 \ldots i_\ell)^{U(j_1 \cdots j_\ell)}$ and is thus θ_ℓ-constant on $U(j_1 \ldots j_\ell)$.

For the converse, we assume f is θ_ℓ-constant on all $U(j_1 \ldots j_\ell)$ for all $0 \le \ell \le m$ and show that $f|_{U(j_1 \cdots j_\ell)} \in \Omega|_{U(j_1 \cdots j_\ell)}$. Then, for $\ell = 0$, we have $f = f|_{U(\epsilon)} \in \Omega|_U = \Omega$ as desired.

If $\ell = m$, then there is a θ_m-class $A(i_1 \ldots i_m)$ such that $f|_{U(j_1 \cdots j_m)} \in A(i_1 \ldots i_m)^{U(j_1 \cdots j_m)}$ since f is θ_m-constant on $U(j_1 \ldots j_m)$. Thus the claim holds for $\ell = m$ since $A(i_1 \ldots i_m)^{U(j_1 \cdots j_m)} \subseteq \Omega|_{U(j_1 \cdots j_m)}$.

Next assume $\ell < m$ and that by the induction hypothesis the claim holds for $\ell + 1$. For a given $U(j_1 \ldots j_\ell)$, there exists $A(i_1 \ldots i_\ell)$ such that $f(u) \in A(i_1 \ldots i_\ell)$ for all $u \in U(j_1 \ldots j_\ell)$ since f is θ_ℓ-constant on $U(j_1 \ldots j_\ell)$. Let f_q denote $f|_{U(j_1 \cdots j_\ell q)}$ for $1 \le q \le s(j_1 \ldots j_\ell)$. Since f is $\theta_{\ell+1}$-constant on $U(j_1 \ldots j_\ell q)$ we have by the induction hypothesis that $f_q \in \Omega|_{U(j_1 \cdots j_\ell q)}$. Also, since each f_q is $\theta_{\ell+1}$-constant, for each q there exists an index p_q with $1 \le p_q \le r(i_1 \ldots i_\ell)$ such that $f_q \in A(i_1 \ldots i_\ell p_q)^{U(j_1 \cdots j_\ell q)}$. Finally, $f = (f_1, \ldots, f_q, \ldots, f_{s(j_1 \cdots j_\ell)}) \in \Omega|_{U(j_1 \cdots j_\ell)}$ by Lemma 3.6. $\square$

The next lemma presents some basic facts about the set $\Omega(\mathbf{A}, C, U)$.

Lemma 3.8. *Suppose $\mathbf{A}$ is a finite algebra, $U \subseteq \operatorname{val}(X, \mathbf{A})$, and $C = \{1_A = \theta_0 > \theta_1 > \cdots > \theta_m > \theta_{m+1} = 0_A\}$ is a chain of congruence relations contained in $\operatorname{Con} \mathbf{A}$. Let Ω denote $\Omega(\mathbf{A}, C, U)$.*

1. *If $x \in X$, then $x^* \in \Omega$ for x^* as in Definition 2.2.*

2. *For all $a \in A$, the constant $|U|$-tuple c^a is in Ω.*

3. $\Omega|_u = A$ *for all* $u \in U$.

4. *If* $v, w \in U$ *and* $(v, w) \in \equiv_\ell \setminus \equiv_{\ell+1}$, *then* $\Omega|_{\{v,w\}} = \theta_\ell$.

5. Ω *is a subuniverse of* $\mathbf{A}^U$.

Proof. For (1), suppose $u, v \in U$ with $u \equiv_\ell v$. We have $(x_i^*(u), x_i^*(v)) = (u(x_i), v(x_i)) \in \theta_\ell$. Hence x_i^* is θ_ℓ-constant on every $\equiv_\ell$ equivalence class. So $x_i^* \in \Omega$ by Lemma 3.7.

To prove (2) it suffices to observe that for all $a \in A$ and every θ_ℓ, we have$(a, a) \in \theta_\ell$ and thus c^a is θ_ℓ-constant on every $\equiv_\ell$ equivalence class.

Claim (3) follows from Claim (2 since $c^a(u) = a$ for every $a \in A$.

For (4), suppose $(v, w) \in \equiv_\ell \setminus \equiv_{\ell+1}$. Let $(a, b) \in \theta_\ell$ be arbitrary. Define $f \in A^U$ by $f(u) = a$ if $u \equiv_{\ell+1} v$ and $f(u) = b$ otherwise. It suffices to show f is $\equiv_q$-constant for every $\theta_q \in C$ on every $\equiv_q$-class. Let U_q be an arbitrary $\equiv_q$-class. If f is constant on U_q or if $q \leq \ell$, then f is $\equiv_q$-constant on U_q. So suppose $q > \ell$ and $u_1, u_2 \in U_q$ with, say, $f(u_1) = a$ and $f(u_2) = b$. Then $u_1 \equiv_{\ell+1} u_2$ since $\equiv_q \subseteq \equiv_{\ell+1}$. But $v \equiv_{\ell+1} u_1$ since $f(u_1) = a$. But then $u_2 \equiv_{\ell+1} v$, which contradicts $f(u_2) = b$.

To prove that Ω is a subuniverse of $\mathbf{A}^U$ it suffices by Lemma 3.7 to show that if t is an arbitrary r-ary term for $\mathbf{A}$ and if $f_1, \ldots, f_r \in A^U$ are such that for every θ_ℓ all the f_j are θ_ℓ-constant on every $\equiv_\ell$-class, then $t^{\mathbf{A}^U}(f_1, \ldots, f_r) \in A^U$ is also θ_ℓ-constant on every $\equiv_\ell$-class. But this follows from the fact that if $A(i_1^1 \ldots i_\ell^1), \ldots, A(i_1^r \ldots i_\ell^r)$ are each θ_ℓ-classes, then $t^{\mathbf{A}}(A(i_1^1 \ldots i_\ell^1), \ldots, A(i_1^r \ldots i_\ell^r))$ is contained in a θ_ℓ-class since t is a term. □

The next result provides the upper bound $|\Omega(\mathbf{A}, C, U)|$ on the cardinality of $\mathbf{Ge}(X, U)$, and it contains a collection of facts that hold whenever this upper bound is obtained.

Corollary 3.9. *Suppose* $\Omega = \Omega(\mathbf{A}, C, U)$ *is as in Lemma 3.8. Then* $\mathbf{Ge}(X, U)$ *is a subalgebra of* $\Omega(\mathbf{A}, C, U)$ *and*

$$|\mathbf{Ge}(X, U)| \leq |\Omega(\mathbf{A}, C, U)|. \tag{3.4}$$

If $\mathbf{A}, C,$ *and* U *are such that* $|\mathbf{Ge}(X, U)| = |\Omega(\mathbf{A}, C, U)|$, *then all of the following statements are true:*

1. *The universe of* $\mathbf{Ge}(X, U)$ *is* Ω.

2. Ω *is generated by* X^* *as a subalgebra of* $\mathbf{A}^U$.

3. $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) = \mathrm{Cg}^{\mathbf{A}}(\overline{vw}) = \Omega|_{\{v,w\}}$ *for all* $v \neq w \in U$.
   If $v, w \in U$ *and* $(v, w) \in \equiv_\ell \setminus \equiv_{\ell+1}$, *then* $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) = \theta_\ell$.

4. *If* $v, w \in U$ *and* $v \preceq w$, *then* $v = w$.

5. *If* h *is an* n*-ary partial operation on* A *with* $\mathrm{supp}(h) = U$ *such that* h *preserves every* $\theta_\ell \in C$, *then there exists* $g \in \mathbf{Ge}(X, U)$ *for which* $g(u) = h(u)$ *for all* $u \in U$.

Proof. By Definition 2.2, $X^* \subseteq A^U$ generates $\mathbf{Ge}(X, U)$. That Ω is a subuniverse of $\mathbf{A}^U$ and $X^* \subseteq \Omega$ follow from Lemma 3.8(5) and Lemma 3.8(1). Hence $\mathbf{Ge}(X, U) \subseteq \Omega(\mathbf{A}, C, U)$ and the inequality (3.4) follows.

If $|\mathbf{Ge}(X, U)| = |\Omega|$, then (1) and (2) are immediate since Ω is finite. Because X^* generates Ω, we have $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) = \Omega|_{\{v,w\}}$. It is always the case for any $v, w \in \mathrm{val}(X, \mathbf{A})$ that $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) \subseteq \mathrm{Cg}^{\mathbf{A}}(\overline{vw})$. Let ℓ be such that $(v, w) \in \equiv_\ell \setminus \equiv_{\ell+1}$. Then $\theta_\ell = \Omega|_{\{v,w\}}$ by Lemma 3.8 (4). Thus,

$$\Omega|_{\{v,w\}} = \mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) \subseteq \mathrm{Cg}^{\mathbf{A}}(\overline{vw}) \subseteq \theta_\ell = \Omega|_{\{v,w\}},$$

which justifies claim (3).

If $v \preceq w$, then there is an endomorphism e of $\mathbf{A}$ such that $e(v(x)) = w(x)$ for all $x \in X$. So $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) = \{(a, e(a)) \mid a \in A\}$. Thus $|\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw})| = |A|$ and hence $\mathrm{Sg}^{\mathbf{A}^2}(\overline{vw}) = \theta_{m+1} = 0_A$ by claim (3). This implies $e(v(x)) = v(x)$ for all $x \in X$. Therefore $v = w$.

For (5), given the n-ary partial operation h with $\mathrm{supp}(h) = U$, let $g \in A^U$ be such that $g = h$. That h preserves every $\theta_\ell \in C$ is equivalent to g being θ_ℓ-constant on every $\equiv_\ell$-class $U(j_1 \dots j_\ell)$. So $g \in \Omega(\mathbf{A}, C, U)$ by Lemma 3.7. Thus $g \in \mathbf{Ge}(X, U) = \Omega$. □

Corollary 3.10. *Let* $K = \{\mathbf{S}_1, \dots, \mathbf{S}_k\}$ *be a finite set of pairwise nonisomorphic finite algebras of the same similarity type. Suppose for each* $1 \leq i \leq k$ *that* C_i *is a chain in* $\mathrm{Con}\,\mathbf{S}_i$ *with* $0_{S_i}, 1_{S_i} \in C_i$. *Let* $U \subseteq \mathrm{val}(X, K)$ *be arbitrary with* $U_i := \{u \in U \mid \mathbf{Alg}(u) = \mathbf{S}_i\}$. *Then for* $\Omega(\mathbf{S}_i, C_i, U_i)$ *as in Definition 3.1*

$$\mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U_1 \cup \dots \cup U_k) \lhd \prod_{i=1}^{k} \mathbf{Ge}(X, U_i) \lhd \prod_{i=1}^{k} \Omega(\mathbf{S}_i, C_i, U_i). \quad (3.5)$$

If U is a minimal transversal with respect to $\preceq$ of $\mathrm{val}(X, K)$*, then*

$$|\mathbf{L}(n, K)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, C_i, U_i)|. \tag{3.6}$$

Proof. The isomorphism and the first embedding in (3.5) are immediate since the U_i are all pairwise disjoint by Definition 2.1(3). The second embedding is from Corollary 3.9. The inequality (3.6) follows from Remark 2.11. $\square$

The next result gives our desired upper bound on the cardinality of an n-generated subdirect power in terms of the set $\Omega(\mathbf{A}, C, U)$ as in Definition 3.1.

Corollary 3.11. *Let $\mathbf{A}$ be a finite algebra whose congruence lattice contains a chain $C = \{1_A = \theta_0 > \theta_1 > \cdots > \theta_{m+1} = 0_A\}$. Then $\mathbf{L}(n, \mathbf{A})$, the largest n-generated subdirect power of $\mathbf{A}$, is isomorphic to a subalgebra of $\Omega(\mathbf{A}, C, U)$, where U is any transversal with respect to $\sim$ of* $\mathrm{val}(X, \mathbf{A})$ *for $X = \{x_1, \ldots, x_n\}$. The resulting upper bound on $|\mathbf{L}(n, \mathbf{A})|$ for this choice of U is*

$$\prod_{j_1=1}^{s(\epsilon)} \sum_{i_1=1}^{r(\epsilon)} \cdots \prod_{j_m=1}^{s(j_1 \ldots j_{m-1})} \sum_{i_m=1}^{r(i_1 \ldots i_{m-1})} |A(i_1 \ldots i_m)|^{|U(j_1 \ldots j_m)|} = |\Omega(\mathbf{A}, C, U)|.$$

Proof. This follows from Corollary 3.10 and the observation in the paragraph following Definition 2.4 that on $\mathrm{val}(X, \mathbf{A})$ the quasi-order $\preceq$ is $\sim$. $\square$

It is important to note that in Corollary 3.11 if U and U' are both transversals with respect to $\sim$ of $\mathrm{val}(X, \mathbf{A})$, then $|\mathbf{Ge}(X, U)| = |\mathbf{Ge}(X, U')|$ by Corollary 2.9. However, the cardinalities of $\Omega(\mathbf{A}, C, U)$ and $\Omega(\mathbf{A}, C, U')$ may differ. Section 6 presents an effective way to choose a particular minimal transversal that may be used in the computational condition of our main theorem.

4 The interpolation condition

This section presents a condition involving the interpolation of a collection of n-ary partial operations on a finite set K of finite algebras of the same similarity type by an n-ary term operation $t(y_1, \ldots, y_n)$ in the language of K. The existence of t is shown to be equivalent to the embedding displayed in (3.5) of Corollary 3.10 being an isomorphism $\prod_{i=1}^{k} \mathbf{Ge}(X, U_i) \cong \prod_{i=1}^{k} \Omega(\mathbf{S}_i, C_i, U_i)$.

Theorem 4.1. *Let $\mathbf{S}_1, \ldots, \mathbf{S}_k$ be a finite set of pairwise nonisomorphic finite algebras of the same similarity type. Suppose for each i that C_i is a linearly ordered subset of* $\operatorname{Con}\mathbf{S}_i$ *that contains both 0_{S_i} and 1_{S_i}. Let $U \subseteq$* $\operatorname{val}(X, \{\mathbf{S}_1, \ldots, \mathbf{S}_k\})$*. For each $1 \leq i \leq k$, let $U_i = \{u \in U \mid \mathbf{Alg}(u) = \mathbf{S}_i\}$. The following two statements are equivalent:*

1. $|\mathbf{Ge}(X, U)| = \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, C_i, U_i)|$.

2. *If for each $1 \leq i \leq k$ an n-ary partial operation $h_i \colon S_i^n \to S_i$ is given with* $\operatorname{supp}(h_i) = U_i$ *and such that each h_i preserves every congruence relation in C_i, then there exists an n-ary term t in the language of the $\mathbf{S}_i$ such that $t^{\mathbf{S}_i}(u) = h_i(u)$ for every $1 \leq i \leq k$ and $u \in U_i$.*

Proof. Assume (1) and suppose the h_i are as given in (2). Let Ω_i denote $\Omega(\mathbf{S}_i, C_i, U_i)$. We have $|\mathbf{Ge}(X, U_i)| \leq |\Omega_i|$ for all i by Corollary 3.9. It is always the case that

$$|\mathbf{Ge}(X, U)| = |\mathbf{Ge}(X, \bigcup_{i=1}^{k} U_i)| \leq \prod_{i=1}^{k} |\mathbf{Ge}(X, U_i)| \leq \prod_{i=1}^{k} |\Omega_i|$$

by Corollary 3.10. Therefore $|\mathbf{Ge}(X, U_i)| = |\Omega_i|$. Apply Corollary 3.9(5) to find an $f_i \in \mathbf{Ge}(X, U_i)$ such that $f_i(u) = h_i(u)$ for each $1 \leq i \leq k$ and each $u \in U_i$. From (1) and the displayed chain of inequalities, it follows that $|\mathbf{Ge}(X, U)| = \prod_{i=1}^{k} |\mathbf{Ge}(X, U_i)|$ and hence $\mathbf{Ge}(X, U) = \prod_{i=1}^{k} \mathbf{Ge}(X, U_i)$. Thus there is an $f \in \mathbf{Ge}(X, U)$ such that $f = (f_1, \ldots, f_k)$. Let $t(y_1, \ldots, y_n)$ be the n-ary term for which $t^{\mathbf{Ge}(X,U)}(x_1^*, \ldots, x_n^*) = f$. Then for all i and for all $u \in U_i$ we have $t^{\mathbf{S}_i}(u) = f_i(u) = h_i(u)$.

Next, assume condition (2) holds. Let $f_i \in \Omega_i$ for $1 \leq i \leq k$ be chosen arbitrarily. By using the notation of Definition 3.3 we have that each f_i is an n-ary partial operation on S_i with support U_i. Each f_i preserves every congruence relation in C_i since each $f_i \in \Omega(\mathbf{S}_i, C, U_i)$. Let t be an n-ary term for which $t^{\mathbf{S}_i}(u) = f(u)$ for every i and $u \in U_i$. Such a term exists by virtue of (2). Now consider $t^{\mathbf{Ge}(X,U)}(x_1^*, \ldots, x_n^*) = g \in \mathbf{Ge}(X, U)$. Then $g|_{U_i} = f_i$ for all i. Hence $\prod_{i=1}^{k} \Omega_i \subseteq \mathbf{Ge}(X, U)$. □

5 The algebraic condition

In this section we present some necessary and sufficient algebraic conditions on an algebra $\mathbf{A}$ and on sets $U \subseteq \operatorname{val}(X, \mathbf{A})$ for the upper bound on $|\mathbf{Ge}(X,U)|$ of Corollary 3.11 to be obtained. To this end we now focus on finite algebras $\mathbf{A}$ for which there exists $m \geq 0$ such that

$$\operatorname{Con}\mathbf{A} = \{1_A = \theta_0 > \theta_1 > \cdots > \theta_m > \theta_{m+1} = 0_A\}.$$

Suppose $U \subseteq \operatorname{val}(X,\mathbf{A})$ is such that

$$|\mathbf{Ge}(X,U)| = |\Omega(\mathbf{A},\operatorname{Con}\mathbf{A},U)|.$$

Since $\mathbf{Ge}(X,U) \subseteq \Omega(\mathbf{A},\operatorname{Con}\mathbf{A},U)$ and is finite, we have

$$\mathbf{Ge}(X,U) = \Omega(\mathbf{A},\operatorname{Con}\mathbf{A},U).$$

A variety $\mathcal{V}$ of algebras is said to have a *majority term* if there is a ternary term $t(x,y,z)$ in the language of $\mathcal{V}$ such that $\mathcal{V} \models t(x,x,y) \approx t(x.y,x) \approx t(y,x,x) \approx x$.

The following is a consequence of the Baker-Pixley Theorem [2]:

Lemma 5.1. *Let $\mathcal{V}$ be a variety that has a majority term $m(x,y,z)$.*

1. *Suppose $\mathbf{A}$ is a finite algebra in $\mathcal{V}$ and $\mathbf{S}$ is a subalgebra of $\mathbf{A}^k$. If $f \in A^k$ is such that for each $1 \leq i < j \leq k$ there exists $s \in S$ such that $(s_i,s_j) = (f_i,f_j)$, then $f \in S$.*

2. *Let $\mathbf{A}_1,\dots,\mathbf{A}_k$ be finite algebras in $\mathcal{V}$. Suppose $\mathbf{G}$ is a subalgebra of $\mathbf{A}_1^{Y_1} \times \cdots \times \mathbf{A}_k^{Y_k}$ for finite, pairwise disjoint index sets $Y_1,\dots,Y_k$. Suppose also that for all $i \neq j$, if $y_i \in Y_i$ and $y_j \in Y_j$, then $G|_{\{y_i,y_j\}} = A_i \times A_j$. If $g_i \in G|_{Y_i}$ for all $1 \leq i \leq k$, then there exists $g \in G$ such that $g|_{Y_i} = g_i$ for all $1 \leq i \leq k$. Thus, in particular, $G = \prod_{i=1}^{k} G|_{Y_i}$.*

The next lemma provides a context for a converse of Corollary 3.9(3) and is the crux of our algebraic condition.

Lemma 5.2. *Let $\mathbf{A}$ be a finite algebra with*

$$\operatorname{Con}\mathbf{A} = \{1_A = \theta_0 > \theta_1 > \cdots > \theta_m > \theta_{m+1} = 0_A\}$$

and let U be any subset of a transversal with respect to $\sim$ of $\operatorname{val}(X,\mathbf{A})$. Suppose that the variety generated by $\mathbf{A}$ has a ternary majority term and suppose that U is such that $\operatorname{Sg}^{\mathbf{A}^2}(\overline{vw}) = \operatorname{Cg}^{\mathbf{A}}(\overline{vw})$ for all $v \neq w \in U$. Then the universe of $\mathbf{Ge}(X,U)$ is $\Omega(\mathbf{A},\operatorname{Con}\mathbf{A},U)$ and thus $|\mathbf{Ge}(X,U)| = |\Omega(\mathbf{A},\operatorname{Con}\mathbf{A},U)|$.

Proof. Let $f \in \Omega(\mathbf{A}, \operatorname{Con}\mathbf{A}, U)$. By Lemma 5.1(1) it suffices to show for all $v, w \in U$ that $f|_{\{v,w\}} \in \mathbf{Ge}(X,U)|_{\{v,w\}}$. We have $f|_{\{v,w\}} \in \operatorname{Cg}^{\mathbf{A}}(\overline{vw})$ by Lemma 3.7. By hypothesis, $\operatorname{Cg}^{\mathbf{A}}(\overline{vw}) = \operatorname{Sg}^{\mathbf{A}^2}(\overline{vw})$. But it is always the case that $\operatorname{Sg}^{\mathbf{A}^2}(\overline{vw}) = \mathbf{Ge}(X,U)|_{\{v,w\}}$. □

If an algebra $\mathbf{A}$ has $\operatorname{Con}\mathbf{A}$ linearly ordered, then all the congruence relations of $\mathbf{A}$ permute and the congruence lattice of $\mathbf{A}$ is a distributive lattice. That is, the congruence lattice of $\mathbf{A}$ is an *arithmetical lattice* of equivalence relations. It is shown in [11] that if a finite algebra $\mathbf{A}$ has an arithmetical congruence lattice, then there is a Pixley operation on A that preserves all the congruence relations of $\mathbf{A}$. We appeal to this result in our proof of the next theorem. In the case that $\operatorname{Con}\mathbf{A}$ is linearly ordered then the operation $t\colon A^3 \to A$ defined by

$$t(a_1,a_2,a_3) = \begin{cases} a_3 & \text{if } \operatorname{Cg}^{\mathbf{A}}(a_1,a_2) < \operatorname{Cg}^{\mathbf{A}}(a_1,a_3) \\ & \text{and } \operatorname{Cg}^{\mathbf{A}}(a_1,a_2) < \operatorname{Cg}^{\mathbf{A}}(a_2,a_3); \\ a_1 & \text{otherwise} \end{cases}$$

can be shown to be a Pixley operation on A and t preserves all the congruence relations of $\mathbf{A}$. Moreover t preserves all subuniverses of $\mathbf{A}$ since it is a conservative operation. As observed in Remark 3.5 we have $\alpha(\theta) = \theta$ for all congruence relations θ and automorphisms α of $\mathbf{A}$. From this it follows that t preserves all the automorphisms of $\mathbf{A}$.

Theorem 5.3. *Suppose $\mathcal{V}$ is a locally finite variety, $X = \{x_1, \ldots, x_n\}$, and $\mathcal{S}_n = \{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ is a transversal with respect to $\cong$ of the n-generated subdirectly irreducible algebras in $\mathcal{V}$. Let U be a minimal transversal of the quasi-ordered set $(\operatorname{val}(X, \mathcal{S}_n), \preceq)$. For each $1 \le i \le k$, let $U_i = \{u \in U \mid \mathbf{Alg}(u) = \mathbf{S}_i\}$. If each $\operatorname{Con}\mathbf{S}_i$ is linearly ordered, then*

$$|\mathbf{L}(n, \mathcal{S}_n)| \le \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i)|, \tag{5.1}$$

and if $n \ge 3$, then the following are equivalent:

1. *The variety $\mathcal{V}$ and the set U are such that equality holds in (5.1).*

2. *$\mathcal{V}$ is congruence permutable and congruence distributive and for all $v, w \in U$*

$$\operatorname{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw}) \supseteq \begin{cases} 0_{\mathbf{Alg}(v)} & \text{if } \mathbf{Alg}(v) = \mathbf{Alg}(w); \\ \mathbf{Alg}(v)\times\mathbf{Alg}(w) & \text{if } \mathbf{Alg}(v) \neq \mathbf{Alg}(w). \end{cases}$$

Proof. The inequality (5.1) follows from Corollary 3.10 and the fact that the free algebra on n free generators in any variety is a subdirect product of the n-generated subdirectly irreducible algebras in the variety.

For (1) implies (2) let U and $\mathcal{S}_n$ be as in the hypotheses and suppose $|\mathbf{Ge}(X,U)| = \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i)|$. Then

$$\mathbf{Ge}(X,U) = \prod_{i=1}^{k} \Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i)$$

and

$$\mathbf{Ge}(X,U_i) = \mathbf{Ge}(X,U)|_{U_i} = \Omega(\mathbf{S}_i, \operatorname{Con} S_i, U_i)$$

since $\mathbf{Ge}(X,U) \subseteq \prod_{i=1}^{k} \Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i)$ by Corollary 3.10 and $\mathbf{Ge}(X,U_i) \subseteq \Omega(\mathbf{S}_i, \operatorname{Con} S_i, U_i)$ by Lemma 3.8(5).

For every algebra $\mathbf{S}_i \in \mathcal{S}_n$ let $t^i \colon S_i^3 \to S_i$ be a Pixley operation on S_i that preserves all the congruence relations of $\mathbf{S}_i$. By [11, Theorem 2.2.5] such t^i exist for each $1 \le i \le k$. Since $n \ge 3$, we may define $q^i \colon S_i^n \to S_i$ by

$$q^i(y_1, \ldots, y_n) = t^i(y_1, y_2, y_3).$$

Then each q^i is an n-ary operation that does not depend on $y_4, \ldots, y_n$ and q^i also preserves all the congruences of $\mathbf{S}_i$. Let h_i denote the n-ary partial operation on S_i obtained by restricting the domain of q^i to U_i. Then by Theorem 4.1 there is an n-ary term h in the language of $\mathcal{V}$ such that

$$h^{\mathbf{S}_i}(u(x_1), \ldots, u(x_n)) = h_i(u(x_1), \ldots, u(x_n)) = t^i(u(x_1), u(x_2), u(x_3)) \quad (5.2)$$

for all i and $u \in U_i \subseteq U$.

We wish to show that

$$y = h^{\mathbf{S}_i}(y, x, x, x, \ldots, x) = h^{\mathbf{S}_i}(x, x, y, y, \ldots, y) = h^{\mathbf{S}_i}(y, x, y, y, \ldots, y) \quad (5.3)$$

for all $\mathbf{S}_i \in \mathcal{S}_n$ and all $x, y \in S_i$. For then the term $p(y_1, y_2, y_3)$ defined as $h(y_1, y_2, y_3, y_3, \ldots, y_3)$ will be a Pixley term for $\mathcal{V}$ since the identities

$$y \approx p(y, x, x) \approx p(x, x, y) \approx p(y, x, y)$$

hold for all algebras in $\mathcal{S}_n$, the set of n-generated subdirectly irreducible algebras in $\mathcal{V}$.

Let $\mathbf{S}$ be an arbitrary member of $\mathcal{S}_n$. (To simplify the notation we suppress the subscript in $\mathbf{S}_i$.) For $\bar{b} = (b_1, b_2, b_3, b_3, \ldots, b_3) \in S^n$ we need to show that $h^{\mathbf{S}}(\bar{b})$ behaves as in (5.3) whenever at least two of the b_i are equal.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{S}$ generated by $\{b_1, b_2, b_3\}$. If $|B| = 1$, then $b_1 = b_2 = b_3 = h^{\mathbf{S}}(\bar{b})$ as desired in (5.3). So we assume $|B| \ge 2$. The algebra $\mathbf{B}$ is a subdirect product of algebras in $\mathcal{S}_n$ since it is at most 3-generated and $n \ge 3$ by hypothesis. Let

$$e \colon \mathbf{B} \to \mathbf{T}_1 \times \cdots \times \mathbf{T}_m$$

be a subdirect embedding where the $\mathbf{T}_j$ are algebras in $\mathcal{S}_n$. (In fact, each $\mathbf{T}_j$ is 3-generated.) The homomorphism $pr_j e$, for pr_j denoting the projection on the j-th coordinate, maps $\mathbf{B}$ onto $\mathbf{T}_j$. Let b_{ij} denote $pr_j e(b_i)$. The algebra $\mathbf{T}_j$ is generated by $\{b_{1j}, b_{2j}, b_{3j}\}$. By w_j we denote the valuation in $\mathrm{val}(X, \mathbf{T}_j)$ for which $w_j(x_1) = b_{1j}, w_j(x_2) = b_{2j}$ and $w_j(x_i) = b_{3j}$ for $3 \leq i \leq n$.

We work with a fixed but arbitrary valuation w_j. The original minimal transversal U contains a valuation v such that $v \preceq w_j$. Suppose $\mathbf{S}_\ell \in \mathcal{S}_n$ is such that $v \in \mathrm{val}(X, \mathbf{S}_\ell)$. So there exists a homomorphism $f \colon \mathbf{S}_\ell \to \mathbf{T}_j$ such that $f(v(x_i)) = w_j(x_i)$ for $1 \leq i \leq n$. Let $a_1, a_2, a_3 \in \mathbf{S}_\ell$ denote $v(x_1), v(x_2), v(x_3)$ respectively. Thus $f(a_i) = b_{ij}$ for $1 \leq i \leq 3$. By (5.2)

$$h^{\mathbf{S}_\ell}(v) = h^{\mathbf{S}_\ell}(v(x_1), \ldots, v(x_n)) = h^{\mathbf{S}_\ell}(a_1, a_2, a_3, \ldots) = t^\ell(a_1, a_2, a_3). \quad (5.4)$$

If f is applied in (5.4) we have

$$h^{\mathbf{T}_j}(w_j) = h^{\mathbf{T}_j}(b_{1j}, b_{2j}, b_{3j}, b_{3j}, \ldots, b_{3j}) = f(t^\ell(a_1, a_2, a_3)) = p^{\mathbf{T}_j}(b_{1j}, b_{2j}, b_{3j}).$$

We wish to show that the equations of (5.3) hold for $h^{\mathbf{T}_j}(b_{1j}, b_{2j}, b_{3j}, b_{3j}, \ldots, b_{3j})$ when at least two of b_{1j}, b_{2j}, b_{3j} are equal.

If say, $b_{2j} = b_{3j}$, then $a_2/\ker f = a_3/\ker f$. Thus,

$$h^{\mathbf{S}_\ell}(v)/\ker f = t^\ell(a_1, a_2, a_3)/\ker f = t^\ell(a_1, a_3, a_3)/\ker f = a_1/\ker f$$

since t^ℓ preserves all the congruence relations of $\mathbf{S}_\ell$. Therefore

$$h^{\mathbf{T}_j}(b_{1j}, b_{3j}, b_{3j}, \ldots, b_{3j}) = b_{1j} = p^{\mathbf{T}_j}(b_{1j}, b_{3j}, b_{3j}).$$

A similar argument applies if $b_{1j} = b_{2j}$ or $b_{1j} = b_{3j}$. So p is a Pixley term for $\mathcal{V}$ and thus $\mathcal{V}$ is congruence permutable and congruence distributive.

Next, suppose $v, w \in U$ with $\mathbf{Alg}(v) = \mathbf{Alg}(w) = \mathbf{S}_i$ for some $1 \leq i \leq k$. For all $a \in \mathbf{Alg}(v)$ we have $(a, a) \in \Omega(\mathbf{Alg}(v), \mathrm{Con}\,\mathbf{S}_i, U_i)|_{\{v,w\}}$ by Lemma 3.8(2). But $\mathbf{Ge}(X, U_i) = \Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i)$ since (1) holds. Thus

$$0_{\mathbf{Alg}(v)} \subseteq \mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw}) = \Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i)|_{\{v,w\}}.$$

If $\mathbf{Alg}(v) \neq \mathbf{Alg}(w)$ with say $v \in U_i$ and $w \in U_j$ for $i \neq j$, and if $a \in S_i$ and $b \in S_j$ are arbitrary, then by Lemma 3.8(2) there is a $c^a \in \Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i) = \mathbf{Ge}(X, U_i)$ and a $c^b \in \Omega(\mathbf{S}_j, \mathrm{Con}\,\mathbf{S}_j, U_j) = \mathbf{Ge}(X, U_j)$ with $c^a|_v = a$ and $c^b|_w = b$. Then $(c^a, c^b) \in \mathbf{Ge}(X, U_i) \times \mathbf{Ge}(X, U_j)$. We have $\mathbf{Ge}(X, U_i \cup U_j) = \mathbf{Ge}(X, U_i) \times \mathbf{Ge}(X, U_j)$ since condition (1) holds. Thus $(a, b) \in \mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw})$.

We next assume that condition (2) holds: $\mathcal{V}$ is congruence permutable and congruence distributive and U is such that for all $v, w \in U$ the subalgebra $\mathrm{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw})$ behaves as in the statement of the Theorem. So $\mathcal{V}$ is arithmetical and therefore has a ternary majority term. It is known (e.g., Theorem 1.2.13 in [11]) that if an algebra $\mathbf{A}$ is in a congruence permutable variety and if B is a subuniverse of $\mathbf{A}^2$ with $0_A \subseteq B$, then B is a congruence relation of $\mathbf{A}$. Thus, for $v, w \in U_i \subseteq U$ we have $\mathrm{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw}) = \mathrm{Cg}^{\mathbf{Alg}(v)}(\overline{vw})$. Then $\mathrm{Ge}(X, U_i) = \Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i)$ by Lemma 5.2 and the hypothesis that $\mathcal{V}$ has a majority term. Now by hypothesis, for $v \in U_i$ and $w \in U_j$ with $i \neq j$ we have $\mathrm{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw}) = \mathbf{S}_i \times \mathbf{S}_j$. Thus Lemma 5.1(2) applies to prove that

$$\mathbf{Ge}(X, U) = \mathbf{Ge}(X, U_1) \times \cdots \times \mathbf{Ge}(X, U_k) = \prod_{i=1}^{k} \Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i).$$

Finally, by combining this equality with the fact that $|\mathbf{L}(n, \mathcal{S}_n)| \geq |\mathbf{Ge}(X, U)|$, we have equality in (5.1). $\square$

Remark 5.4. The condition (2) in Theorem 5.3 can be replaced by:
(2') $\mathcal{V}$ has a majority term and for all $v, w \in U$,

$$\mathrm{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw}) = \begin{cases} \mathrm{Cg}^{\mathbf{Alg}(v)}(\overline{vw}) & \text{if } \mathbf{Alg}(v) = \mathbf{Alg}(w); \\ \mathbf{Alg}(v) \times \mathbf{Alg}(w) & \text{if } \mathbf{Alg}(v) \neq \mathbf{Alg}(w). \end{cases}$$

For we have (1) implies (2') since (2) implies (2'). That (2') implies (1) can be argued by suitably modifying the final paragraph of the proof of Theorem 5.3.

6 The computational condition

For the inequality

$$|\mathbf{L}(n, \mathcal{S}_n)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i)|, \tag{6.1}$$

presented in Theorem 5.3, a natural question is if there is a minimal transversal of $(\mathrm{val}(X, \mathcal{S}_n), \preceq)$ for which equality holds. Although any two such minimal transversals U and U' will be $\sim$-equivalent by Lemma 2.5 and will have $|\mathbf{Ge}(X,U)| = |\mathbf{Ge}(X,U')|$ by virtue of Corollary 2.9, the values of $|\Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i)|$ and $|\Omega(\mathbf{S}_i, \operatorname{Con}\mathbf{S}_i, U_i')|$ can be different. The computational condition we present is a set of numerical conditions for a given n, $\mathcal{S}_n$, and minimal transversal U that are based on the values of some basic parameters involving the algebras in $\mathcal{S}_n$ and the valuations in U, which, if satisfied, imply that equality holds in (6.1). The next result provides a method to select a minimal transversal W such that if equality were to hold in (6.1) for a minimal transversal U, then it must hold for W as well. Thus, there exists a minimal transversal U for which equality holds in (6.1) if and only if equality holds in (6.1) for the minimal transversal W.

Theorem 6.1. *Suppose* $\mathbf{A}$ *is a finite algebra with* $\operatorname{Con}\mathbf{A}$ *a chain*

$$1_A = \theta_0 \succ \theta_1 \succ \cdots \succ \theta_m \succ \theta_{m+1} = 0_A.$$

Let W' *be any* $\sim$*-independent subset of* $\operatorname{val}(X, \mathbf{A})$*. Then a set* $W \subseteq \operatorname{val}(X, \mathbf{A})$ *can be constructed such that* $W \sim W'$ *and for all* $0 \leq q \leq m+1$*, for all* $w, w' \in W$*, and for all* $u, u' \in \operatorname{val}(X, \mathbf{A})$ *if* $u \sim w$ *and* $u' \sim w'$ *with* $u \equiv_q u'$*, then* $w \equiv_q w'$*.*

Proof. We work with orbits of the automorphism group $\operatorname{Aut}\mathbf{A}$ acting on $\operatorname{val}(X, \mathbf{A})$. So let $\mathcal{O}$ be the set of such orbits. Each orbit is an equivalence class of the relation $\sim$. Consider a binary relation $\equiv_\ell$ defined on $\mathcal{O}$ by $O \equiv_\ell O'$ if and only if there exist valuations $u \in O$ and $u' \in O'$ such that $u \equiv_\ell u'$. It is easily checked using Remark 3.5(3) that the relation $\equiv_\ell$ on $\mathcal{O}$ is an equivalence.

The proof will consist of a construction of successive $\sim$-independent sets $W_{m+1}, W_m, \ldots, W_0$ with the properties that

- $W_{m+1} = W'$;
- $W_{\ell+1} \sim W_\ell$ for all $m \geq \ell \geq 0$;
- if $W_\ell = \{w_1, \ldots, w_p\}$ with $w_r \in O_r \subseteq \mathcal{O}$ for $1 \leq r \leq p$, then for all $q \geq \ell$ and $1 \leq r, s \leq p$, if $O_r \equiv_q O_s$, then $w_r \equiv_q w_s$.

Thus, W_0 is the desired set W.

For $\ell = m+1$ let $W_\ell = W' = \{w_1, \ldots, w_p\}$ with $w_r \in O_r$ for $1 \leq r \leq p$. For valuations u and v we have $u \equiv_{m+1} v$ if and only if $u = v$. It follows that if $O_r \equiv_{m+1} O_s$, then $O_r \cap O_s \neq \emptyset$, and thus $r = s$ and so $w_r \equiv_{m+1} w_s$.

Next, let $\ell < m+1$. Suppose $W_{\ell+1} = \{u_1, \dots, u_p\}$ with $u_r \in O_r$ for all $1 \le r \le p$ is such that for all $q \ge \ell+1$ and for all $1 \le r, s \le p$, if $O_r \equiv_q O_s$, then $u_r \equiv_q u_s$. We construct W_ℓ from $W_{\ell+1}$.

Note that if $q \ge \ell$ and if $O_r \equiv_q O_s$, then $O_r \equiv_\ell O_s$ since $\equiv_q \subseteq \equiv_\ell$. So to construct W_ℓ we may consider each equivalence class C of $\equiv_\ell$ on the set $\mathcal{O}$ of orbits separately.

Let C be an arbitrary $\equiv_\ell$-class of $\mathcal{O}$ and let $B_1, \dots, B_t$ be the $\equiv_{\ell+1}$ classes contained in C. For each i, with $1 \le i \le t$, let $B_i = \{O_{i_1}, \dots, O_{i_{k_i}}\}$. Then $O_r \equiv_{\ell+1} O_s$ for all $O_r, O_s \in B_i$. So $u_r \equiv_{\ell+1} u_s$ by hypothesis. For each r with $O_r \in B_1$, we define $w_r \in W_\ell$ to be u_r. Thus, for all $q \ge \ell$ and for all $O_r, O_s, \in B_1$, if $O_r \equiv_q O_s$, then $w_r \equiv_q w_s$. Without loss of generality we assume $O_1 \in B_1$.

Consider an $\equiv_{\ell+1}$-class B_j, for $2 \le j \le t$, and choose an orbit $O_{j_0} \in B_j$. Since $O_1 \equiv_\ell O_{j_0}$, there exist valuations $y \in O_1$ and $z \in O_{j_0}$ such that $y \equiv_\ell z$. Let $\beta \in \operatorname{Aut} \mathbf{A}$ be such that $\beta(y) = w_1$. Since both y and w_1 are in the same orbit O_1, such an automorphism exists. Then $\beta(z) \equiv_\ell \beta(y) = w_1$ by Remark 3.5(3). Both $\beta(z)$ and u_{j_0} are in O_{j_0} so there exists $\alpha \in \operatorname{Aut} \mathbf{A}$ such that $\alpha(u_{j_0}) = \beta(z)$. For each $O_s \in B_j$ let w_s denote $\alpha(u_s)$. Since all $u_s \in O_s \in B_j$ are $\equiv_{\ell+1}$ related, it follows that all the w_s for $O_s \in B_j$ are $\equiv_{\ell+1}$ related by virtue of Remark 3.5(3). However, $w_1 \equiv_\ell \beta(z) = \alpha(u_{j_0}) \equiv_{\ell+1} \alpha(u_s) = w_s$ for all $O_s \in B_j$. Therefore $w_1 \equiv_\ell w_s$. Hence all the $w_s \in O_s \in C$ are $\equiv_\ell$ related. Let W_ℓ be the set of all w_s constructed in this way as C ranges over the $\equiv_\ell$ classes of $\mathcal{O}$. The set W_ℓ is clearly a transversal with respect to $\sim$ of $\operatorname{val}(X, \mathbf{A})$. Moreover, if $w_r, w_s \in W_0$ and if $u \sim w_r$ and $u' \sim w_s$ with $u \equiv_q u'$ for a $q \ge 0$, then $O_r \equiv_q O_s$ by definition and hence $w_r \equiv_q w_s$ by construction. □

One property of the $\sim$-independent set W constructed in Theorem 6.1 is that for all $w_r, w_s \in W$, if $u \sim w_r$ and $v \sim w_s$, then $\operatorname{Cg}^{\mathbf{A}}(\overline{w_r w_s}) \le \operatorname{Cg}^{\mathbf{A}}(\overline{uv})$. Indeed, if $\operatorname{Cg}^{\mathbf{A}}(\overline{uv}) = \theta_\ell$, then $u \equiv_\ell v$ and hence $w_r \equiv_\ell w_s$ and $\operatorname{Cg}^{\mathbf{A}}(\overline{w_r w_s}) \le \theta_\ell$.

This property motivates the next definition and corollary.

Definition 6.2. *An $\sim$-independent set W contained in* $\operatorname{val}(X, \mathbf{A})$ *is called* congruence-reduced *if for all $w, w' \in W$ and for all $u, u' \in \operatorname{val}(X, \mathbf{A})$ if $u \sim w$ and $u' \sim w'$, then $\operatorname{Cg}^{\mathbf{A}}(\overline{ww'}) \le \operatorname{Cg}^{\mathbf{A}}(\overline{uu'})$.*

Corollary 6.3. *Suppose K is a finite set of finite algebras with each algebra in K having a congruence lattice that is a chain. Then there exists a minimal* *transversal with respect to* $\preceq$ *of* $(\mathrm{val}(X, K), \preceq)$ *that is congruence-reduced.*

Proof. Without loss of generality the algebras in K are pairwise nonisomorphic. Let U be any minimal transversal of $(\mathrm{val}(X, K), \preceq)$. For each $\mathbf{A} \in K$ let $U_{\mathbf{A}}$ denote $\{u \in U \mid \mathbf{Alg}(u) = \mathbf{A}\}$. Each nonvoid $U_{\mathbf{A}}$ is an $\sim$-independent subset of $\mathrm{val}(X, \mathbf{A})$. For each such $U_{\mathbf{A}}$ let $W_{\mathbf{A}}$ be constructed from $U_{\mathbf{A}}$ as in Theorem 6.1. Then $U_{\mathbf{A}} \sim W_{\mathbf{A}}$ and $W_{\mathbf{A}}$ is congruence-reduced. Let W denote the union of all the $W_{\mathbf{A}}$. Then W is a minimal transversal of $(\mathrm{val}(X, K), \preceq)$ since $U \sim W$. Moreover, W is congruence-reduced since each $W_{\mathbf{A}}$ is. $\square$

7 Main theorem

The following is our extension of the Primal Cluster Theorem presented in the Introduction.

Theorem 7.1. *Suppose* $\mathcal{V}$ *is a locally finite variety,* $X = \{x_1, \ldots, x_n\}$, *and* $\mathcal{S}_n = \{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ *is a transversal with respect to* $\cong$ *of the* n*-generated subdirectly irreducible algebras in* $\mathcal{V}$. *Let* U *be a minimal transversal of the quasi-ordered set* $(\mathrm{val}(X, \mathcal{S}_n), \preceq)$. *For each* $1 \leq i \leq k$, *let* $U_i = \{u \in U \mid \mathbf{Alg}(u) = \mathbf{S}_i\}$. *Then*

$$\mathbf{F}_{\mathcal{V}}(n) \cong \mathbf{L}(n, \mathcal{S}_n) \cong \mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U_1 \cup \cdots \cup U_k) \lhd \prod_{i=1}^{k} \mathbf{Ge}(X, U_i). \tag{7.1}$$

If C_i, *for each* i, *is a linearly ordered subset of* $\mathrm{Con}\,\mathbf{S}_i$ *containing both* 0_{S_i} *and* 1_{S_i}, *then*

$$|\mathbf{L}(n, \mathcal{S}_n)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, C_i, U_i)|, \tag{7.2}$$

and thus if each congruence lattice $\mathrm{Con}\,\mathbf{S}_i$ *is linearly ordered, then*

$$|\mathbf{L}(n, \mathcal{S}_n)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \mathrm{Con}\,\mathbf{S}_i, U_i)|. \tag{7.3}$$

Moreover, if each $\mathrm{Con}\,\mathbf{S}_i$ *is linearly ordered and* $n \geq 3$, *then the first three of the following conditions are equivalent and are implied by the fourth.*

1. **(bound obtained):** *The variety* $\mathcal{V}$ *and the minimal transversal* U *are such that equality holds in (7.3).*

2. **(interpolation):** *If given, for each $1 \leq i \leq k$, an n-ary partial operation $h_i\colon S_i^n \to S_i$ with $\operatorname{supp}(h_i) = U_i$ such that h_i preserves every congruence relation of $\mathbf{S}_i$, then there exists an n-ary term t in the language of the $\mathbf{S}_i$ such that $t^{\mathbf{S}_i}(u) = h_i u)$ for every $1 \leq i \leq k$ and $u \in U_i$.*

3. **(algebraic):** *$\mathcal{V}$ is congruence permutable and congruence distributive and U has the property that for all $v, w \in U$*

$$\operatorname{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw}) \supseteq \begin{cases} 0_{\mathbf{Alg}(v)} & \text{if } \mathbf{Alg}(v) = \mathbf{Alg}(w); \\ \mathbf{Alg}(v) \times \mathbf{Alg}(w) & \text{if } \mathbf{Alg}(v) \neq \mathbf{Alg}(w). \end{cases}$$

4. **(computational):** *Suppose $X' = \{x_1, x_2, x_3\}$ with $\mathcal{S}_3 = \{\mathbf{S}'_1, \ldots, \mathbf{S}'_{k'}\}$ a transversal with respect to $\cong$ of the 3-generated subdirectly irreducible algebras in $\mathcal{V}$. Let V be any minimal transversal of the quasi-ordered set $(\operatorname{val}(X', \mathcal{S}_3), \preceq)$ that is congruence-reduced. For each $1 \leq i \leq k'$ let $V_i = \{v \in V \mid \mathbf{Alg}(v) = \mathbf{S}'_i\}$. Suppose that the minimal transversal U is congruence-reduced. Then for this V and this U*

$$|\mathbf{Ge}(\{x_1, x_2, x_3\}, V)| = \prod_{i=1}^{k'} |\Omega(\mathbf{S}'_i, \operatorname{Con}\mathbf{S}'_i, V_i)| \tag{7.4}$$

and for all $v, w \in U$

$$|\operatorname{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw})| = \begin{cases} |\operatorname{Cg}^{\mathbf{Alg}(v)}(\overline{vw})| & \text{if } \mathbf{Alg}(v) = \mathbf{Alg}(w); \\ |\mathbf{Alg}(v) \times \mathbf{Alg}(w)| & \text{if } \mathbf{Alg}(v) \neq \mathbf{Alg}(w). \end{cases} \tag{7.5}$$

Proof. The chain of isomorphisms in (7.1) is Remark 2.11. The embedding in (7.1) is immediate since the U_i are pairwise disjoint. The inequality (7.2) follows from Remark 2.11 and Corollary 3.9. The inequality (7.3) is from (7.2). It remains to prove the 'Moreover'.

The equivalence of (1) and (2) is Theorem 4.1. The equivalence of (1) and (3) is Theorem 5.3.

We complete the proof by showing that if the computational condition (4) holds, then the algebraic condition (3) follows. From Corollary 6.3 we know that the congruence-reduced minimal transversals U and V that appear in condition (4) can always be constructed. We have $\mathcal{S}_3 \subseteq \mathcal{S}_n$ since $n \geq 3$. So every $\mathbf{S}'_i \in \mathcal{S}_3$ has a congruence lattice that is a finite linearly ordered set. If the equality (7.4) holds for V, then it follows that equality holds in the inequality (7.3) when $n = 3$. Thus $\mathcal{V}$ is congruence permutable and congruence distributive by virtue of the equivalence for $n = 3$ of the bound obtained condition (1) and the algebraic condition (3). If the equality in (7.5) holds for U, then the second part of the algebraic condition (3) holds for all $v, w \in U$ since the set $\mathrm{Sg}^{\mathbf{Alg}(v)\times\mathbf{Alg}(w)}(\overline{vw})$ is always contained in $\mathbf{Alg}(v)\times\mathbf{Alg}(w)$ and it is also contained in $\mathrm{Cg}^{\mathbf{Alg}(v)}(\overline{vw})$ if $\mathbf{Alg}(v) = \mathbf{Alg}(w)$. Then the equality of cardinalities implies the equality of the sets since the sets are all finite. Thus the algebraic condition (3) holds for $\mathcal{V}$ and all $v, w \in U$ since $0_{\mathbf{Alg}(v)}$ is contained in every congruence relation on $\mathbf{Alg}(v)$. $\square$

Remark 7.2. If the bound obtained condition (1) holds for every n, then all four conditions in Theorem 7.1 are equivalent since in this case the two transversals U and V of the computational condition (4) will exist and have the desired properties.

8 An Example

This section illustrates for a particular 3-element algebra $\mathbf{A}$ and for the variety $\mathsf{V}(\mathbf{A})$ generated by $\mathbf{A}$ the computation of upper bounds for $|\mathbf{L}(n, \mathbf{A}|$ and $|\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)|$ by means of the formula for $|\Omega(\mathbf{A}, \mathrm{Con}\,\mathbf{A}, \mathrm{val}(X, \mathbf{A}))|$. The section also illustrates how the interpolation, algebraic, and computational conditions of Theorem 7.1 may be applied to show that these upper bounds are obtained for this $\mathbf{A}$.

Let $A = \{0, 1, 2\}$ and suppose that θ is the equivalence relation given by the partition $01|2$, B is the subset $\{2\}$, and α is the permutation $(01)(2)$ that interchanges 0 and 1 and leaves 2 fixed. We investigate $\Omega(\mathbf{A}, \mathrm{Con}\,\mathbf{A}, U)$ if $\mathbf{A}$ were an algebra with universe A, with congruence lattice $\{1_A \succ \theta \succ 0_A\}$, with subalgebra lattice $\{\mathbf{A} \succ \mathbf{B}\}$, with $\mathrm{Aut}\,\mathbf{A} = \{\iota, \alpha\}$, and with U a transversal with respect to $\sim$ of $\mathrm{val}(X, \mathbf{A})$.

It is easily seen that if $|X| = n$, then $\mathrm{val}(X, \mathbf{A}) = A^X \setminus B^X$ and that the cardinality of the transversal U is $(3^n - 1)/2$. In the notation of Definition 3.1 we have $r(\epsilon) = 2$. Let $A(1) = \{0, 1\}$ and $A(2) = \{2\}$. Here $m = 1$ and we write θ for θ_1. The value of $s(\epsilon)$ is the number of n-tuples $(A(p_1), \ldots, A(p_n))$ with each $p_q \in \{1, 2\}$ and with at least one p_q not 2. Thus, $s(\epsilon) = 2^n - 1$. Then $U(1), \ldots, U(2^n - 1)$ are the $\equiv_1$ equivalence classes in U. If, say, $U(j) \subseteq A(p_1) \times \cdots \times A(p_n)$, then $|U(j)| = 2^e 1^{n-e}/2 = 2^{e-1}$, where e is the number of p_q that have value 1.

With these values for $r(\epsilon), s(\epsilon), |A(i)|$ and $|U(j)|$ we have from Remark 3.2 that

$$|\Omega(\mathbf{A}, \operatorname{Con}\mathbf{A}, U)| = \prod_{j=1}^{s(\epsilon)} \sum_{i=1}^{r(\epsilon)} |A(i)|^{|U(j)|} = \prod_{j=1}^{2^n-1} \sum_{i=1}^{2} |A(i)|^{|U(j)|} = \prod_{e=1}^{n} (2^{2^{e-1}}+1)^{\binom{n}{e}} \tag{8.1}$$

where $\binom{n}{e}$ is the number of $U(j) \subseteq \prod_{q=1}^{n} A(p_q)$ that have e coordinates that are in $A(1)$.

If $\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)$ denotes the free algebra on n free generators for the variety $\mathsf{V}(\mathbf{A})$ generated by the algebra $\mathbf{A}$, then $|\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)| = |\mathbf{L}(n, \mathbf{A})|$ since the free algebras in $\mathsf{V}(\mathbf{A})$ are subdirect products of $\mathbf{A}$ and the proper nontrivial subalgebras of $\mathbf{A}$. From formula (8.1) and Corollary 3.11 we have

$$|\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)| = |\mathbf{L}(n, \mathbf{A})| \leq \prod_{e=1}^{n} (2^{2^{e-1}} + 1)^{\binom{n}{e}}. \tag{8.2}$$

A question which remains is whether there exists an algebra $\mathbf{A}$ with universe $\{0, 1, 2\}$, $\operatorname{Con}\mathbf{A} = \{0_A, \theta, 1_A\}$, $\operatorname{Sub}\mathbf{A} = \{\mathbf{B}, \mathbf{A}\}$, and $\operatorname{Aut}\mathbf{A} = \{\iota, \alpha\}$. And if there is such an algebra for which the upper bound of (8.1) is obtained.

It is easily checked that the algebra $\mathbf{A}_0 = \langle\{0, 1, 2\}, u_1, u_2\rangle$ in which u_1 is the unary operation α and u_2 is the unary constant operation with value 2 has the desired congruence lattice, subalgebra lattice, and automorphism group. But, of course, the upper bound in (8.1) is far too large for this $\mathbf{A}_0$. A natural candidate for an algebra $\mathbf{A}$ that obtains this upper bound would be the algebra whose term operations are all finitary operations on $\{0, 1, 2\}$ that preserve θ, B, and α. This $\mathbf{A}$ would be an expansion of $\mathbf{A}_0$ and thus would have the desired congruence lattice, subalgebra lattice, and automorphism group. We show that the upper bound of (7.1) is indeed obtained for $\mathsf{V}(\mathbf{A})$ when the n-ary term operations of $\mathbf{A}$ are all n-ary operations on A that preserve θ, B, and α.

Our arguments do not always make full use of the very simple form of $\theta, \mathbf{B}$, and α in this example since we wish to illustrate techniques that may be used for more complex algebras.

One method to show that equality holds in (8.2) for this algebra $\mathbf{A}$ is to apply Theorem 4.1 with $k = 1$ and U a transversal with respect to $\sim$ of $\operatorname{val}(X, \mathbf{A})$. We sketch an argument to show that the interpolation condition (2) of this theorem holds. To this end let h be an n-ary partial operation on A with $\operatorname{supp}(h) = U$ that preserves θ. To show that condition (2) holds it suffices to find an n-ary operation f on A that preserves θ, B, and α and for which $f(u) = h(u)$ for all $u \in U$. For each $\overline{a} \in A^n$ define f by

$$f(\overline{a}) = \begin{cases} h(\overline{a}) & \text{if } \overline{a} \in U; \\ \alpha(h(\alpha^{-1}(\overline{a}))) & \text{if } \overline{a} \in \alpha(U); \\ 2 & \text{if } \overline{a} \in B^n. \end{cases}$$

Since A^X is the disjoint union of $\mathrm{val}(X, \mathbf{A})$ and B^X, and $\mathrm{val}(X, \mathbf{A})$ is the disjoint union of U and $\alpha(U)$, it is immediate that f is an n-ary operation on A that preserves B. A straightforward argument that f preserves θ can be given using the facts that in this example B is an equivalence class of θ and $(a, \alpha(a)) \in \theta$ for all $a \in A$. Likewise, an easy argument that f preserves α can be given using the facts that $\alpha(B) = B$ and $\alpha^2 = \iota$. Thus, for this $\mathbf{A}$ the inequality in (8.2) may be replaced with equality by virtue of Theorem 4.1.

Another way to show that equality holds in (8.2) is to argue that the algebraic condition (2) of Theorem 5.3 holds for this $\mathbf{A}$ with $n \geq 3$. The Pixley operation displayed immediately before Theorem 5.3 preserves the congruence relation θ, the automorphism α, and every subset of A. Hence it is a term operation of $\mathbf{A}$. So it is a Pixley term for $\mathbf{A}$ and for the variety $\mathcal{V} = \mathsf{HSP}\mathbf{A}$. Thus $\mathcal{V}$ is congruence permutable and congruence distributive. The only subdirectly irreducible algebras in $\mathcal{V}$ are $\mathbf{A}$ and $\mathbf{A}/\theta$ by Jónsson's Theorem. It is easily seen that for every valuation w on $\mathbf{A}/\theta$ there is a valuation $u \in \mathrm{val}(X, \mathbf{A})$ such that $u \preceq w$. So a minimal transversal U with respect to $\preceq$ of $\mathrm{val}(X, \{\mathbf{A}, \mathbf{A}/\theta\})$ is any transversal with respect to $\sim$ of $\mathrm{val}(X, \mathbf{A})$.

To complete the argument we show that $(c, c) \in \mathrm{Sg}^{\mathbf{A}^2}(\overline{vw})$ for every $c \in A$ and $v, w \in U$. For each $c \in A$ consider an operation $f_c \colon A^n \to A$ given by

$$f_c(\overline{a}) = \begin{cases} \delta(c) \text{ if } \overline{a} \in \delta(U) \text{ for a } \delta \in \mathrm{Aut}\,\mathbf{A}; \\ a_1 \text{ if } \overline{a} \neq u \text{ for all } u \in \mathrm{val}(X, \mathbf{A}). \end{cases}$$

Each f_c is indeed an n-ary operation since, by Remark 3.5(5), $\mathrm{val}(X, \mathbf{A}) = \bigcup_{\delta \in \mathrm{Aut}\,\mathbf{A}} \delta(U)$ and $A^X = \mathrm{val}(X, \mathbf{A}) \cup B^X$ are both disjoint unions. Each f_c preserves θ since in $\mathbf{A}$ the subuniverse B is a congruence class of θ and $\delta(a)/\theta = a/\theta$ for all $a \in A$ and $\delta \in \mathrm{Aut}\,\mathbf{A}$. It is immediate that f_c preserves subuniverses and automorphisms of $\mathbf{A}$. So f_c is a term operation of $\mathbf{A}$. Hence $(c, c) \in \mathrm{Sg}^{\mathbf{A}^2}(\overline{vw})$ for all $v, w \in U$. Therefore the algebraic condition of Theorem 7.1 holds for this $\mathbf{A}$, and in particular, equality holds in (8.2).

We next use computational methods to explicitly construct an algebra $\mathbf{C}$ with universe $C = \{0,1,2\}$, $\operatorname{Con}\mathbf{C} = \{0_A, \theta, 1_A\}$, $\operatorname{Sub}\mathbf{C} = \{\mathbf{B}, \mathbf{A}\}$, $\operatorname{Aut}\mathbf{C} = \{\iota, \alpha\}$, and having a transversal U with respect to $\sim$ of $\operatorname{val}(X, \mathbf{C})$ for which

$$|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(n)| = |\Omega(\mathbf{C}, \operatorname{Con}\mathbf{C}, U)| = \prod_{e=1}^{n} (2^{2^{e-1}} + 1)^{\binom{n}{e}}, \tag{8.3}$$

In particular, we find a small set of fundamental operations for $\mathbf{C}$. Our presentation illustrates the use of the computational condition in Theorem 7.1 and of the software package UACalculator [10].

We have observed in (8.2) that the expression in (8.3) is an upper bound. Thus $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(1)|$, $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(2)|$, and $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(3)|$ are bounded above by 3, 45, and 57,375 respectively. By inspection we can see that for the given θ, B, and α the only unary operations that could be term operations for $\mathbf{C}$ are the constant function $\overline{2}$, the identity function ι, and the permutation α. We start to build $\mathbf{C}$ by choosing $\overline{2}$ and α as its fundamental unary term operations. As already observed it is easily seen that the congruence lattice, subalgebra lattice, and automorphism group of the algebra $\langle C, \alpha, \overline{2}\rangle$ are as desired. This could also be verified using UACalculator.

All binary operations that preserve θ, B, and α must conform to this template:

$\cdot$	0	1	2
0	a	b	c
1	$\alpha(b)$	$\alpha(a)$	$\alpha(c)$
2	d	$\alpha(d)$	2

Here c and d are arbitrary elements of C and (a,b) can be any ordered pair in θ. Thus there are $3^2 \cdot 5$ possible binary term operations for $\mathbf{C}$. By selecting binary operations that conform to this template and by including them with the unary terms α and $\overline{2}$ as fundamental term operations of $\mathbf{C}$, and then using UACalculator to compute the free algebra on 2 free generators for the variety generated by $\mathbf{C}$ it is possible to find two binary operations, say $b1$ and $b2$, such that $\mathbf{C} = \langle C, \alpha, \overline{2}, b1, b2\rangle$ has $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(1)| = 3$ and $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(2)| = 45$. Finally, we include the Pixley operation $t(a_1, a_2, a_3)$ displayed immediately before Theorem 5.3 as a fundamental term operation of $\mathbf{C}$. If we attempt to compute the cardinality of the free algebra on 3 free generators for this $\mathbf{C}$, then after approximately two hours 57,375 elements are enumerated. The estimated time that UACalculator predicts for completion of this computation is impossibly long. However, since 57,375 is also the upper bound on the size of $\mathbf{F}_{\mathsf{V}(\mathbf{C})}(3)$ we can conclude for the algebra $\mathbf{C} = \langle C, \alpha, \overline{2}, b1, b2, t\rangle$ that $|\mathbf{F}_{\mathsf{V}(\mathbf{C})}(3)| = |\mathbf{Ge}(\{x_1, x_2, x_3\}, U)| = |\Omega(\mathbf{C}, \operatorname{Con}\mathbf{C}, U)|$. Thus (7.4) in the computational condition of Theorem 7.1 holds for the variety $\mathsf{V}(\mathbf{C})$. In order to show that (7.5) holds it suffices to show that every subalgebra of $\mathbf{C}^2$ that has cardinality greater than $|C|$ is a congruence relation of $\mathbf{C}$. This is easily checked using UACalculator. Thus, the computational condition of Theorem 7.1 holds for the variety generated by $\mathbf{C}$ and for all values of $n \geq 3$. Therefore, all four conditions of that theorem hold. We conclude that $|\mathbf{L}(n, \{\mathbf{C}, \mathbf{C}/\theta\})| = |\mathbf{F}_{\mathsf{V}(\mathbf{C})}(n)| = |\Omega(\mathbf{C}, \operatorname{Con}\mathbf{C}, U)| = \prod_{e=1}^{n}(2^{2^{e-1}} + 1)^{\binom{n}{e}}$ for all n.

9 Applications

Throughout this section $\mathbf{A}$ is a finite algebra and C is a linearly ordered subset of $\operatorname{Con}\mathbf{A}$ that contains 0_A and 1_A. The sets $U \subseteq \operatorname{val}(X, \mathbf{A})$ and $\Omega(\mathbf{A}, C, U) \subseteq A^U$ are as in Definition 3.1. We consider methods for determining the cardinality of $\Omega(\mathbf{A}, C, U)$ and we present some applications of Theorem 7.1.

We first provide a version of Theorem 7.1 for semisimple varieties. Next come formulas based on inclusion-exclusion arguments for determining the cardinality of $\Omega(\mathbf{A}, C, U)$. We then consider in detail the case of an arbitrary finite algebra that is rigid, has no proper subalgebras, and has a congruence lattice that is a chain. The section concludes with an analysis of $\mathbf{L}(n, \mathbf{A})$ and $|\Omega(\mathbf{A}, \operatorname{Con}\mathbf{A}, \operatorname{val}(X, \mathbf{A}))|$ when $\mathbf{A}$ is congruence uniform.

9.1 Simple algebras and semisimple varieties

We call a variety *n-semisimple* if every n-generated subdirectly irreducible algebra in it is simple. A variety is *semisimple* if every subdirectly irreducible algebra in it is simple.

Proposition 9.1. *Suppose $|X| = n$ and $\mathbf{A}$ is a finite algebra with U a transversal with respect to $\sim$ of* $\operatorname{val}(X, \mathbf{A})$. *Then*

$$|\mathbf{Ge}(X, U)| = |\mathbf{L}(n, \mathbf{A})| \leq |\Omega(\mathbf{A}, \{0_A, 1_A\}, U)| = |A|^{|U|} = |A|^{\frac{|\operatorname{val}(X, \mathbf{A})|}{|\operatorname{Aut}\mathbf{A}|}},$$

and $\mathbf{Ge}(X, U)$ *is subdirectly embedded in* $\Omega(\mathbf{A}, \{0_A, 1_A\}, U)$.

Proof. The equalities and inequality follow from Remark 2.11, Corollary 3.10, Remark 3.2, and Lemma 3.5(5). $\square$

The upper bound on $\mathbf{L}(n, \mathbf{A})$ in Proposition 9.1 depends only on the cardinality of the automorphism group and on certain numerical parameters involving the lattice of subuniverses of $\mathbf{A}$. It can be argued that if equality holds in this proposition for $n = 1 + |A|$, then $\mathbf{A}$ is simple.

The next result is a version of Theorem 7.1 restricted to n-semisimple varieties. In these varieties we show that all four conditions in the main theorem are equivalent. A proof of the equivalence of the bound obtained condition (1) and the algebraic condition (3) for n-semisimple locally finite varieties has been given in [4].

Theorem 9.2. *Suppose* $\mathcal{V}$ *is a locally finite variety,* $X = \{x_1, \dots, x_n\}$, *and* $\mathcal{S}_n = \{\mathbf{S}_1, \dots, \mathbf{S}_k\}$ *is a transversal with respect to* $\cong$ *of the* n*-generated subdirectly irreducible algebras in* $\mathcal{V}$. *Let* U *be a minimal transversal of the quasi-ordered set* $(\mathrm{val}(X, \mathcal{S}_n), \preceq)$. *For each* $1 \leq i \leq k$, *let* $U_i = \{u \in U \mid \mathbf{Alg}(u) = \mathbf{S}_i\}$. *Then*

$$\mathbf{L}(n, \mathcal{S}_n) \cong \mathbf{Ge}(X, U) \cong \mathbf{Ge}(X, U_1 \cup \cdots \cup U_k) \lhd \prod_{i=1}^{k} \mathbf{Ge}(X, U_i).$$

$$|\mathbf{L}(n, \mathcal{S}_n)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \{0_{S_i}, 1_{S_i}\}, U_i)| = \prod_{i=1}^{k} |S_i|^{|U_i|}. \tag{9.1}$$

If each $\mathbf{S}_i$ *is a simple algebra, then*

$$|\mathbf{L}(n, \mathcal{S}_n)| \leq \prod_{i=1}^{k} |\Omega(\mathbf{S}_i, \mathrm{Con}\, \mathbf{S}_i, U_i)| = \prod_{i=1}^{k} |S_i|^{\frac{|\mathrm{val}(X, \mathbf{S}_i)|}{|\mathrm{Aut}\, \mathbf{S}_i|}}. \tag{9.2}$$

Moreover, if $n \geq 3$ *and* $\mathcal{V}$ *is* n*-semisimple, then the following are equivalent:*

1. **(bound obtained):** *The variety* $\mathcal{V}$ *is such that equality holds in (9.2).*
2. **(interpolation):** *If given, for each* $1 \leq i \leq k$, *an* n*-ary partial operation* $h_i \colon S_i^n \to S_i$, *with* $\mathrm{supp}(h_i) = U_i$ *then there exists an* n*-ary term* t *in the language of* $\mathcal{V}$ *such that* $t^{\mathbf{S}_i}(u) = h_i(u)$ *for every* $1 \leq i \leq k$ *and* $u \in U_i$.

3. **(algebraic):** *$\mathcal{V}$ is congruence permutable and congruence distributive.*

4. **(computational):** *Suppose $X' = \{x_1, x_2, x_3\}$ with $\mathcal{S}_3 = \{\mathbf{S}'_1, \ldots, \mathbf{S}'_{k'}\}$ a transversal with respect to $\cong$ of the 3-generated subdirectly irreducible algebras in $\mathcal{V}$. Let U' be a transversal with respect to $\sim$ of* $\mathrm{val}(X', \mathcal{S}_3)$. *Then,*

$$|\mathbf{Ge}(X', U')| = \prod_{i=1}^{k'} |S'_i|^{\frac{|\mathrm{val}(X', \mathbf{S}'_i)|}{|\mathrm{Aut}\, \mathbf{S}'_i|}}.$$

Proof. The first display is Corollary 2.10 and Remark 2.11. The inequality in (9.1) is Corollary 3.10 with $C = \{0_{S_i}, 1_{S_i}\}$. The equality is from Remark 3.2.

If each $\mathbf{S}_i$ is simple, then (9.2) follows from (9.1) and Proposition 9.1 since a minimal transversal with respect to $\preceq$ of $(\mathrm{val}(X, \mathcal{S}_n), \preceq)$ is a transversal with respect to $\sim$ of $\mathrm{val}(X, \mathcal{S}_n)$.

We use Theorem 7.1 for the proof of the 'Moreover'. (1) implies (2) as in Theorem 7.1 since every n-ary partial operation preserves 0_{S_i} and 1_{S_i}. That (2) implies (3) is immediate from the corresponding parts of Theorem 7.1. For (4) implies (3) we observe that if (4) holds, then (1) of Theorem 7.1 holds for $n = 3$. Thus (3) of Theorem 7.1 holds, which gives (3).

We conclude the proof by showing (3) implies (4). Let X', $\mathcal{S}_3$, and U' be as in the computational condition (4). Then $\mathbf{Ge}(X', U')$ is subdirectly embedded in

$$\prod_{i=1}^{k'} |S'_i|^{\frac{|\mathrm{val}(X, \mathbf{S}'_i)|}{|\mathrm{Aut}\, \mathbf{S}'_i|}}$$

as in Proposition 9.1. By hypothesis $\mathcal{V}$ is congruence permutable so by a result of Foster and Pixley [9, Theorem 2.4] for all $v \neq w \in U'$ the algebra $\mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw})$ is isomorphic to $\mathbf{Alg}(v) \times \mathbf{Alg}(w)$ or $\mathbf{Alg}(v)$ or $\mathbf{Alg}(w)$. If, say, $\mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw}) \cong \mathbf{Alg}(w)$, then by Lemma 2.6 there is homomorphism from $\mathbf{Alg}(w)$ onto $\mathbf{Alg}(v)$ mapping w to v, which is impossible since $\mathbf{Alg}(w)$ is simple and U' is a transversal with respect to $\sim$. Hence

$$\mathbf{Ge}(X', U')|_{\{v,w\}} = \mathrm{Sg}^{\mathbf{Alg}(v) \times \mathbf{Alg}(w)}(\overline{vw}) = \mathbf{Alg}(v) \times \mathbf{Alg}(w)$$

for all $v \neq w$ in U'. By hypothesis the variety $\mathcal{V}$ is arithmetical and thus has a majority term. Then

$$\mathbf{Ge}(X', U') \cong \prod_{i=1}^{k'} |S'_i|^{\frac{|\mathrm{val}(X, \mathbf{S}'_i)|}{|\mathrm{Aut}\, \mathbf{S}'_i|}}$$

by the Baker-Pixley Theorem as in Lemma 5.1. $\square$

9.2 Valuations and inclusion-exclusion

This subsection is devoted to the actual computation of the cardinality of $\Omega(\mathbf{A}, C, U)$ by means of the formula in Remark 3.2. In this formula we have $|X| = n$. The index ℓ runs from 0 to $m+1$. The four main parameters are the cardinalities of the congruence classes $A(i_1 \ldots i_\ell)$ of the congruence relation θ_ℓ; the cardinalities of the equivalence classes $U(j_1 \ldots j_s)$ of the equivalence relation $\equiv_\ell$; the number $r(i_1 \ldots r_{\ell-1})$ of θ_ℓ congruence classes contained in $A(i_1 \ldots i_{\ell-1})$; and the number $s(j_1 \ldots j_{\ell-1})$ of equivalence classes of $\equiv_\ell$ contained in $U(j_1 \ldots j_{\ell-1})$.

The congruence relations θ_ℓ, for $0 \leq \ell \leq m+1$, as well as the congruence classes $A(i_1 \ldots i_\ell)$ of each θ_ℓ are considered as given as are the cardinalities of these congruence classes. The value of each $r(i_1 \ldots i_\ell)$, which is the number of congruence classes of $\theta_{\ell+1}$ contained in the congruence class $A(i_1 \ldots i_\ell)$, is also considered as given. The total number of congruence classes of θ_0 is 1, of θ_1 is $r(\epsilon)$, and for $2 \leq \ell \leq m+1$ is easily seen to be

$$\sum_{i_1=1}^{r(\epsilon)} \sum_{i_2=1}^{r(i_1)} \cdots \sum_{i_{\ell-1}=1}^{r(i_1 \ldots i_{\ell-2})} r(i_1 \ldots i_{\ell-1}).$$

In order to consider the cardinalities of the $U(j_1 \ldots j_\ell)$ we introduce some notation for arguments involving inclusion-exclusion.

For finite sets S and $T_1, \ldots, T_k$ let $\mathcal{T}$ denote $\{T_1, \ldots, T_k\}$ and let $\mathrm{Inx}(S, \mathcal{T})$ denote the set $S \setminus \bigcup_{i=1}^{k} T_i$. The cardinality of $\mathrm{Inx}(S, \mathcal{T})$ is denoted $\mathrm{inx}(S, \mathcal{T})$. For $\emptyset \neq D \subseteq \{1, \ldots, k\}$ let $T_D = (\bigcap_{d \in D} T_d) \cap S$ and let $T_\emptyset = S$. In particular, $T_{\{d\}} = T_d \cap S$. With this notation the inclusion-exclusion formula provides the following expression for the cardinality of $\mathrm{Inx}(S, \mathcal{T})$:

$$\mathrm{inx}(S, \mathcal{T}) = \sum_{D \subseteq \{1, \ldots, k\}} (-1)^{|D|} |T_D|.$$

Thus, the value of $\mathrm{inx}(S, \mathcal{T})$ is determined by the cardinalities of S and all of the $\bigcap_{d \in D} T_d \cap S$.

In what follows, $\mathcal{M}$ denotes the set $\{M_1, \ldots, M_k\}$ of universes of maximal proper subalgebras of a finite algebra $\mathbf{A}$. Suppose $w \in W \subseteq A^X$. Then $w \in W$ is in $\mathrm{val}(X, \mathbf{A})$ if and only if $w \notin M_i^n$ for all $1 \leq i \leq k$. Let

$\mathcal{M}_n = \{M_1^n, \ldots, M_k^n\}$. Thus $W \cap \mathrm{val}(X, \mathbf{A}) = \mathrm{Inx}(W, \mathcal{M}_n)$ and the number of valuations in W is $\mathrm{inx}(W, \mathcal{M}_n)$.

Suppose $U(j_1 \ldots j_\ell) \subseteq U \subseteq \mathrm{val}(X, \mathbf{A})$ is an equivalence class of $\equiv_\ell$ and let $h \in \Omega(\mathbf{A}, C, U)_{U(j_1 \ldots j_\ell)}$. Then h is θ_ℓ-constant by Lemma 3.7 and there exist congruence classes $A(i_{p1} \ldots i_{p\ell})$ of θ_ℓ for $1 \leq p \leq n$ such that

$$\Omega(\mathbf{A}, C, U)|_{U(j_1 \ldots j_\ell)} \subseteq \prod_{p=1}^{n} A(i_{p1} \ldots i_{p\ell}) \setminus \bigcup_{q=1}^{k} M_q^X.$$

Thus an upper bound for $|U(j_1 \ldots j_\ell)|$ is $\mathrm{inx}(\prod_{p=1}^n A(i_{p1} \ldots i_{p\ell}), \mathcal{M}_n)$.

If U is all of $\mathrm{val}(X, \mathbf{A})$ and if $v \in \prod_{p=1}^n A(i_{p1} \ldots i_{p\ell})$ is a valuation, then $v \in U(j_1 \ldots j_\ell)$. In this case

$$\begin{aligned} |U(j_1 \ldots j_\ell)| = \mathrm{inx}\big(\prod_{p=1}^{n} A(i_{p1} \ldots i_{p\ell}), \mathcal{M}_n\big) = \\ \sum_{D \subseteq \{1, \ldots k\}} (-1)^{|D|} \big|\big(\bigcap_{d \in D} M_d^n\big) \bigcap \prod_{p=1}^{n} A(i_{p1} \ldots i_{p\ell})\big| \\ = \sum_{D \subseteq \{1, \ldots k\}} (-1)^{|D|} \prod_{p=1}^{n} \big|\big(\bigcap_{d \in D} M_d\big) \bigcap A(i_{p1} \ldots i_{p\ell})\big|. \end{aligned} \tag{9.3}$$

We now consider for $U \subseteq \mathrm{val}(X, \mathbf{A})$ the number $s(j_1 \ldots j_\ell)$ of $\equiv_{\ell+1}$ classes contained in $U(j_1 \ldots j_\ell)$. Suppose $U(j_1 \ldots j_\ell) \subseteq \prod_{p=1}^n A(i_{p1} \ldots i_{p\ell})$. The $\equiv_{\ell+1}$ equivalence classes in $U(j_1 \ldots j_\ell)$ are $U(j_1 \ldots j_\ell j_{\ell+1})$ for $1 \leq j_{\ell+1} \leq s(j_1 \ldots j_\ell)$. For each p, with $1 \leq p \leq n$, there exists $i_{p,\,\ell+1}$ with $1 \leq i_{p,\,\ell+1} \leq r(i_{p1} \ldots i_{p\ell})$ such that $U(j_1 \ldots j_\ell j_{\ell+1}) \subseteq \prod_{p=1}^n A(i_{p1} \ldots i_{p\ell} i_{p,\,\ell+1})$. Thus

$$s(j_1 \ldots j_\ell) \leq \prod_{p=1}^{n} r(i_{p1} \ldots i_{p\ell}).$$

This upper bound can be improved upon by including in the count only those $\prod_{p=1}^n A(i_{p1} \ldots i_{p\ell}\, i_{p,\,\ell+1})$ that contain at least one valuation. That is, those for which $\mathrm{inx}(\prod_{p=1}^n A(i_{p1} \ldots i_{p\ell}\, i_{p,\,\ell+1}), \mathcal{M}_n) \geq 1$. Let sgn denote the signum function, that is, $\mathrm{sgn}(x) = 1, 0,$ or -1 if x is positive, zero, or negative, respectively. Suppose $U(j_1 \ldots j_\ell) \subseteq \prod_{p=1}^n A(i_{p1} \ldots i_{p\ell})$. Then

$$s(j_1 \ldots j_\ell) \leq \sum_{\substack{(i_{1,\,\ell+1}, \ldots, i_{n,\,\ell+1}) \\ 1 \leq i_{p,\,\ell+1} \leq r(i_{p1} \ldots i_{p\ell})}} \mathrm{sgn}\Big(\mathrm{inx}\big(\prod_{p=1}^{n} A(i_{p1} \ldots i_{p\ell} i_{p,\,\ell+1}), \mathcal{M}_n\big)\Big). \tag{9.4}$$

In the event that $U = \mathrm{val}(X, \mathbf{A})$ the inequality in this expression becomes an equality.

An upper bound for the cardinality of $\Omega(\mathbf{A}, C, U)$ may be obtained using the upper bounds on $U(j_1 \ldots j_\ell)$ and $s(j_1 \ldots j_\ell)$ given above. But further sharpening of this upper bound would, in general, require additional details about the set $U \subseteq \mathrm{val}(X, \mathbf{A})$. However, if U consists of all of $\mathrm{val}(X, \mathbf{A})$, then the value of $|U(j_1 \ldots j_m)|$ is presented in (9.3) and the value of $s(j_1 \ldots j_\ell)$ is given by the upper bound in (9.4). Together, they provide an expression for the cardinality of $\Omega(\mathbf{A}, C, \mathrm{val}(X, \mathbf{A}))$ when used in (3.1) of Remark 3.2.

As an example, consider the case of a finite rigid algebra $\mathbf{A}$ with $\mathrm{Con}\,\mathbf{A} = \{1_A = \theta_0 > \theta_1 > \theta_2 = 0_A\}$. Suppose $\mathcal{M} = \{M_1, \ldots, M_k\}$ is the set of universes of the maximal proper subalgebras of $\mathbf{A}$. A transversal with respect to $\sim$ of $\mathrm{val}(X, \mathbf{A})$ is all of $\mathrm{val}(X, \mathbf{A})$ since $\mathbf{A}$ is rigid. Then

$$|\mathbf{L}(n, \mathbf{A})| \leq |\Omega(\mathbf{A}, \mathrm{Con}\,\mathbf{A}, \mathrm{val}(X, \mathbf{A}))| = \prod_{j_1=1}^{s(\epsilon)} \sum_{i_1=1}^{r(\epsilon)} |A(i_1)|^{|U(j_1)|} =$$

$$\prod_{\substack{(q_1,\ldots,q_n)\in\{1,\ldots,r(\epsilon)\}^n \\ \mathrm{inx}(\prod_{p=1}^n A(q_p),\mathcal{M}_n)>0}} \sum_{i_1=1}^{r(\epsilon)} |A(i_1)|^{\mathrm{inx}(\prod_{p=1}^n A(q_p),\mathcal{M}_n)}.$$

9.3 Rigid algebras with no proper subalgebras

The computation of $|\Omega(\mathbf{A}, C, U)|$ is of particular interest in the case that $U = A^X$. This would be the case if U were the transversal with respect to $\sim$ for an algebra $\mathbf{A}$ that is rigid and has no proper subalgebras. For such an algebra $|\Omega(\mathbf{A}, C, A^X)|$ may be computed as follows: If $U(j_1 \ldots j_\ell) \subseteq \prod_{p=1}^n A(i_{p1} \ldots i_{p\ell})$, then equality must hold and the cardinality of $U(j_1 \ldots j_\ell)$ is $\prod_{p=1}^n |A(i_{p1} \ldots i_{p\ell})|$. The number $s(j_1 \ldots j_l)$ of $\equiv_{\ell+1}$ classes in $U(j_1 \ldots j_\ell)$

is $\prod_{p=1}^{n} r(i_{p1}\dots i_{p\ell})$. Then

$$|\Omega(\mathbf{A},C,A^X)| = \prod_{\substack{(i_{11},\dots,i_{n1})\\ 1\le i_{p1}\le r(\epsilon)\\ 1\le p\le n}} \sum_{i_1=1}^{r(\epsilon)} \prod_{\substack{(i_{12},\dots,i_{n2})\\ 1\le i_{p2}\le r(i_{p1})\\ 1\le p\le n}} \sum_{i_2=1}^{r(i_1)} \prod_{\substack{(i_{13},\dots,i_{n3})\\ 1\le i_{p3}\le r(i_{p1}i_{p2})\\ 1\le p\le n}} \sum_{i_3=1}^{r(i_1i_2)} \cdots$$

$$\prod_{\substack{(i_{1m},\dots,i_{nm})\\ 1\le i_{pm}\le r(i_{p1}i_{p2}\dots i_{p,m-1})\\ 1\le p\le n}} \sum_{i_m=1}^{r(i_1i_2\dots i_{m-1})} |A(i_1\dots i_m)|^{\prod_{p=1}^{n}|A(i_{p1}\dots i_{pm})|}. \quad (9.5)$$

If $\mathbf{A}$ is an algebra whose set of term operations consists of all finitary operations on A that preserve every equivalence relation $\theta_\ell \in C$, then $\mathbf{A}$ is rigid and has no proper subalgebras. Since $\mathbf{A}$ has no proper subalgebras, $\mathbf{L}(n,\mathbf{A}) \cong \mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)$ for all n. The congruence lattice of $\mathbf{A}$ is the chain C by [15]. The algebra is hence *congruence primal* (also called *hemiprimal*). Section 3.4 of [11] contains some fundamental results on congruence primal and related algebras. As observed in the paragraph before Theorem 5.3, there is a Pixley operation that preserves every equivalence relation in C. By Jónsson's Theorem the subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A})$ are the algebras $\mathbf{A}/\theta_\ell$ for $1 \le \ell \le m+1$. Hence the hypotheses of Theorem 7.1 hold for $\mathsf{V}(\mathbf{A})$.

Condition 2 of Theorem 4.1 holds with $U = A^X$ and all n since every partial operation with support U is in fact a total operation on A. Therefore by Remark 7.2 the four equivalent conditions of 7.1 hold for this variety and we conclude that the values of $|\mathbf{L}(n,\mathbf{A})|$ and $|\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)|$ are given by the expression in (9.5).

For example, if $\mathbf{A}$ is the algebra whose term operations consist of all operations on A that preserve a single equivalence relation θ_1 of A, then (9.5) applies with $m = 1$ and yields

$$|\mathbf{L}(n,\mathbf{A})| = |\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)| = \prod_{(q_1,\dots,q_n)\in\{1,\dots,r(\epsilon)\}^n} \sum_{i=1}^{r(\epsilon)} |A(i)|^{|A(q_1)|\times\cdots\times|A(q_n)|}.$$

Let r denote $r(\epsilon)$. For each r-tuple $(n_1,\dots,n_r)$ of non-negative integers that sum to n assign all those n-tuples $(q_1,\dots,q_n) \in \{1,\dots,r\}^n$ for which the number of coordinates that have value q is n_q. If $(q_1,\dots,q_n)$ and $(q_1',\dots,q_n')$

are assigned to the same $(n_1, \ldots, n_r)$, then $|A(q_1)| \times \cdots \times |A(q_n)| = |A(q_1')| \times \cdots \times |A(q_n')|$. Since there are $\binom{n}{n_1 \ldots n_r}$ n-tuples assigned to $(n_1, \ldots, n_r)$, the expression may be rewritten as

$$\prod_{\substack{n_1+\cdots+n_r=n \\ 0 \le n_q \le n}} \left(\sum_{i=1}^{r} |A(i)|^{\prod_{q=1}^{r} |A(q)|^{n_q}} \right)^{\binom{n}{n_1 \ldots n_r}}.$$

This formula for the free spectrum of $\mathsf{V}(\mathbf{A})$ is given in [7].

We next consider $\Omega(\mathbf{A}, C, U)$ in the case that the algebra $\mathbf{A}$ is congruence uniform. Then for each $0 \le \ell \le m$ there are integers c_ℓ and r_ℓ such that $|A(i_1 \ldots i_\ell)| = c_\ell$ for all θ_ℓ classes and $r(i_1 \ldots i_\ell) = r_\ell$. In particular, $c_0 = |A|$ and $c_{m+1} = 1$. We have $r_\ell = c_\ell / c_{\ell+1}$ and r_ℓ is the index of the equivalence relation $\theta_{\ell+1}$ when restricted to a θ_ℓ class. Note that $c_m = r_m$.

Suppose for this algebra $\mathbf{A}$ and for every $0 \le \ell \le m$ there are integers d_ℓ and s_ℓ for which all $\equiv_\ell$ classes $U(j_1 \ldots j_\ell)$ have cardinality d_ℓ and have $s(j_1 \ldots j_\ell) = s_\ell$. Then s_ℓ is in fact $d_\ell / d_{\ell+1}$ and $d_m = s_m$. Note that the product $s_0 \ldots s_\ell = d_0 / d_{\ell+1}$, which is the index of the equivalence relation $\equiv_{\ell+1}$ in the set U. Then from Remark 3.2 we have

$$\begin{aligned} |\Omega(\mathbf{A}, C, U)| &= (r_0(r_1(\ldots(r_{m-2}(r_{m-1}c_m^{d_m})^{s_{m-1}})^{s_{m-2}} \ldots)^{s_2})^{s_1})^{s_0} = \\ & r_0^{s_0} r_1^{s_0 s_1} \ldots r_{m-2}^{s_0 \ldots s_{m-2}} r_{m-1}^{s_0 \ldots s_{m-1}} c_m^{d_m s_0 \ldots s_{m-1}} = \prod_{\ell=0}^{m} r_\ell^{s_0 \ldots s_\ell} = \prod_{\ell=0}^{m} r_\ell^{d_0/d_{\ell+1}}. \end{aligned} \tag{9.6}$$

An example of an algebra $\mathbf{A}$ for which the d_ℓ an s_ℓ exist as in (9.6) would be a congruence uniform algebra that is rigid and has no proper subalgebras.

Theorem 9.3. *Let A be a finite set and suppose*

$$C = \{1_A = \theta_0 > \theta_1 > \cdots > \theta_m > \theta_{m+1} = 0_A\}$$

is a chain of uniform equivalence relations on A with the blocks of θ_ℓ having cardinality c_ℓ for each $0 \le \ell \le m+1$. If $\mathbf{A}$ is any algebra on A with $\operatorname{Con} \mathbf{A} = C$ *that is rigid and has no proper subalgebras, then for every n*

$$|\mathbf{L}(n, \mathbf{A})| = |\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)| \le |\Omega(\mathbf{A}, C, A^X)| = \prod_{\ell=0}^{m} (c_\ell / c_{\ell+1})^{(|A|/c_{\ell+1})^n}.$$

Moreover, there exist algebras $\mathbf{A}$ with $\operatorname{Con} \mathbf{A} = C$ *for which $|\mathbf{L}(n, \mathbf{A})| = |\Omega(\mathbf{A}, C, A^X)|$ holds for every $n \ge 0$ and hence for which all four conditions of Theorem 7.1 hold.*

Proof. Any transversal with respect to $\sim$ of $\mathrm{val}(X, \mathbf{A})$ is A^X since $\mathbf{A}$ is rigid and has no proper subalgebras. Hence both $\mathbf{L}(n, \mathbf{A})$ and $\mathbf{F}_{\mathsf{V}(\mathbf{A})}(n)$ are isomorphic to $\mathbf{Ge}(X, A^X)$. Therefore the first equality holds. The inequality is from Corollary 3.11. Since $\mathbf{A}$ is congruence uniform and the equivalence relations $\equiv_\ell$ partition the entire set A^X, it follows that each $\equiv_\ell$ is also uniform with $|U(j_1 \dots j_\ell)| = c_\ell^n$. So the second equality follows from (9.6). As observed in the two paragraphs following (9.5), if $\mathbf{A}$ is an algebra whose term operations are all finitary operations on A that preserve all the $\theta_i \in C$, then all four conditions of Theorem 7.1 hold for the variety $\mathsf{V}(\mathbf{A})$ and for all n. $\square$

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