Algebras defined from ordered sets and the varieties they generate

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Abstract

We investigate ways of representing ordered sets as algebras and how the order relation is reflected in the algebraic properties of the variety (equational class) generated by these algebras. In particular we consider two different but related methods for constructing an algebra with one binary operation from an arbitrary ordered set with a top element. The two varieties generated by all these algebras are shown to be well-behaved in that they are locally finite, finitely based, and have an equationally definable order relation. We exhibit a bijection between the subdirectly irreducible algebras in each variety and the class of all ordered sets with top element. We determine the structure and cardinality of the free algebra on n free generators and provide sharp bounds on the number of n-generated algebras in each variety. These enumeration results involve the number of quasi-orders on an n-element set.

Keywords: ordered set, quasi-order, semilattice, equationally definable order relation, algebra, subdirectly irreducible, free algebra, Hilbert algebra, order algebra.

1 Introduction

We are interested in algebraic interpretations of ordered sets. For a familiar example, if the ordered set $\mathbf{S} = \langle A, \leq \rangle$ is a join semilattice, then an algebraic counterpart is $\mathbf{A} = \langle A, \vee \rangle$ and the order relation $\leq$ of $\mathbf{S}$ and the algebraic operation $\vee$ of $\mathbf{A}$ are related by

$$x \leq y \quad \text{if and only if} \quad y = x \vee y.$$

The class of algebras that arise in this way form the equational class, or variety, of semilattices and thus the class is closed under the formation of homomorphisms, subalgebras and direct products. Every algebra in the class is determined by and determines a unique semilattice order.

A more recent example of this phenomenon is the class of *order algebras* presented by Freese, Ježek et al. in [8]. Here, for every ordered set $\mathbf{P} = \langle A, \leq \rangle$ an algebra $\langle A, \cdot \rangle$ is formed where the binary operation $\cdot$ is defined by

$$a \cdot b = \begin{cases} a & \text{if } a \leq b, \\ b & \text{if } a \not\leq b. \end{cases}$$

This algebra is called the *order algebra* obtained from $\mathbf{P}$. Let $\mathcal{O}$ be the class of all order algebras. Unlike the join semilattice example, the class $\mathcal{O}$ is not a variety. However, $\mathcal{V} = \mathsf{V}(\mathcal{O}) = \mathsf{HSP}(\mathcal{O})$, the variety of algebras generated by $\mathcal{O}$, has the property that every algebra $\mathbf{A} \in \mathcal{V}$ has an equationally definable order relation given by $x \leq y$ if and only if $x \cdot y = x$ for all $x, y \in A$.

The coding of ordered sets can also start on the algebraic side. Let $\mathbf{A}$ be an algebra with universe A and operation symbols in a language $\mathcal{L}$. Suppose $E = \{p_i(x, y) \approx q_i(x, y),\ i \in I\}$ is a set of equations involving binary terms of $\mathcal{L}$ such that the binary relation $\leq$ defined on A by

$$a \leq b \quad \text{iff} \quad p_i^{\mathbf{A}}(a, b) = q_i^{\mathbf{A}}(a, b) \quad i \in I \tag{1}$$

is an order. In this case we say E defines the order $\leq$ on A and that $\leq$ is an *equationally defined order* for the algebra $\mathbf{A}$. If $\mathcal{K}$ is a class of algebras in the language $\mathcal{L}$ and E defines an order $\leq$ on every algebra in $\mathcal{K}$, then we say $\leq$ is an equationally defined order for the class $\mathcal{K}$. Moreover, in this case the order $\leq$ is equationally defined for the quasivariety $\mathsf{Q}(\mathcal{K})$ generated by $\mathcal{K}$, that is, for the class $\mathsf{SPP_U}(\mathcal{K})$, since reflexivity, antisymmetry, and transitivity can all be formulated as universal Horn sentences.

In the following we will endow classes of ordered sets with a binary operation, defined by conditions involving the order relation and possibly some additional constants, in such a way that the order relation becomes equationally definable. In the quasivariety generated by the resulting algebras the order relation is then definable as well, by the same set of equations; the quasivariety will however contain algebras in which the binary operation is not defined by the original condition.

More precisely, let $\mathcal{K}$ be a class of ordered sets, possibly with constants, and suppose there is a universal formula $\varphi(x, y, z)$ (in the language containing $\leq$ together with the constant symbols of $\mathcal{K}$) which defines on every $\mathbf{P} \in \mathcal{K}$ a binary operation $\cdot$ on P. Let $\mathrm{Al}(\mathbf{P})$ denote the algebra $\langle P, \cdot \rangle$, and $\mathrm{Al}(\mathcal{K})$ the class of algebras $\mathrm{Al}(\mathbf{P})$ for $\mathbf{P} \in \mathcal{K}$. Suppose furthermore that the relation $\leq$ of $\mathbf{P}$ is equationally definable in the algebra $\mathrm{Al}(\mathbf{P})$, i.e., there are terms $p_i(a, b) = q_i(a, b)$, $i \in I$ in the language of $\mathcal{K}$ such that for all $\mathbf{P}$ and all $a, b \in P$ we have

$$a \leq b \quad \text{iff} \quad p_i^{\mathrm{Al}(\mathbf{P})}(a, b) = q_i^{\mathrm{Al}(\mathbf{P})}(a, b), \quad i \in I. \tag{2}$$

We will call such a class, as well as its members, *pure*—a pure algebra $\mathrm{Al}(\mathbf{P})$ is thus determined completely by the ordered set $\mathbf{P}$. Observe that pure classes are closed under the formation of subalgebras.

Let $\mathrm{Al}(\mathcal{K})$ be a class of pure algebras. As previously observed the quasivariety $\mathsf{Q}(\mathrm{Al}(\mathcal{K}))$ generated by $\mathrm{Al}(\mathcal{K})$ also has an equationally definable order, determined by the same condition (2). But the operation $\cdot$ is no longer determined by the order of the underlying ordered set. For example, the class $\mathcal{O}$ of order algebras is pure but if $\mathbf{P}$ is a nontrivial ordered set and $\mathbf{A}$ is the order algebra obtained from $\mathbf{P}$, then the algebra $\mathbf{A}^2$ is not pure although it does have an equationally definable order.

We are interested in varieties generated by families of pure algebras. Of special concern are the finitely generated free algebras in such varieties.

Definition 1.1. *Let $\mathcal{V}$ be a locally finite variety with $\mathbf{F}_{\mathcal{V}}(n)$ denoting the free algebra for $\mathcal{V}$ on n free generators. The sequence of cardinalities $|\mathrm{F}_{\mathcal{V}}(n)|$ for $n = 0, 1, 2, \ldots$ is called the* free spectrum *of $\mathcal{V}$. The number of pairwise nonisomorphic homomorphic images of $\mathbf{F}_{\mathcal{V}}(n)$, which is also the number, up to isomorphism, of at most n-generated algebras in $\mathcal{V}$, is denoted $\mathrm{G}_{\mathcal{V}}(n)$. The* G-spectrum *of $\mathcal{V}$ is the sequence $\mathrm{G}_{\mathcal{V}}(n)$, for $n = 0, 1, 2, \ldots$.*

In this paper we consider the class $\mathcal{P}$ of all ordered sets with a top element

1. For each $\mathbf{P} = \langle P, \leq, 1\rangle \in \mathcal{P}$ we investigate two similar binary operations $\rightarrow$ and $\cdot$ that can be defined on P:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a \not\leq b; \end{cases} \qquad \text{and} \qquad a \cdot b = \begin{cases} b & \text{if } a \leq b, \\ 1 & \text{if } a \not\leq b. \end{cases}$$

Thus, for every $\mathbf{P}$ the algebras $\langle P, \rightarrow\rangle$ and $\langle P, \cdot\rangle$ are pure algebras in the sense previously described.

We let $\mathcal{H}$ and $\mathcal{J}$ denote the varieties generated by all $\langle P, \rightarrow\rangle$ and by all $\langle P, \cdot\rangle$ as $\mathbf{P}$ ranges over $\mathcal{P}$. In Section 2 we investigate some important varietal properties of $\mathcal{H}$ and in Section 3 we do the same for $\mathcal{J}$. We show that both varieties $\mathcal{H}$ and $\mathcal{J}$ are locally finite and have finite equational bases. We completely characterize the subdirectly irreducible algebras in each variety and show that every subdirectly irreducible algebra is pure and the subdirectly irreducible algebras in $\mathcal{H}$ and $\mathcal{J}$ are precisely the class of pure algebras obtained from ordered sets of the form $\mathbf{P} \oplus 1$ for $\mathbf{P} \in \mathcal{P}$. That is, there are natural bijections between the class $\mathcal{P}$ of ordered sets and the classes of subdirectly irreducible algebras in $\mathcal{H}$ and in $\mathcal{J}$. Despite these structural similarities between $\mathcal{H}$ and $\mathcal{J}$ there are some surprising differences. We show that whereas $\mathcal{H}$ has $2^{\aleph_0}$ subvarieties and is not finitely generated, the variety $\mathcal{J}$ has only one proper nontrivial subvariety and $\mathcal{J}$ is generated by a 3-element pure algebra. Section 4 describes a general method for decomposing free algebras in arbitrary varieties into canonically defined subalgebras. This decomposition provides an inclusion-exclusion type formula for the cardinality of the free algebra on n free generators in a variety. We then consider n-generated algebras in both $\mathcal{H}$ and $\mathcal{J}$. For each we obtain a detailed description of the free algebras $\mathbf{F}_{\mathcal{H}}(n)$ and $\mathbf{F}_{\mathcal{J}}(n)$ and sharp bounds on the free spectrum and on the G-spectrum of these two varieties. Section 5 contains this analysis for $\mathcal{J}$ and Section 6 deals with $\mathcal{H}$.

There are various ways ordered sets manifest themselves algebraically in this setting. As already mentioned, there are one-to-one correspondences between members of $\mathcal{P}$ and the subdirectly irreducibles in $\mathcal{H}$ and in $\mathcal{J}$. In the case of $\mathcal{H}$, which is a congruence distributive variety, finite subdirectly irreducible algebras generate distinct subvarieties and each such subvariety has a finite equational basis. Thus, to every finite ordered set $\mathbf{P} \in \mathcal{P}$ there is a unique subvariety of $\mathcal{H}$ and a set of identities in the language $\{\rightarrow, 1\}$ associated with $\mathbf{P}$. For the variety $\mathcal{J}$ there is no such correspondence. However, we exhibit a natural connection between elements of the free algebra $\mathbf{F}_{\mathcal{J}}(n)$ and quasi-orders defined on subsets of an n-element set. It is by means of this correspondence that we describe the structure of $\mathbf{F}_{\mathcal{J}}(n)$ and provide a formula for $|\mathrm{F}_{\mathcal{J}}(n)|$ in terms of the values of q_k, the number of distinct quasi-orders on a k-element set.

In general our terminology and notation follow that of [5] or [14]. For a class $\mathcal{K}$ of algebras, by $\mathcal{K}_{FIN}$ and $\mathcal{K}_{SI}$ we denote all finite algebras and all subdirectly irreducible algebras in $\mathcal{K}$. The equational class or variety generated by $\mathcal{K}$ is denoted $\mathsf{V}(\mathcal{K})$ and the quasivariety is $\mathsf{Q}(\mathcal{K})$. By Birkhoff's Theorem $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$ where the operators H, S and P form all homomorphic images, subalgebras and direct products of algebras in the class. We also have $\mathsf{Q}(\mathcal{K}) = \mathsf{SPP_U}(\mathcal{K})$, where $\mathsf{P_U}(\mathcal{K})$ is the class of all ultraproducts of algebras in $\mathcal{K}$.

We use the phrases *ordered sets* and *partially ordered sets* and *posets* interchangeably.

2 Hilbert algebras

Let $\mathcal{P}$ denote the set of all posets $\mathbf{P} = \langle P, \leq, 1\rangle$ with largest element 1. For $\mathbf{P} \in \mathcal{P}$ we will write $\mathrm{Hi}(\mathbf{P})$ for the pure algebra $\langle P, \to, 1\rangle$ where $\to$ is the binary operation defined by

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a \nleq b. \end{cases}$$

For any $\mathbf{P} \in \mathcal{P}$ the algebra $\mathrm{Hi}(\mathbf{P})$ is a Hilbert algebra; i.e., a $\to$-subreduct of a Heyting algebra. It is known that the class of Hilbert algebras can be defined as the class of algebras $\mathbf{A} = \langle A, \to, 1\rangle$ satisfying the identities and quasi-identity

(i) $x \to (y \to x)) \approx 1$,

(ii) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \approx 1$,

(iii) if $x \to y \approx 1$ and $y \to x \approx 1$, then $x \approx y$.

In fact, it has been shown in [6] that the class of all Hilbert algebras forms a variety. From identity (i) it follows that the constant 1 in any Hilbert algebra is equationally definable. Thus it is not critical that 1 is specified in the similarity type, i.e., we may write a Hilbert algebra $\mathbf{A}$ as $\langle A, \to\rangle$ or $\langle A, \to, 1\rangle$.

It is easy to see that for any poset $\mathbf{P}$ with 1 the algebra $\mathrm{Hi}(\mathbf{P})$ satisfies (i), (ii) and (iii); i.e., $\mathrm{Hi}(\mathbf{P})$ is a Hilbert algebra.

Definition 2.1. *Let* $\mathcal{HI}$ *denote the class of all Hilbert algebras,* $\mathcal{HI}_p$ *the class of all pure Hilbert algebras, and* $\mathcal{H}$ *the variety generated by all pure Hilbert algebras.*

In any Hilbert algebra $\mathbf{A}$ the relation $x \leq y$ if and only if $x \to y \approx 1$ is a partial order, with largest element 1. We will denote this poset by $\mathrm{Po}(\mathbf{A})$. Not every Hilbert algebra is pure, however: in general $x \nleq y$ does not imply $x \to y = y$. The smallest example of a non-pure Hilbert algebra is the algebra $\mathbf{2}^2$, where $\mathbf{2}$ is the $\{\to, 1\}$-reduct of the 2-element Boolean algebra. Note that for any poset $\mathbf{P}$ with 1 we have $\mathrm{Po}(\mathrm{Hi}(\mathbf{P})) = \mathbf{P}$, while for Hilbert algebras $\mathbf{A}$ we have $\mathrm{Hi}(\mathrm{Po}(\mathbf{A})) = \mathbf{A}$ if and only if $\mathbf{A}$ is pure.

The variety $\mathcal{HI}$ is locally finite, congruence distributive, and has the Congruence Extension Property (CEP) — hence, it has Equationally Definable Principal Congruences (EDPC). $\mathcal{HI}$ is also 1-regular, i.e., a congruence Θ on a Hilbert algebra $\mathbf{A}$ is determined by the class $1/\Theta$. The classes of the form $1/\Theta$ are *filters*, i.e., subsets $F \subseteq A$ containing 1 and closed under 'modus ponens': if $a, a \to b \in F$, then $b \in F$; filters are in particular always upwardly closed. Conversely, every filter F of a Hilbert algebra $\mathbf{A}$ is the 1-class of a (unique) congruence $\Theta(F)$ of $\mathbf{A}$; in fact, $\Theta(F) = \{(a, b) : a \to b \in F, b \to a \in F\}$. From this description of the congruences it can be easily seen that a Hilbert algebra $\mathbf{A}$ is subdirectly irreducible if and only if there is an element $e \in A$ such that $e < 1$, and for all $x \in A$, $x \neq 1$, we have $x \leq e$. Let $\mathcal{P}^{\oplus 1}$ denote the class of all posets with 1 that contain an element $e < 1$ such that $x \leq e$ for all $x \neq 1$.

In a pure Hilbert algebra $\mathrm{Hi}(\mathbf{P})$ every non-empty upwardly closed set is a filter. Given such an upwardly closed set $F \subseteq P$, the congruence $\Theta(F)$ of $\mathrm{Hi}(\mathbf{P})$ has at most one non-trivial block, viz. the filter F. Note that $\mathrm{Hi}(\mathbf{P})/\Theta(F)$ is the pure Hilbert algebra on the poset $\mathbf{P}/F$. The class of pure Hilbert algebras is thus closed under the operation H of forming homomorphic images.

Given a poset $\mathbf{P}$ with 1, every subposet $\mathbf{Q}$ (with 1) gives rise to a subalgebra of $\mathrm{Hi}(\mathbf{P})$ with domain Q, and the subalgebra is pure as well. Conversely, as we have already observed, every subalgebra $\mathbf{Q}$ of a pure algebra $\mathrm{Hi}(\mathbf{P})$ is pure, with underlying poset $\mathbf{Q}$ a subposet of $\mathbf{P}$. The class of pure Hilbert algebras is thus also closed under the operation S of forming subalgebras.

We have already seen that the class of pure Hilbert algebras is not closed under the operation of forming products: the algebra $\mathbf{2}$ is pure, but $\mathbf{2}^2$ is not. However, since it is an elementary class, the class of pure Hilbert algebras is closed under the operation $\mathsf{P_U}$ of forming ultraproducts.

Theorem 2.2. *The subdirectly irreducible algebras in $\mathcal{H}$ are pure Hilbert algebras.*

Proof. Since $\mathcal{H} = \mathsf{V}(\mathcal{H}I_p)$, and $\mathcal{H}I_p$ is included in the congruence distributive variety $\mathcal{H}I$, by Jónsson's Lemma the subdirectly irreducible algebras in $\mathcal{H}$ are in $\mathsf{HSP_U}(\mathcal{H}I_p)$. By the above remarks, $\mathsf{HSP_U}(\mathcal{H}I_p) \subseteq \mathcal{H}I_p$. ☐

For a semilattice or Heyting algebra $\mathbf{A}$, let $\mathbf{A}^{\rightarrow}$ be the Hilbert algebra reduct of $\mathbf{A}$. Let $\mathbf{B}$ denote the Brouwerian semilattice in Figure 1.

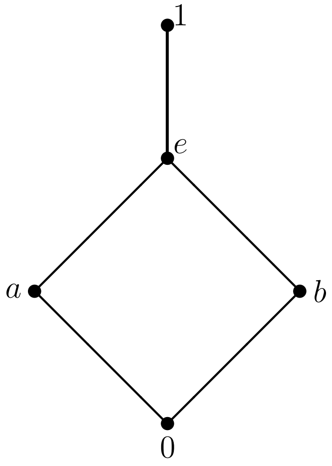

Figure 1

Since in $\mathbf{B}$ we have $a \not\leq 0$, but $a \rightarrow 0 = b \neq 0$, it follows that $\mathbf{B}^{\rightarrow}$ is not pure. Thus by the above theorem $\mathbf{B} \in (\mathcal{H}I \setminus \mathcal{H})_{SI}$, and thus $\mathcal{H}$ is a proper subvariety of $\mathcal{H}I$.

The class $\mathcal{H}I_p$ consists of all Hilbert algebras satisfying the first order sentence:

$$\forall x \forall y[(x \rightarrow y \approx 1) \vee (x \rightarrow y \approx y)],$$

and forms thus an elementary positive universal class. In [4, Theorem 2.5] an algorithm is presented that produces for any elementary positive universal class of algebras included in a variety with EDPC a finite set of equations that axiomatizes the variety generated by the class. Since the variety of Hilbert algebras has EDPC, and $\mathcal{H}$ is the variety generated by the elementary positive universal class $\mathcal{H}I_p$, the result is applicable here and we have

Theorem 2.3. *The variety $\mathcal{H}$ is finitely based.*

Given a variety $\mathcal{V} \subseteq \mathcal{H}$, let $\mathcal{V}^p = \{\mathrm{Po}(\mathbf{A}) : \mathbf{A} \in \mathcal{V}_{SI}, \mathbf{A} \text{ finite } \} \subseteq \mathcal{P}^{\oplus 1}_{FIN}$. Note that $\mathcal{V} = \mathsf{V}(\mathcal{V}^p)$, so the map $\mathcal{V} \mapsto \mathcal{V}^p$ is 1-1 on the collection of subvarieties of $\mathcal{H}$ into the 'powerset' of $\mathcal{P}^{\oplus 1}$. We say that a class $\mathcal{K} \subseteq \mathcal{P}^{\oplus 1}$ is *closed under the formation of $\mathcal{P}^{\oplus 1}$-substructures* if every substructure of a poset $\mathbf{P} \in \mathcal{K}$ that belongs to $\mathcal{P}^{\oplus 1}$ belongs also to $\mathcal{K}$.

Theorem 2.4. *The lattice of subvarieties of $\mathcal{H}$ is isomorphic to the lattice of subclasses of $\mathcal{P}^{\oplus 1}_{FIN}$ closed under the formation of $\mathcal{P}^{\oplus 1}$-substructures.*

Proof. Clearly for every subvariety $\mathcal{V}$ of $\mathcal{H}$ the class $\mathcal{V}^p$ is closed under the formation of $\mathcal{P}^{\oplus 1}$-substructures. Conversely, if $\mathcal{K} \subseteq \mathcal{P}^{\oplus 1}_{FIN}$ is closed under $\mathcal{P}^{\oplus 1}$-substructures, then for $\mathcal{V} = \mathsf{V}(\mathrm{Hi}(\mathcal{K})) \subseteq \mathcal{H}$ we have $\mathcal{V}^p = \mathcal{K}$. Firstly, clearly $\mathcal{V}^p \supseteq \mathcal{K}$. Conversely, if $\mathbf{P} \in \mathcal{V}^p$, then $\mathbf{P} = \mathrm{Po}(\mathbf{A})$ for some finite subdirectly irreducible algebra $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A} \in \mathsf{HSP_U}(\mathrm{Hi}(\mathcal{K}))$ by Jónsson's lemma, but since $\mathcal{H}I$ has EDPC we even have $\mathbf{A} \in \mathsf{HS}(\mathrm{Hi}(\mathcal{K}))$. From earlier remarks it follows that $\mathsf{HS}(\mathrm{Hi}(\mathcal{K})) = \mathsf{S}(\mathrm{Hi}(\mathcal{K}))$. Since $\mathcal{K}$ is closed under $\mathcal{P}^{\oplus 1}$-substructures we conclude, again by earlier remarks, that $\mathsf{S}(\mathrm{Hi}(\mathcal{K})) \subseteq \mathrm{Hi}(\mathcal{K})$. Thus the finite subdirectly irreducible algebras of $\mathcal{V}$ are in $\mathrm{Hi}(\mathcal{K})$, and hence $\mathcal{V}^p \subseteq \mathcal{K}$. □

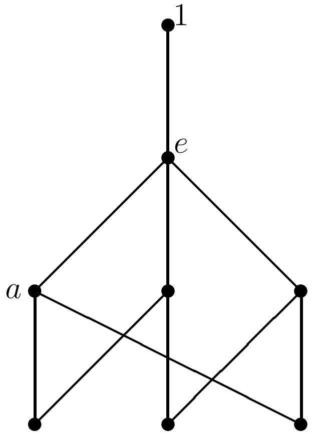

Figure 2

Corollary 2.5. *$\mathcal{H}$ has $2^{\aleph_0}$ subvarieties.*

Proof. Let $\mathbf{C}_n$ denote the crown on $2n$ elements with the two element chain $\{e < 1\}$ added on top. Figure 2 represents $\mathbf{C}_3$. Note that if $n < m$ then $\mathbf{C}_n \notin \mathsf{S}(\mathbf{C}_m)$. For I a set of natural numbers let $\mathcal{C}_I = \{\mathbf{C}_n : n \in I\}$. Then for sets of natural numbers I, J such that $I \neq J$ we have $\mathsf{S}(\mathcal{C}_I) \cap \mathcal{P}^{\oplus 1}_{FIN} \neq \mathsf{S}(\mathcal{C}_J) \cap \mathcal{P}^{\oplus 1}_{FIN}$. Thus there is a continuum of subsets of $\mathcal{P}^{\oplus 1}_{FIN}$ closed under $\mathcal{P}^{\oplus 1}$-substructures, and hence, by Theorem 2.4 a continuum of subvarieties of $\mathcal{H}$. $\square$

3 Join algebras

We continue to let $\mathcal{P}$ denote the class of all posets $\mathbf{P} = \langle P, \leq, 1\rangle$ with largest element 1. For $\mathbf{P} \in \mathcal{P}$ define a binary operation $\cdot$ on $\mathbf{P}$ by

$$a \cdot b = \begin{cases} b & \text{if } a \leq b, \\ 1 & \text{if } a \nleq b. \end{cases}$$

We write $\mathrm{Jo}(\mathbf{P})$ for the algebra $\langle P, \cdot\rangle$. Let $\mathcal{J}$ be the variety generated by all such algebras $\mathrm{Jo}(\mathbf{P})$ and we call each $\mathbf{A} \in \mathcal{J}$ a *join algebra.* A join algebra $\mathbf{A} \in \mathcal{J}$ is *pure* if it is of the form $\mathrm{Jo}(\mathbf{P})$, for some $\mathbf{P} \in \mathcal{P}$.

For example, if we let $\mathbf{P}$ be the 2-element chain, then the pure join algebra $\mathrm{Jo}(\mathbf{P})$ is the 2-element join semilattice. The variety generated by $\mathrm{Jo}(\mathbf{P})$ is the variety of all join semilattices, which is a subvariety of $\mathcal{J}$. An easy argument shows that the only join semilattices that are pure join algebras are those of the form $\mathrm{Jo}(\mathbf{Q} \oplus 1)$ where $\mathbf{Q}$ is an antichain.

If $\mathbf{A}$ is a pure join algebra and $S \subseteq A$, then the subuniverse of $\mathbf{A}$ generated by S is either S or $S \cup \{1\}$. From this observation it follows that every pure join algebra is locally finite and that $\mathcal{J}$ is a locally finite variety.

Every pure join algebra $\mathrm{Jo}(\mathbf{P})$ satisfies each of the following identities:

(1) $x1 \approx 1x \approx 1$

(2) $xx \approx x$

(3) $(xy)(yx) \approx (yx)(xy)$

(4) $(xy)z \approx x((xy)z)$

(5) $x(xy) \approx xy$

(6) $y(xy) \approx xy$

(7) $(xz)((xy)(yz)) \approx ((xy)z)(yz)$

We prove (7); the proofs of the others are similar. In any pure join algebra the left side of (7) evaluates to 1 or z as does the right side. The left side will evaluate to z if and only if $y \leq z, x \leq y$ and $x \leq z$. The right side will evaluate to z if and only if $y \leq z$ and $x \leq y$. But then by transitivity, $x \leq z$ as well. So the left side evaluates to z if and only if the right side does.

These identities also hold for every $\mathbf{A} \in \mathcal{J}$ since the variety $\mathcal{J}$ is generated by its pure members. Note that 1 is not in the similarity type of $\mathcal{J}$, however.

We define a binary relation $\leq$ on algebras $\mathbf{A} \in \mathcal{J}$ as follows. For all $a, b \in A$,

$$a \leq b \quad \text{iff} \quad a \cdot b = b.$$

The relation $\leq$ is indeed a partial order. To see this, observe that $\leq$ is reflexive since $\mathcal{J} \models xx \approx x$. Transitivity follows from (4) since if $a \leq b$ and $b \leq c$, then $ab = b$, $bc = c$ and thus $c = bc = (ab)c = a((ab)c) = ac$. A similar argument using (3) shows that $\leq$ is antisymmetric.

For $\mathbf{A} \in \mathcal{J}$ we let $\text{Po}(\mathbf{A})$ denote the ordered set $\langle A, \leq \rangle$.

The order $\leq$ on $\text{Po}(\mathbf{A})$ is equationally definable on $\mathbf{A}$ by $x \leq y$ iff $x \cdot y = y$, which is also the equational definition of order for join semilattices. This definition of the order on join algebras is another way that the variety $\mathcal{J}$ may be viewed as a natural extension of the variety of semilattices.

Note that for an arbitrary join algebra $\mathbf{A}$, if $a \not\leq b$, then all we can say about the value of $a \cdot b$ is that it is not b. However, if $\mathbf{A}$ is a pure join algebra $\text{Jo}(\mathbf{P})$, then the operation $\cdot$ of $\mathbf{A}$ is completely determined by the underlying poset $\mathbf{P}$.

Lemma 3.1. *Let a, b, c be elements in a join algebra $\mathbf{A}$.*

1. $a \leq ab$
2. $b \leq ab$
3. *If* $a \leq b$, *then* $ac \leq bc$

Proof. The first two claims follow from identities (5) and (6). For the third, replace x, y and z in identity (7) by a, b and c. By use of $ab = b$, (5) and (2) the left side becomes $(ac)(bc)$ and the right side simplifies to bc. So $bc = (ac)(bc)$, which gives $ac \leq bc$. ☐

Lemma 3.2. *Let* $\mathbf{A}$ *be a join algebra with* U *a nonempty, upwardly closed subset of* $\mathrm{Po}(\mathbf{A}) = \langle A, \leq \rangle$*. Then* U *is a subuniverse of* $\mathbf{A}$ *and the equivalence relation* θ *defined on* A *whose blocks are* U *and singleton* a *for each* $a \in A - U$ *is a congruence relation of* $\mathbf{A}$*.*

Proof. If $a, b \in U$ and $c \in A$, then by Lemma 3.1 the elements ac, bc, ca and cb will also be in U. □

Congruence relations on pure join algebras are especially well-behaved. If $\mathbf{A}$ is pure and $\theta \in \mathsf{Con}\,\mathbf{A}$ with $(a, b) \in \theta$ for $a \nleq b$, then $(aa, ab) = (a, 1) \in \theta$. So θ has at most one congruence class that is not a singleton, and such a class will be $1/\theta$. Thus, the congruence lattice of a pure join algebra is isomorphic to the lattice of nonempty, upwardly closed subsets of the ordered set $\mathrm{Po}(\mathbf{A}) = \langle A, \leq \rangle$.

Suppose $\mathbf{A} \in \mathcal{J}$ is a subdirectly irreducible algebra with monolith μ and $(c, d) \in \mu$ with $c \neq d$. For each $a \in A$, let $\theta_a \in \mathsf{Con}\,\mathbf{A}$ be the congruence relation in which the upwardly closed subset $\{x \in A : x \geq a\}$ is one congruence class and all other congruence classes are singletons. If a is not the top element of $\langle A, \leq \rangle$, then $(c, d) \in \theta_a$, that is, $a \leq c$ and $a \leq d$. From this it follows that $\{c, d\} = \{1, e\}$, where $e \prec 1$ and $b \leq e$ for all $b < 1$. Moreover, $\mu = \theta_e$. So if $\mathbf{A} \in \mathcal{J}$ is subdirectly irreducible, then the poset $\mathrm{Po}(\mathbf{A})$ is of the form $P \oplus 1$ for some $P \in \mathcal{P}$. We will eventually argue that every subdirectly irreducible algebra in $\mathcal{J}$ is pure.

The description of subdirectly irreducible algebras in $\mathcal{J}$ given in the previous paragraph allows us to determine the set of types, in the sense of tame congruence theory [12], that appear in finite algebras in $\mathcal{J}$. If $\mathbf{A} \in \mathcal{J}$ is a finite subdirectly irreducible algebra having monolith μ, then the type of the covering pair $0 \prec \mu$ is $\mathbf{5}$ since $\{1, e\}$ is the only nontrivial congruence class of μ and $\{1, e\}$ is polynomially equivalent to a semilattice. It follows that $\mathrm{typ}\{\mathcal{J}\} = \{\mathbf{5}\}$. One consequence of this is that the variety $\mathcal{J}$ satisfies no nontrivial congruence identity but $\mathcal{J}$ is congruence meet semi-distributive.

Every subalgebra of a pure join algebra is pure. The product of two nontrivial join algebras will not be pure. The order relation on the product of two join algebras is the product of the two orders on the factors. If θ is a congruence relation on a pure join algebra $\mathbf{A}$, then the set $(A - 1/\theta) \cup \{1\}$ is the universe of a subalgebra $\mathbf{B}$ of $\mathbf{A}$ for which $\mathbf{B}$ is isomorphic to $\mathbf{A}/\theta$. So every homomorphic image of a pure join algebra is pure. In fact, if $\mathcal{K}$ is a class of pure join algebras closed under isomorphism, then $\mathsf{HS}(\mathcal{K}) \subseteq \mathsf{S}(\mathcal{K})$.

Lemma 3.3. *Let* $\mathbf{A}$ *be a finite join algebra.*

1. *For each congruence relation* θ *on* $\mathbf{A}$ *and every* $a \in A$ *the congruence class* a/θ *is a convex subset of* $\langle A, \leq \rangle$ *and has a largest element* $m(a, \theta)$.
2. *The function* $\theta \mapsto \{m(a, \theta) : a \in A\}$ *is one-to-one from* $\mathsf{Con}\,\mathbf{A}$ *into* 2^A.

Proof. Let $a \in A$ and $\theta \in \mathsf{Con}\,\mathbf{A}$ be arbitrary. For any $b, c \in a/\theta$ we have $bc \in a/\theta, bc \geq b$, and $bc \geq c$. From these facts it follows that the product of all the elements of a/θ, regardless of how they are associated, will be the largest element of a/θ. For $a \in A$ let $m(a, \theta)$ denote the largest element in a/θ. If θ is understood from the context we write m_a for $m(a, \theta)$.

We show that for every $a \in A$, if $a \leq c \leq m_a$, then $c \in a/\theta$. Suppose, to the contrary, that there are $a \in A$ for which this is not the case. Choose a so that m_a is maximal among all such m_a. There exists $a < c < m_a$ with $(a, m_a) \in \theta$ but $(a, c) \notin \theta$. Since θ is a congruence relation, $(ac, m_a c) \in \theta$. From $ac = c$ and $m_a c > m_a$ we have $(c, m_a c) \in \theta$ with $c < m_a < m_a c$ and $(c, m_a) \notin \theta$. Thus $m_a < m_a c \leq m_c$, which violates the maximality of m_a. So all the sets a/θ are convex.

For every congruence relation θ the map $a \mapsto m(a, \theta)$ uniquely determines θ; indeed θ is the kernel of this map. Let $M_\theta = \{m(a, \theta) : a \in A\}$. We show that the set M_θ determines the map $a \mapsto m(a, \theta)$. To prove this, it suffices to show that for every $a \in A$ the element $m(a, \theta)$ is the greatest lower bound of $\{m \in M_\theta : a \leq m\}$. If $a, b \in A$ with $a \leq m_b$, then $m_b = am_b \overset{\theta}{\equiv} m_a m_b$. This implies $m_b = m_a m_b$, so $m_a \leq m_b$, that is, $m_a \leq m$ for all $m \in M_\theta$ with $a \leq m$. Thus, the function $\theta \mapsto M_\theta$ is one-to-one. ☐

Let $\mathbf{S} = \langle \{1, e, a\}, \cdot \rangle$ be the pure join algebra in which $a < e < 1$.

Theorem 3.4. *Every pure subdirectly irreducible algebra in* $\mathcal{J}$ *is in* $\mathsf{HSP}(\mathbf{S})$.

Proof. Let $\mathbf{A} \in \mathcal{J}$ be subdirectly irreducible. We have $\langle A, \leq \rangle = \mathbf{P} \oplus m \oplus 1$ for some poset $\mathbf{P}$. The subset $\{a, e\}$ of S is a 2-element chain in $\langle S, \leq \rangle$. Thus, there exists an index set I and an order embedding h of $P \cup \{m\}$ into $\{a, e\}^I$ with $h(m) = \bar{e}$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{S}^I$ generated by $h(P) \cup \{\bar{e}, \bar{1}\}$. Note that if $x, y \in h(P)$, then either $xy = y$ or xy has at least one coordinate with value 1. Also, $\bar{e}x$ has at least one coordinate 1 and $x\bar{e} = \bar{e}$. Let U be the set of all $b \in B$ that have at least one coordinate with value 1. Then U is an upwardly closed subset of B. Let θ denote the congruence relation on $\mathbf{B}$ whose only nonsingleton congruence class is U. If

$\mathbf{C}$ denotes $\mathbf{B}/\theta$, then the elements of $\mathbf{C}$ are U, $\{\bar{e}\}$ and $\{h(r)\}$ for each $r \in P$. In $\mathbf{C}$ we have $\{h(r)\} < \{\bar{e}\} < U$ for every $r \in P$. If $r, s \in P$, then in $\mathbf{C}$ the value of $\{h(r)\} \cdot \{h(s)\}$ is $\{h(s)\}$ if $r \leq s$ and is U otherwise. Hence $\mathbf{C}$ is isomorphic to $\mathbf{A}$. ☐

Corollary 3.5. $\mathcal{J} = \mathsf{HSP}(\mathbf{S})$.

Proof. The variety $\mathcal{J}$ is generated by pure join algebras. As observed immediately before Lemma 3.3, every homomorphic image of a pure join algebra is pure. Hence every pure join algebra is a subdirect product of pure subdirectly irreducible algebras in $\mathcal{J}$. An application of Lemma 3.4 completes the proof. ☐

For a directed graph $G = \langle V, E \rangle$ form an algebra $\mathbf{A}_G = \langle V \cup \{\infty\}, \cdot \rangle$ in which the binary operation $\cdot$ is given by $xy = y$ if $(x, y) \in E$ and $xy = \infty$ otherwise. $\mathbf{A}_G$ is called a *directed graph algebra*. Every pure join algebra is a directed graph algebra with the top element serving as the element ∞. B. Walter has shown in [16] that the variety of directed graph algebras generated by $\mathbf{S}$ has a finite basis for its equational theory. Actually, Walter works with directed graph algebras in which the element ∞ is a constant in the language, but he points out that his results carry over to the algebras in which ∞ is not in the language. Thus by the previous corollary we have

Corollary 3.6. *The variety $\mathcal{J}$ is finitely based.*

We next investigate subdirectly irreducible join algebras and quasivarieties contained in $\mathcal{J}$.

Theorem 3.7. *Every subdirectly irreducible algebra in $\mathcal{J}$ that has at least three elements contains a subalgebra isomorphic to* $\mathbf{S}$.

Proof. Let $\mathbf{A} \in \mathcal{J}$ be subdirectly irreducible, $|A| \geq 3$. We know $\mathbf{A}$ has a top element 1 which covers an element e, and $a \leq e$ for all $a < 1$. Let $b < e$ be arbitrary. For every $x \in A$ we have $xx = x, 1x = x1 = 1$, and $xy = y$ if and only if $x \leq y$. These facts allow us to fill in the table for $\cdot$ when restricted to $\{1, e, b\}$ as follows:

$\cdot$	1	e	b
1	1	1	1
e	1	e	
b	1	e	b

The entry eb is the only entry that is not determined. We do know that eb must have value e or 1. If $eb = 1$, then we have a 3-element subalgebra isomorphic to $\mathbf{S}$. So suppose that for all $b < e$ we have $eb = e$. For all $b \leq e$ and $c \leq e$ we have $bc \leq ec = e$ by Lemma 3.1. That is, if $b, c \leq e$, then $bc \leq e$. From this it follows that if θ is the equivalence relation on A with equivalence classes $\{x : x \leq e\}$ and $\{1\}$, then θ will be a congruence relation on $\mathbf{A}$. But $(1, e) \notin \theta$, which violates the assumption that $\mathbf{A}$ is subdirectly irreducible. $\square$

Corollary 3.8. *The only proper, nontrivial subvariety of* $\mathcal{J}$ *is the variety of semilattices.*

Lemma 3.9. *If* $\mathbf{A} \in \mathcal{J}$ *is a finite subdirectly irreducible algebra, then* $\mathbf{A}$ *is pure.*

Proof. From Corollary 3.5 there is an integer n and a subalgebra $\mathbf{B}$ of $\mathbf{S}^n$ such that $\mathbf{A}$ is isomorphic to $\mathbf{B}/\alpha$ for some congruence relation α on $\mathbf{B}$. For the remainder of the proof we identify $\mathbf{A}$ with $\mathbf{B}/\alpha$. Recall that the elements of $\mathbf{S}$ are ordered by $a < e < 1$. Let $d \in B$ be such that $d/\alpha \prec 1/\alpha$ in $\mathbf{A}$ and d is the largest element in d/α. Such a d exists by virtue of the form of subdirectly irreducible algebras in $\mathcal{J}$ and Lemma 3.3. The subdirectly irreducible algebra $\mathbf{A} = \mathbf{B}/\alpha$ is $(1/\alpha, d/\alpha)$-irreducible. We have $1/\alpha = \{b \in B : b \nleq d\}$

Let $\Delta = \{i : d_i \neq 1\}$.

We have $b \nleq d$ if and only if there exists $1 \leq i \leq n$ with $i \in \Delta$ and $b_i > d_i$.

Consider the following equivalence relation γ on $\mathbf{B}$:

(i) if $r \nleq d$, then $(r, s) \in \gamma$ iff $s \nleq d$, and

(ii) if $r \leq d$, then $(r, s) \in \gamma$ iff $r_i = s_i$ for all $i \in \Delta$.

So for all $r \in B$ we have

(i) $r \nleq d$ implies $r \in 1/\gamma$, and

(ii) $r \leq d$ implies $r/\gamma = \{s \in B : r_i = s_i$ for all $i \in \Delta\}$.

Note that $1/\gamma = 1/\alpha$.

We argue that γ is a congruence relation on $\mathbf{B}$. Let $(r, s) \in \gamma$ and $b \in B$. If $(r, 1) \in \gamma$, then $rb \geq r$ and $br \geq r$ by Lemma 3.1. So $br \equiv rb \equiv 1(\gamma)$ since $1/\gamma$ is upwardly closed. Hence (rb, sb) and (br, bs) are in γ. If $r \leq d$ and $s \leq d$, then $(br)_i = (bs)_i$ for all $i \in \Delta$ and $(rb)_i = (sb)_i$ for all $i \in \Delta$. So again (rb, sb) and (br, bs) are in γ. So γ is indeed a congruence relation.

If $\alpha \vee \gamma > \alpha$, then in $\mathsf{Con}\,\mathbf{A}$ we have $(\alpha \vee \gamma)/\alpha > 0_A$. However, the class $1/(\alpha \vee \gamma) = 1/\alpha$, which violates the $(1/\alpha, d/\alpha)$-irreducibility of $\mathbf{A}$. Thus, $\alpha \vee \gamma = \alpha$.

Finally, suppose $\mathbf{A}$ is not pure. Then there exist $r, s \in B$ such that $r/\alpha \cdot s/\alpha \notin \{s/\alpha, 1/\alpha\}$. In particular, $r \not\leq s$ for otherwise $rs = s$ would imply $r/\alpha \cdot s/\alpha = s/\alpha$.

We have $r/\alpha \cdot s/\alpha \leq d/\alpha$, so $r \leq d$ and $s \leq d$. We argue that $(rs, s) \in \gamma$ and that $(rs, s) \notin \alpha$, thereby contradicting $\alpha \vee \gamma = \alpha$. So $\mathbf{A}$ must be pure.

Clearly $(rs, s) \notin \alpha$ since $rs/\alpha = r/\alpha \cdot s/\alpha \neq s/\alpha$. If it were the case that $(rs, s) \notin \gamma$, then there would exist $i \in \Delta$ with $r_i s_i \neq s_i \leq d_i \leq e$. If $s_i = e$, then $r_i = e$ or $r_i = a$. In either case, $r_i \cdot s_i = s_i$. Similarly if $s_i = a$. So no such i can exist, that is, $(rs, s) \in \gamma$. □

Theorem 3.10. *Every subdirectly irreducible algebra in $\mathcal{J}$ is pure.*

Proof. The variety $\mathcal{J}$ of join algebras is locally finite. Let ϕ be a quasi-identity that fails in $\mathcal{J}$. Then it fails in a finite join algebra, say $\mathbf{A}$. Write $\mathbf{A}$ as a subdirect product of subdirectly irreducible algebras $\mathbf{A}_i$. These are all in $\mathcal{J}$, and hence (being finite and subdirectly irreducible) they are pure by the previous lemma. The quasi-identity ϕ must fail in one of these algebras $\mathbf{A}_i$, for if it did not, it would hold in all of them, and hence also in $\mathbf{A}$. This shows that $\mathcal{J}$ is the quasivariety generated by the class $\mathcal{K}$ of all finite subdirectly irreducible algebras in $\mathcal{J}$, that is, $\mathcal{J} = \mathsf{SPP_U}(\mathcal{K})$. Every member of $\mathcal{K}$ is pure. An arbitrary subdirectly irreducible $\mathbf{A} \in \mathcal{J}$ is therefore in $\mathsf{SPP_U}(\mathcal{K})$, hence in $\mathsf{SP_U}(\mathcal{K})$, and thus is pure since subalgebras and ultraproducts of pure algebras are pure. □

Note that $\mathcal{J}$ is finitely generated as a variety by the 3-element algebra $\mathbf{S}$ but as a quasivariety the class $\mathcal{J}$ is not finitely generated since $\mathcal{J}$ contains arbitrarily large finite subdirectly irreducible algebras.

Although $\mathcal{J}$ contains only one proper nontrivial subvariety – the variety of semilattices – it does have a rich collection of subquasivarieties.

Theorem 3.11. *The variety $\mathcal{J}$ contains $2^{\aleph_0}$ quasivarieties.*

Proof. We use the ordered sets $\mathbf{C}_n$ of Corollary 2.5. For every positive n let $\mathbf{A}_n = \mathrm{Jo}(\mathbf{C}_n)$. Then $\mathbf{A}_n$ is a subdirectly irreducible algebra in $\mathcal{J}$. If I is a set of natural numbers with $m \notin I$, then $\mathbf{A}_m \notin \mathsf{SPP_U}(\{\mathbf{A}_n : n \in I\})$. So for arbitrary sets I and K of natural numbers with $I \neq K$, the quasivarieties $\mathsf{Q}(\{\mathbf{A}_n : n \in I\})$ and $\mathsf{Q}(\{\mathbf{A}_n : n \in K\})$ are distinct.

□

4 Decomposition of free algebras

We describe a method for decomposing free algebras in an arbitrary variety into overlapping canonically defined subalgebras and show how the cardinality of the free algebras can be obtained using an inclusion-exclusion argument involving these subalgebras. In subsequent sections we apply this general decomposition method to the varieties $\mathcal{H}$ and $\mathcal{J}$. For these two varieties the order structure of these canonical subalgebras of the free algebras are amenable to analysis and by means of this analysis we obtain sharp results concerning the free spectra and G-spectra of $\mathcal{H}$ and $\mathcal{J}$.

Let X be a set and $\mathcal{L}$ a set of operation symbols. By $\mathrm{T}_{\mathcal{L}}(X)$ we denote the set of all terms that can be built recursively from X using the symbols in $\mathcal{L}$. If $\mathcal{L}$ is clear from the context we write $\mathrm{T}(X)$. The free term algebra for $\mathcal{L}$ generated by X has universe $\mathrm{T}_{\mathcal{L}}(X)$ and is denoted $\mathbf{T}_{\mathcal{L}}(X)$. We usually consider a fixed positive integer n with $X = \{x_1, \ldots, x_n\}$ and in this case we may write $\mathrm{T}_{\mathcal{L}}(n)$ and $\mathrm{T}(n)$ for $\mathrm{T}_{\mathcal{L}}(X)$ and $\mathrm{T}(X)$ respectively. For $t \in \mathrm{T}_{\mathcal{L}}(X)$ we denote by $\mathrm{var}(t)$ the set of all elements of X that appear in t.

Let $\mathcal{V}$ be a variety having $\mathcal{L}$ as its set of operation symbols. Elements of the free algebra in $\mathcal{V}$ on n free generators may be represented as $\bar{s}$ where $s \in \mathrm{T}_{\mathcal{L}}(n)$ and for $s, t \in \mathrm{T}_{\mathcal{L}}(n)$ we have

$$\bar{s} = \bar{t} \text{ if and only if } \mathcal{V} \models s \approx t.$$

For $1 \leq n < \omega$ let $\mathbf{F}_{\mathcal{V}}(n)$ denote the free algebra in $\mathcal{V}$ freely generated by the elements $\bar{x}_1, \ldots, \bar{x}_n$. The universe of $\mathbf{F}_{\mathcal{V}}(n)$ is denoted $\mathrm{F}_{\mathcal{V}}(n)$ and consists of $\{\bar{s} : s \in \mathrm{T}_{\mathcal{L}}(n)\}$. Thus for $s \in \mathrm{T}_{\mathcal{L}}(n)$ we have

$$s^{\mathbf{F}_{\mathcal{V}}(n)}(\bar{x}_1, \ldots, \bar{x}_n) = \overline{s(x_1, \ldots, x_n)}.$$

Definition 4.1. *A* valuation *into an algebra* $\mathbf{A}$ *is a map* $v : \{x_1, \ldots, x_n\} \to A$ *or its natural homomorphic extension* $v : \mathbf{T}_{\mathcal{L}}(n) \to \mathbf{A}$ *with the property that the set* $\{v(x_1), \ldots, v(x_n)\}$ *generates* $\mathbf{A}$.

Valuations are related to identities by the fact that in any variety $\mathcal{V}$ and for terms $s, t \in \mathrm{T}(X)$

$$\mathcal{V} \models s \approx t \text{ iff } v(s) = v(t) \text{ for every valuation } v \text{ into } \mathbf{A} \in \mathcal{V}.$$

For the next series of definitions we require that $\mathcal{L}$ not contain any nullary symbols. For any term $s \in \mathrm{T}_{\mathcal{L}}(X)$ we define the *right-most variable of* s, denoted $\mathrm{rv}(s)$, inductively as follows:

$$\mathrm{rv}(s) = \begin{cases} x_i & \text{if } s = x_i \\ \mathrm{rv}(t_m) & \text{if } s = f(t_1, \dots, t_m) \text{ for } f \in \mathcal{L} \text{ and } t_1, \dots, t_m \in \mathrm{T}_{\mathcal{L}}(X). \end{cases}$$

For $1 \leq i \leq n$ let

$$\mathrm{T}_i(n) = \{s \in \mathrm{T}_{\mathcal{L}}(n) : \mathrm{rv}(s) = x_i\}$$

and

$$\overline{\mathrm{T}}_i(n) = \{\bar{s} \in \mathrm{F}_{\mathcal{V}}(n) : s \in \mathrm{T}_i(n)\}.$$

The set $\overline{\mathrm{T}}_i(n)$ is a subuniverse of $\mathbf{F}_{\mathcal{V}}(n)$ and by $\overline{\mathbf{T}}_i(n)$ we denote the corresponding subalgebra. For $1 \leq \ell \leq n$ we write

$$\overline{\mathrm{T}}_{\leq \ell}(n) = \bigcap_{1 \leq i \leq \ell} \overline{\mathrm{T}}_i(n) \quad \text{and} \quad \overline{\mathbf{T}}_{\leq \ell}(n) = \bigcap_{1 \leq i \leq \ell} \overline{\mathbf{T}}_i(n).$$

An earlier version of the following theorem is given in [1].

Theorem 4.2. *Let $\mathcal{V}$ be a locally finite variety in a language that has no nullary operation symbols. For $1 \leq n < \omega$ we have*

$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{(\ell-1)} \binom{n}{\ell} |\overline{\mathrm{T}}_{\leq \ell}(n)|. \tag{3}$$

Proof. The set $\mathrm{F}_{\mathcal{V}}(n)$ is finite since $\mathcal{V}$ is locally finite. There are no nullary constant symbols so every term in $\mathrm{T}(n)$ has a right-most variable. Hence $\mathrm{F}_{\mathcal{V}}(n) = \bigcup_{i=1}^{n} \overline{\mathrm{T}}_i(n)$. Applying the inclusion-exclusion principle we see that

$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{\substack{1 \leq \ell \leq n \\ 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n}} (-1)^{\ell-1} \, |\overline{\mathrm{T}}_{i_1}(n) \cap \overline{\mathrm{T}}_{i_2}(n) \cap \cdots \cap \overline{\mathrm{T}}_{i_\ell}(n)|. \tag{4}$$

Every permutation of X extends to an automorphism of $\mathbf{F}_{\mathcal{V}}(n)$ so all of the subalgebras $\overline{\mathbf{T}}_i(n)$ are isomorphic. Likewise for every choice of $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$ the algebra $\overline{\mathbf{T}}_{i_1}(n) \cap \overline{\mathbf{T}}_{i_2}(n) \cap \cdots \cap \overline{\mathbf{T}}_{i_\ell}(n)$ is isomorphic to $\overline{\mathbf{T}}_{\leq \ell}(n)$. Therefore

$$|\overline{\mathrm{T}}_{i_1}(n) \cap \overline{\mathrm{T}}_{i_2}(n) \cap \cdots \cap \overline{\mathrm{T}}_{i_\ell}(n)| = |\overline{\mathrm{T}}_{\leq \ell}(n)|.$$

For $1 \leq \ell \leq n$ there are $\binom{n}{\ell}$ choices for $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$. Hence from (4) we obtain (3). □

Theorem 4.2 is completely general. Its usefulness depends on the variety $\mathcal{V}$ and how much structure can be found for the algebras $\overline{\mathrm{T}}_i(n)$ and their intersections. We next show that for $\mathcal{H}$ and $\mathcal{J}$ the structure of the $\overline{\mathrm{T}}_i(n)$ can be determined sufficiently so as to successfully apply Theorem 4.2.

5 Free join algebras

In this section we describe $\mathbf{F}_{\mathcal{J}}(n)$, the free algebra on n free generators for the variety of join algebras. We find an explicit formula for the cardinality of $\mathbf{F}_{\mathcal{J}}(n)$ in terms of the number q_n of quasi-orders on an n-element set. We use known bounds on q_n to obtain sharp upper and lower bounds on the free spectrum and on the G-spectrum, $\mathrm{G}_{\mathcal{J}}(n)$, of $\mathcal{J}$. In the next section we provide a similar analysis for the variety $\mathcal{H}$.

We apply the material of Section 4 to the variety $\mathcal{J}$ of join algebras. Throughout this section the language $\mathcal{L} = \{\cdot\}$ consists of one binary operation symbol and as usual we write $T(n)$ for $T_{\mathcal{L}}(n)$. We describe the subalgebras $\overline{\mathbf{T}}_i(n)$ of $\mathbf{F}_{\mathcal{J}}(n)$ and determine the cardinality of $\mathbf{F}_{\mathcal{J}}(n)$ by means of Theorem 4.2.

Definition 5.1. *Let t be an arbitrary member of $T(n)$. The binary relation $\mathrm{re}(t)$ on $\mathrm{var}(t)$ is defined inductively by:*

$$\mathrm{re}(t) = \begin{cases} \{(x_i, x_i)\} & \text{if } t = x_i \\ \mathrm{re}(p) \cup \mathrm{re}(q) \cup \{(\mathrm{rv}(p), \mathrm{rv}(q))\} & \text{if } t = p \cdot q. \end{cases}$$

The transitive closure of $\mathrm{re}(t)$ is denoted $\mathrm{qo}(t)$.

A *quasi-order* on a set Z is any reflexive transitive binary relation on Z. For a quasi-order σ on Z and $a \in Z$ let

$$a/\sigma = \{b \in Z : (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}.$$

If σ is a quasi-order on Z, then the quotient Z/σ is always an ordered set in which $a/\sigma \leq b/\sigma$ if and only if $(a, b) \in \sigma$. The ordered set Z/σ has a top element if and only if there exists $z \in Z$ such that $(a, z) \in \sigma$ for all $a \in Z$. For every $\mathcal{L}$ consisting of one binary operation $\cdot$ and for every $t \in \mathrm{T}(n)$, the relation $\mathrm{qo}(t)$ is a quasi-order on the set $\mathrm{var}(t)$ in which $\mathrm{rv}(t)/\mathrm{qo}(t)$ is the top element of the ordered set $\mathrm{var}(t)/\mathrm{qo}(t)$.

Lemma 5.2. *Let τ be a quasi-order on a set $Y \subseteq X$ such that Y/τ has a top element. Then there exists $t \in \mathrm{T}(X)$ such that $\mathrm{qo}(t) = \tau$.*

Proof. Let $(x_{i_1}, x_{j_1}), (x_{i_2}, x_{j_2}), (x_{i_3}, x_{j_3}), \ldots, (x_{i_k}, x_{j_k})$ be any list of the elements of τ subject only to the constraint that x_{j_1}/τ is the top element of Y/τ. Let

$$t = (x_{i_k} x_{j_k})(\ldots(x_{i_3} x_{j_3})((x_{i_2} x_{j_2})(x_{i_1} x_{j_1}))\ldots).$$

Clearly $\mathrm{var}(t) = Y$. It is immediate from the recursive definition of the operator qo that $\mathrm{qo}(t) = \tau$. $\square$

Definition 5.1 and Lemma 5.2 apply to any $T_{\mathcal{L}}(n)$ for a language $\mathcal{L}$ consisting of one binary operation symbol. We now show how the equational theory of join algebras can be represented using the quasi-orders $\mathrm{qo}(t)$ as t ranges over join algebra terms. We exhibit a bijection between the collection of quasi-orders on subsets of X and the elements of the free join algebra $\mathbf{F}_{\mathcal{J}}(X)$.

Lemma 5.3. *Let v be a valuation into a pure join algebra $\mathbf{A}$ and let $t \in \mathrm{T}(n)$.*

1. $v(t) = v(\mathrm{rv}(t))$ *or* 1.
2. *If* $t = p \cdot q$, *then* $v(t) = v(rv(p)) \cdot v(\mathrm{rv}(q))$ *or* 1.
3. $v(t) < 1$ *iff* $v(x_i \cdot x_j) < 1$ *for all* $(x_i, x_j) \in \mathrm{qo}(t)$.

Proof. The first claim is an easy consequence of the fact that in any pure join algebra, if $ab \neq 1$, then $ab = b$. The second claim follows from the first and from that $v(p \cdot q) = v(p) \cdot v(q)$ for any terms p and q.

Suppose $v(t) < 1$. Then for every $x_i \in \mathrm{var}(t)$ we have $v(x_i) < 1$ and therefore $v(x_i \cdot x_i) = v(x_i) < 1$. If $(x_i, x_j) \in \mathrm{re}(t)$ with $i \neq j$, then there exists a subterm $p \cdot q$ of t with $\mathrm{rv}(p) = x_i$ and $\mathrm{rv}(q) = x_j$. Since $v(t) < 1$ we have $v(p \cdot q) < 1$. By (2) we have $v(x_i \cdot x_j) < 1$. If $(x_i, x_j) \in \mathrm{qo}(t)$ with $i \neq j$, then there exist $x_{k_0} \dots x_{k_m}$ with $x_i = x_{k_0}$, $x_j = x_{k_m}$ and $(x_{k_\ell}, x_{k_{\ell+1}}) \in \mathrm{re}(t)$ for $0 \leq \ell < m$. We have already seen that $v(x_{k_\ell} \cdot x_{k_{\ell+1}}) < 1$ for each ℓ, so $v(x_{k_\ell}) \leq v(x_{k_{\ell+1}})$. The transitivity of the order on $\mathbf{A}$ gives $v(x_i) \leq v(x_j)$ and therefore $v(x_i \cdot x_j) < 1$.

Conversely, suppose $v(x_i \cdot x_j) < 1$ for all $(x_i, x_j) \in \mathrm{qo}(t)$. We argue by induction that $v(s) < 1$ for every subterm s of t. If $x_i \in \mathrm{var}(t)$, then $v(x_i) < 1$ since $v(x_i \cdot x_i) < 1$. Let $p \cdot q$ be a subterm of t and suppose $v(p) < 1$ and $v(q) < 1$. By part (1) we have $v(p) = v(\mathrm{rv}(p))$ and $v(q) = v(\mathrm{rv}(q))$. Moreover, $(\mathrm{rv}(p), \mathrm{rv}(q)) \in \mathrm{qo}(t)$ so $v(\mathrm{rv}(p) \cdot \mathrm{rv}(q)) < 1$. Therefore $v(p \cdot q) = v(p) \cdot v(q) = v(\mathrm{rv}(p)) \cdot v(\mathrm{rv}(q)) = v(\mathrm{rv}(p) \cdot \mathrm{rv}(q)) < 1$. $\square$

Lemma 5.4. *Let $s, t \in \mathrm{T}(n)$.*

1. $\mathcal{J} \models s \approx t$ *iff* $\mathrm{qo}(s) = \mathrm{qo}(t)$.
2. $\mathcal{J} \models st \approx t$ *iff* $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$.

Proof. Suppose $\mathcal{J} \not\models s \approx t$. There must exist a valuation v into either $\mathbf{S}$ or the 2-element semilattice for which, say, $v(s) = 1$ and $v(t) < 1$. But then $\mathrm{qo}(s) = \mathrm{qo}(t)$ would contradict Lemma 5.3 (3). For the other direction in (1), suppose $\mathrm{qo}(s) \neq \mathrm{qo}(t)$. If $\mathrm{var}(s) \neq \mathrm{var}(t)$, then we must have $\mathcal{J} \not\models s \approx t$. So we may as well assume at the outset that $\mathrm{var}(s) = \mathrm{var}(t) = Y \subseteq X$. Suppose $(x_\ell, x_m) \in \mathrm{qo}(s) - \mathrm{qo}(t)$. Form the poset $\mathbf{P}_t = (Y/\mathrm{qo}(t)) \oplus 1$ and the pure join algebra $\mathbf{A}_t = \mathrm{Jo}(\mathbf{P}_t)$. Let v be the valuation into $\mathbf{A}_t$ for which $v(x_i) = x_i/\mathrm{qo}(t)$ for all $x_i \in Y$. For every $(x_i, x_j) \in \mathrm{qo}(t)$ we have $x_i/\mathrm{qo}(t) \leq x_j/\mathrm{qo}(t) < 1$ in $\mathbf{P}_t$. Hence $v(x_i \cdot x_j) = x_j/\mathrm{qo}(t) < 1$. Therefore $v(t) < 1$ by Lemma 5.3 (3). But $v(x_\ell) \nleq v(x_m)$ in $\mathbf{P}_t$, which implies that $v(x_\ell \cdot x_m) = 1$. So $v(s) \neq v(t)$ in $\mathbf{A}_t$ and therefore $\mathcal{V} \not\models s \approx t$.

If $\mathcal{J} \models st \approx t$, then $\mathrm{qo}(st) = \mathrm{qo}(t)$ by (1). Since $\mathrm{qo}(st)$ is the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t) \cup \{(\mathrm{rv}(s), \mathrm{rv}(t))\}$, we have $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$. On the other hand, if $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$, then $\mathrm{var}(s) \subseteq \mathrm{var}(t)$ and therefore $(\mathrm{rv}(s), \mathrm{rv}(t)) \in \mathrm{qo}(t)$. So the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t) \cup \{(\mathrm{rv}(s), \mathrm{rv}(t))\}$ is $\mathrm{qo}(t)$. Therefore $\mathrm{qo}(st) = \mathrm{qo}(t)$, which from (1) gives $\mathcal{J} \models st \approx t$. □

Recall that $\overline{\mathrm{T}}_i(n) = \{\bar{s} \in \mathrm{F}_{\mathcal{J}}(n) : \mathrm{rv}(s) = x_i\}$ and that $\overline{\mathbf{T}}_i(n)$ is the subalgebra of $\mathbf{F}_{\mathcal{J}}(n)$ with universe $\overline{\mathrm{T}}_i(n)$. Note that if $\mathcal{J} \models s \approx t$, then $\mathrm{var}(s) = \mathrm{var}(t)$. In particular, if $\bar{s} \in \overline{\mathrm{T}}_i(n)$, then $x_i \in \mathrm{var}(s)$.

For $1 \leq i \leq n$ define

$$\begin{aligned} \mathrm{Q}_i(n) = \{\sigma :\ & \exists Y \subseteq X,\ x_i \in Y,\ \sigma \text{ is a quasi-order on } Y, \\ & \text{and } x_i/\sigma \text{ is the top element of } Y/\sigma\}. \end{aligned}$$

For σ_1 and $\sigma_2 \in \mathrm{Q}_i(n)$ with σ_k defined on $Y_k \subseteq X$ for $k = 1, 2$, let $\sigma_1 \cdot \sigma_2$ denote the smallest quasi-order on $Y_1 \cup Y_2$ that contains both σ_1 and σ_2. Then necessarily $\sigma_1 \cdot \sigma_2 \in \mathrm{Q}_i(n)$ and $\sigma_1 \cdot \sigma_2$ is the transitive closure of $\sigma_1 \cup \sigma_2$. Let $\mathbf{Q}_i(n)$ denote the algebra $\langle \mathrm{Q}_i(n), \cdot \rangle$.

Theorem 5.5. *The map* $\bar{t} \mapsto \mathrm{qo}(t)$ *is an isomorphism from* $\overline{\mathbf{T}}_i(n)$ *onto* $\mathbf{Q}_i(n)$.

Proof. Lemmas 5.4 and 5.2 show the map is well-defined, one-to-one and onto $\mathrm{Q}_i(n)$. It remains to show that $\mathrm{qo}(s \cdot t) = \mathrm{qo}(s) \cdot \mathrm{qo}(t)$ for all $\bar{s}$ and $\bar{t}$ in $\overline{\mathrm{T}}_i(n)$. Let $\mathrm{rv}(s) = x_\ell$ and $\mathrm{rv}(t) = x_m$. By the definition of qo we have that $\mathrm{qo}(s \cdot t)$ is the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t) \cup \{(x_\ell, x_m)\}$. Clearly $(x_\ell, x_i) \in \mathrm{qo}(s)$ since $\bar{s} \in \overline{\mathrm{T}}_i(n)$. Also $(x_i, x_m) \in \mathrm{qo}(t)$ since $t \in \overline{\mathrm{T}}_i(n)$ and $\mathrm{rv}(t) = x_m$. So (x_ℓ, x_m) is in the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t)$. Hence the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t) \cup \{(x_\ell, x_m)\}$ is the same as the transitive closure of $\mathrm{qo}(s) \cup \mathrm{qo}(t)$, which is $\mathrm{qo}(s) \cdot \mathrm{qo}(t)$. $\square$

From Theorem 5.5 it follows that $\mathbf{Q}_i(n)$ is a join algebra. The induced order on $\mathrm{Q}_i(n)$ is that of containment. Therefore, $\bar{s} \leq \bar{t}$ in $\overline{\mathbf{T}}_i(n)$ if and only if $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$. Actually, from Lemma 5.4 we see that for arbitrary $\bar{s}, \bar{t} \in \mathrm{F}_{\mathcal{J}}(n)$ it is the case that $\bar{s} \leq \bar{t}$ if and only if $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$.

Let q_k denote the number of quasi-orders on $\{1, 2, \ldots, k\}$. It is known that q_k is also equal to the number of topologies on $\{1, 2, \ldots, k\}$. For example, the values of q_k for $k = 1, 2, 3, 4$ are 1, 4, 29, 355 respectively. The paper [10] by Heitzig and Reinhold contains background and recently obtained values for the sequence q_k. We let $q_0 = 1$.

Theorem 5.6.

$$|\mathrm{F}_{\mathcal{J}}(n)| = \sum_{i=0}^{n-1} q_i \binom{n}{i} (2^{n-i} - 1).$$

Proof. We have

$$|\mathbf{F}_{\mathcal{J}}(n)| = \sum_{\ell=1}^{n} (-1)^{(\ell-1)} \binom{n}{\ell} |\overline{\mathrm{T}}_{\leq \ell}(n)|$$

by Theorem 4.2. The cardinality of $\overline{\mathrm{T}}_{\leq \ell}(n)$ is equal to $|\mathrm{Q}_1(n) \cap \cdots \cap \mathrm{Q}_\ell(n)|$ by Theorem 5.5. Let σ be a quasi-order on a set $Y \subseteq X$ for which Y/σ has a top element. Let $Z \subseteq Y$ be the set of all x_i for which x_i/σ is the top element in Y/σ. Then $\sigma = \tau \cup (Y \times Z)$ where τ is a quasi-order on $Y - Z$. Conversely, if $\emptyset \neq Z \subseteq Y \subseteq X$ and τ is a quasi-order on $Y - Z$, then $\sigma = \tau \cup (Y \times Z)$ is a quasi-order on Y for which Y/σ has a maximal element consisting of x_k/σ for every $x_k \in Z$. For $\sigma \in \mathrm{Q}_1(n) \cap \cdots \cap \mathrm{Q}_\ell(n)$ there are $\binom{n-\ell}{i}$ choices for the set $Y - Z$ if $|Y - Z| = i$. For a given i, with $0 \leq i \leq n - \ell$, there are q_i choices for a quasi-order on $Y - Z$. The set Z must contain $\{x_1, \ldots, x_\ell\}$ so there are $2^{n-\ell-i}$ ways to choose the remaining elements of Z. Therefore

$$|\mathrm{Q}_1(n) \cap \cdots \cap \mathrm{Q}_\ell(n)| = \sum_{i=0}^{n-\ell} q_i \binom{n-\ell}{i} 2^{n-\ell-i}.$$

This implies

$$|\mathbf{F}_{\mathcal{J}}(n)| = \sum_{\ell=1}^{n} (-1)^{(\ell-1)} \binom{n}{\ell} \left(\sum_{i=0}^{n-\ell} q_i \binom{n-\ell}{i} 2^{n-\ell-i} \right).$$

We interchange the order of summation and use the formula $\binom{n}{\ell}\binom{n-\ell}{i} = \binom{n}{i}\binom{n-i}{n-\ell-i}$ to obtain

$$|\mathbf{F}_{\mathcal{J}}(n)| = \sum_{i=0}^{n-1} q_i \binom{n}{i} \left(\sum_{\ell=1}^{n-i} (-1)^{\ell-1} \binom{n-i}{n-\ell-1} 2^{n-\ell-i} \right).$$

The inner summation simplifies to $2^{n-i} - 1$. Thus,

$$|\mathbf{F}_{\mathcal{J}}(n)| = \sum_{i=0}^{n-1} q_i \binom{n}{i} (2^{n-i} - 1).$$

□

For example, if we use the previously mentioned values of q_i for $1 \le i \le 4$, then we see that the values of $|\mathrm{F}_{\mathcal{J}}(n)|$ for $1 \le n \le 5$ are 1, 5, 28, 231 and 3031 respectively.

Let p_n denote the number of partially ordered sets on $\{1, 2, \ldots, n\}$. For example, the values of p_n for $1 \le n \le 4$ are 1, 3, 19, and 219 respectively. Clearly $p_n \le q_n$ for all n. Erné [7, Satz 4.6] shows that p_n and q_n are asymptotically equal functions. In [13] Kleitman and Rothschild show that there exists a positive constant c such that for all sufficiently large n

$$\frac{n^2}{4} + \frac{3n}{2} - c \lg n \le \lg(p_n) \le \frac{n^2}{4} + \frac{3n}{2} + c \lg n.$$

Here lg denotes log base 2. From Erné's result we have similar bounds on $\lg(q_n)$. That is, there is a positive constant c such that for all large n,

$$\frac{n^2}{4} + \frac{3n}{2} - c \lg n \le \lg(q_n) \le \frac{n^2}{4} + \frac{3n}{2} + c \lg n. \tag{5}$$

For $0 \le i \le n-1$ let $t(n, i)$ denote the term $q_i \binom{n}{i}(2^{n-i} - 1)$ that appears in the formula for $|\mathrm{F}_{\mathcal{J}}(n)|$ in Theorem 5.6.

Lemma 5.7. *For all sufficiently large n and all $1 \le i \le n-1$ the inequality $t(n, i-1) \le t(n, i)$ holds.*

Proof. For $i \ge 2$ and any quasi-order σ on $\{1, \ldots, i-1\}$ we can form three distinct quasi-orders on $\{1, \ldots, i\}$:

$$\sigma \cup \{(k, i) : 1 \le k \le i\}, \quad \sigma \cup \{(i, k) : 1 \le k \le i\} \quad \text{and} \quad \sigma \cup \{(i, i)\}.$$

Since σ can be uniquely recovered from each of these quasi-orders, we get $q_i \geq 3q_{i-1}$. For all $1 \leq i \leq n/2$ we have $\binom{n}{i-1} \leq \binom{n}{i}$ and $2^{n-(i-1)} - 1 \leq 3(2^{n-i} - 1)$. Since $q_{i-1} \leq q_i/3$ we have $t(n, i-1) \leq t(n, i)$ for $i \leq n/2$. Next, we consider the ratio $t(n, n-i)/t(n, n-(i+1))$ for $1 \leq i \leq n/2$ and for n sufficiently large that the inequalities in (5) hold. This gives

$$\frac{t(n, n-i)}{t(n, n-(i+1))} = \frac{q_{n-i}}{q_{n-(i+1)}} \frac{\binom{n}{i}}{\binom{n}{i+1}} \frac{2^i - 1}{2^{i+1} - 1} \geq \frac{q_{n-i}}{q_{n-(i+1)}} \frac{(i+1)}{(n-i)} \frac{1}{3} \geq \frac{q_{n-i}}{q_{n-(i+1)}} \frac{1}{3(n-i)}.$$

We have from (5)

$$\begin{aligned} \lg(\frac{q_{n-i}}{q_{n-(i+1)}}) &\geq \frac{n}{2} - \frac{i}{2} + \frac{5}{4} - c(\lg(n-(i+1)) + \lg(n-i)) \\ &\geq \frac{n-i}{2} - c(\lg(n-(i+1)) + \lg(n-i)). \end{aligned}$$

For n large enough so that

$$\frac{n-i}{2} \geq c(\lg(n-(i+1)) + \lg(n-i)) + \lg(3(n-i))$$

we have $t(n, n-i)/t(n-(i+1)) \geq 1$, as desired. □

It follows that

$$nq_{n-1} = t(n, n-1) \leq |\mathrm{F}_{\mathcal{J}}(n)| = \sum_{i=0}^{n-1} t(n, i) \leq n\, t(n, n-1) = n^2 q_{n-1} \qquad (6)$$

for all sufficiently large n.

If we apply the bounds in (5) to q_{n-1} and use them in (6), then we obtain the following bounds for the size of $\mathbf{F}_{\mathcal{J}}(n)$.

Theorem 5.8. *There exists a positive constant c such that for all sufficiently large n,*

$$2^{\frac{n^2}{4}+n-c\lg n} \leq |\mathrm{F}_{\mathcal{J}}(n)| \leq 2^{\frac{n^2}{4}+n+c\lg n}.$$

We next investigate the G-spectrum of the variety of join algebras and provide upper and lower bounds for $\mathrm{G}_{\mathcal{J}}(k)$.

For any variety $\mathcal{V}$ an upper bound for $\mathrm{G}_{\mathcal{V}}(k)$ is the cardinality of the congruence lattice of the free algebra for $\mathcal{V}$ on k free generators. If $\mathbf{A}$ is any finite join algebra, then by part (2) of Lemma 3.3 we have $|\mathsf{Con}\,\mathbf{A}| \leq 2^{|A|}$. In what follows we write $f(n)$ for the cardinality of $\mathbf{F}_{\mathcal{J}}(n)$. Thus, we have $\mathrm{G}_{\mathcal{J}}(k) \leq 2^{f(k)}$.

We next consider a lower bound for the G-spectrum of $\mathcal{J}$. For every element $\bar{t}$ of $\mathbf{F}_{\mathcal{J}}(n)$ we have from Definition 5.1 the quasi-order $\mathrm{qo}(t)$ on $\mathrm{var}(t)$. For $\bar{s}, \bar{t} \in \mathrm{F}_{\mathcal{J}}(n)$, by Lemma 5.4, $\bar{s} \leq \bar{t}$ if and only if $\mathrm{qo}(s) \subseteq \mathrm{qo}(t)$. So $n(n-1)$ is an upper bound for the length of the longest chain in $\mathbf{F}_{\mathcal{J}}(n)$. If $\bar{s}$ and $\bar{t}$ are distinct elements of $\mathbf{F}_{\mathcal{J}}(n)$ such that $\mathrm{qo}(s)$ and $\mathrm{qo}(t)$ have the same number of elements, then they are incomparable. Therefore, there is an antichain Z in the poset of $\mathbf{F}_{\mathcal{J}}(n)$ of cardinality at least $f(n)/n^2$. If $S \subseteq Z$, then let θ_S be the congruence relation on $\mathbf{F}_{\mathcal{J}}(n)$ having $\{\bar{t} \in \mathrm{F}_{\mathcal{J}}(n) : \bar{t} \geq \bar{s}$ for some $\bar{s} \in S\}$ as one congruence class and all other congruence classes of θ_S are singletons. Then $\bar{x}_i/\theta_S = \{\bar{x}_i\}$ for each free generator $\bar{x}_i$. Therefore, the number of congruence relations on $\mathbf{F}_{\mathcal{J}}(n)$ for which each free generator is in a singleton congruence class is at least 2^Z, which is at least $2^{f(n)/n^2}$. Each $\bar{x}_i$ in $\mathbf{F}_{\mathcal{J}}(n)$ has the property that if $\bar{p} \cdot \bar{q} = \bar{x}_i$, then $\bar{p} = \bar{q} = \bar{x}_i$. From these observations an argument as in [3] or in [8] shows that there are at least $2^{f(n)/n^2}/n!$ congruences on $\mathbf{F}_{\mathcal{J}}(n)$ having pairwise nonisomorphic quotients. Thus, for all n

$$\frac{2^{\frac{f(n)}{n^2}}}{n!} \leq \mathrm{G}_{\mathcal{J}}(n) \leq 2^{f(n)}.$$

The bounds for $f(n)$ given in Theorem 5.8 also apply to $f(n)/n^2$, possibly with a different value of c. By adjusting the value of c to absorb the $n!$ divisor in the lower bound for $\mathrm{G}_{\mathcal{J}}(n)$ we obtain the following.

Theorem 5.9. *There exists a positive constant c such that for all sufficiently large n,*

$$2^{2^{\frac{n^2}{4}+n-c\lg n}} \leq \mathrm{G}_{\mathcal{J}}(n) \leq 2^{2^{\frac{n^2}{4}+n+c\lg n}}.$$

Thus, for the variety of join algebras the G-spectrum is one level of exponentiation higher than that of the free spectrum. For the subvariety of semilattices there is a similar gap between the G-spectra and free spectra, e.g. [2].

6 Free $\mathcal{H}$ algebras

In this section we consider the free spectra and G-spectra for $\mathcal{H}$ and some of its subvarieties. The structure and cardinality of finitely generated free algebras for the variety $\mathcal{HI}$ of all Hilbert algebras have been investigated in papers such as Diego [6], Urquhart [15], and Hendriks [11]. In [9], F. Guzmán and C. Lynch consider free algebras in varieties of Hilbert algebras generated by a single finite pure Hilbert algebra. They provide an explicit formula for the cardinality of the free algebra in the variety generated by a pure Hilbert algebra that is a finite chain. Our paper [1] investigates the structure of free algebras for some varieties of Hilbert algebras.

Throughout this section we use the notation developed in Section 4 and we apply Theorem 4.2. In order to satisfy the condition that there not be any constant symbols in the similarity type we consider Hilbert algebras using the language $\mathcal{L} = \{\to\}$. We can represent 1 by the term $x_i \to x_i$ so there is no loss of generality in suppressing the constant symbol 1. For a variety $\mathcal{V}$ of Hilbert algebras we write $\mathcal{V} \models s \leq t$ if $\mathcal{V} \models st \approx 1$.

In a variety $\mathcal{V}$ of Hilbert algebras the subuniverses $\overline{\mathrm{T}}_i(n)$ of $\mathbf{F}_{\mathcal{V}}(n)$ can have significant overlap. For example, if $t = (x_1 \to x_2) \to ((x_2 \to x_1) \to x_1)$, then $\bar{t}$ is in $\overline{\mathrm{T}}_1(n)$ but $\bar{t}$ is also in $\overline{\mathrm{T}}_2(n)$ since $\mathcal{HI} \models t \approx (x_1 \to x_2) \to ((x_2 \to x_1) \to x_2)$. Indeed, in any pure Hilbert algebra $\mathbf{A}$, we have $t^{\mathbf{A}}(a, b) = 1$ if $a \neq b$ and $t^{\mathbf{A}}(a, b) = a$ if $a = b$. Note that the term $\bar{x}_k \to \bar{x}_k$ is in $\overline{\mathrm{T}}_i$ for all $1 \leq k \leq n$.

The following facts are easily established.

Lemma 6.1. *Let v be a valuation into a pure Hilbert algebra $\mathbf{A}$ and t a Hilbert term with $\mathrm{rv}(t) = x_i$.*

1. *$v(t) \in \{1, v(\mathrm{rv}(t))\}$.*
2. *If $\bar{t} \in \overline{\mathrm{T}}_i \cap \overline{\mathrm{T}}_j$ and $v(t) \neq 1$, then $v(x_i) = v(x_j)$.*
3. *$\mathcal{HI} \models \mathrm{rv}(t) \leq t$.*
4. *For every $a, b \in A$*

$$(a \to b) \to b = \begin{cases} b & \text{if } a \leq b, \\ 1 & \text{if } a \not\leq b. \end{cases}$$

Although $\mathrm{rv}(t) \leq t$ holds for all terms t, the converse, that $x_i \leq t$ implies $\bar{t} \in \overline{\mathrm{T}}_i$, does not hold in general. For example, $\bar{x}_1 \leq ((\bar{x}_1 \to \bar{x}_2) \to \bar{x}_2) \notin \overline{\mathrm{T}}_1$. However, for the subvariety of $\mathcal{H}$ generated by the 2-element implication algebra, it is the case that $\bar{x}_i \leq \bar{t}$ if and only if $\bar{t} \in \overline{\mathrm{T}}_i$.

For every $t \in \mathrm{T}(n)$ we have $\mathcal{H} \models \mathrm{rv}(t) \leq t$. Thus every $\bar{x}_i$ is minimal in the order of $\mathbf{F}_{\mathcal{V}}(n)$, and for every element $\bar{t}$ of $\mathrm{F}_{\mathcal{V}}(n)$ there is an $\bar{x}_i \leq \bar{t}$.

Definition 6.2. *If* $v : \{x_1, \ldots, x_n\} \to \mathbf{A}$ *and* $v' : \{x_1, \ldots, x_n\} \to \mathbf{A}'$ *are two valuations into subdirectly irreducible algebras* $\mathbf{A}$ *and* $\mathbf{A}'$ *in a subvariety* $\mathcal{V}$ *of* $\mathcal{H}$*, we say that* v *and* v' *are* equivalent *if there exist homomorphisms* $h : \mathbf{A} \to \mathbf{A}'$ *and* $h' : \mathbf{A}' \to \mathbf{A}$ *such that* $v' = hv$ *and* $v = h'v'$*. Let* $E = E(n, \mathcal{V})$ *be a set containing precisely one valuation from each equivalence class of this equivalence relation.*

Recall that a subdirectly irreducible Hilbert algebra $\mathbf{A}$ has an element $e \prec 1$ with $a \leq e$ for all $a \in A, a \neq 1$. If v is a valuation mapping X to the subdirectly irreducible algebra $\mathbf{A}$, then it is easily seen that that $e \in v(X)$.

Definition 6.3. *For* $1 \leq \ell \leq n$ *define* $E_{\leq \ell}$ *to be the set of all valuations* $v \in E$ *that map* $\{x_1, \ldots, x_n\}$ *into a subdirectly irreducible algebra* $\mathbf{A}$ *such that* $v(x_i) = e$ *for* $1 \leq i \leq \ell$*.*

The next result is a consequence of Theorem 2.9 in [1].

Lemma 6.4. *For any variety* $\mathcal{V}$ *of Hilbert algebras the algebra* $\overline{\mathbf{T}}_{\leq \ell}(n)$ *is isomorphic to* $\mathbf{C}_2^{|E_{\leq \ell}|}$ *where* $\mathbf{C}_2 = \mathrm{Hi}(\mathbf{P})$ *for* $\mathbf{P}$ *the 2-element chain* $e < 1$*.*

Theorem 4.2 and Lemma 6.4 provide the following expression for the cardinality of finitely generated free algebras in $\mathcal{V}$.

Theorem 6.5. *For every variety* $\mathcal{V} \subseteq \mathcal{H}$

$$|\mathrm{F}_{\mathcal{V}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{|E_{\leq \ell}|}.$$

We use Theorem 6.5 to determine the cardinality of the finitely generated free algebras for $\mathcal{V} = \mathcal{H}$. We first describe $E_{\leq \ell}$. Let $v \in E_{\leq \ell}$ with $v : X \to \mathbf{A}$. Then $\mathbf{A}$ is a subdirectly irreducible algebra and is therefore pure by Theorem 2.2. The algebra $\mathbf{A} \in \mathcal{H}$ is generated by the element e and the $v(x_j)$, for $\ell + 1 \leq j \leq n$, and has at most $n - \ell$ elements strictly below e. Let $R = \{x_j \in X : v(x_j) < e\}$ and $T = \{x_j \in X : v(x_j) = 1\}$. For this valuation v we form a quasi-order σ_v consisting of the union of these sets:

- $R \times (X - R)$,
- a quasi-order ρ on R given by $\rho = \{(x_j, x_k) \in R^2 : v(x_j) \leq v(x_k)\}$,
- $X \times T$,

- the diagonal $\{(x_j, x_j) : x_j \in X\}$.

If $|R| = r$ and q_r is the number of quasi-orders on an r-element set, then the number of ways the set R and the quasi-order ρ on R can be chosen is $\binom{n-\ell}{r} q_r$. Having chosen r of the $v(x_j)$ to have value strictly less than e there are at most $2^{n-\ell-r}$ ways to choose the set T. Hence

$$|E_{\leq \ell}| \leq \sum_{r=0}^{n-\ell} \binom{n-\ell}{r} q_r 2^{n-\ell-r}.$$

This upper bound is actually obtained since for every set $R \subseteq \{x_{\ell+1}, \ldots, x_n\}$ of cardinality r and every quasi-order ρ defined on R, and for every choice of a set $T \subseteq \{x_{\ell+1}, \ldots, x_n\} - R$ with the quasi-order τ on X given by

$$\begin{aligned}\tau = {} & \{(x_j, x_k) : x_j \in R, x_k \in X - R\} \\ & \cup \{(x_j, x_k) : x_j \in X, x_k \in T\} \\ & \cup \{(x_j, x_j) : x_j \in X\}.\end{aligned}$$

there is a subdirectly irreducible algebra $\mathbf{A} = \mathrm{Hi}(\mathbf{P})$ for $\mathbf{P}$ the poset $X/(\rho \cup \tau)$. For this $\mathbf{A}$, the element 1 is $T/(\rho \cup \tau)$, the element e is $(X - (R \cup T))/(\rho \cup \tau)$, and each $x_j/(\rho \cup \tau)$ for $x_j \in R$ is strictly below e. If $v : X \to \mathbf{A}$ is defined by $v(x_i) = x_i/(\rho \cup \tau)$, then $v(X)$ generates $\mathbf{A}$, $v \in E_{\leq \ell}$, and $\sigma_v = \rho \cup \tau$. Thus,

$$|E_{\leq \ell}| = \sum_{r=0}^{n-\ell} \binom{n-\ell}{r} q_r 2^{n-\ell-r}. \tag{7}$$

Theorem 6.5 gives the following.

Theorem 6.6. *For every $n \geq 1$*

$$|\mathrm{F}_{\mathcal{H}}(n)| = \sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{n}{\ell} 2^{\sum_{r=0}^{n-\ell} \binom{n-\ell}{r} q_r 2^{n-\ell-r}}.$$

The values for $n = 1$, 2, and 3 are 2, 14 and 12266 respectively.

Note that $\mathbf{F}_{\mathcal{H}}(n)$ and $\mathbf{F}_{\mathcal{H}I}(n)$ are the same for $n \leq 2$. However, $\mathbf{F}_{\mathcal{H}I}(3)$ is known to have cardinality $3 * 2^{23} - 3 * 2^3 + 2^1 = 25,165,802$ (see [11], or [1]).

Theorem 6.7. *For every nontrivial subvariety $\mathcal{V}$ of $\mathcal{H}$ the cardinality of $\mathbf{F}_{\mathcal{V}}(n)$ is asymptotic to $n2^{|E_{\leq 1}|}$. That is,*

$$\lim_{n \to \infty} \frac{|\mathrm{F}_{\mathcal{V}}(n)|}{n 2^{|E_{\leq 1}|}} = 1.$$

Proof. For every $1 < \ell \leq n$ we have $|E_{\leq 1}| - |E_{\leq \ell}| \geq 2^{n-2}$ since every valuation v with $v(x_1) = e$ and $v(x_2) = 1$ is in $E_{\leq 1}$ but not in $E_{\leq \ell}$. Therefore

$$\frac{2^{|E_{\leq \ell}|}}{2^{|E_{\leq 1}|}} \leq 2^{-2^{n-2}}.$$

Thus, for large n,

$$\frac{\binom{n}{\ell} 2^{|E_{\leq \ell}|}}{n 2^{|E_{\leq 1}|}} \leq 2^{-2^{n-3}}.$$

Hence by Theorem 6.5,

$$\lim_{n\to\infty} \frac{|\mathrm{F}_{\mathcal{V}}(n)|}{n 2^{|E_{\leq 1}|}} = 1 + \lim_{n\to\infty} \sum_{\ell=2}^{n} (-1)^{\ell-1} \frac{\binom{n}{\ell} 2^{|E_{\leq \ell}|}}{n 2^{|E_{\leq 1}|}} = 1.$$

□

We next consider $\mathrm{G}_{\mathcal{H}}(n)$, the G-spectrum of the variety $\mathcal{H}$.

Recall that for every finite $\mathbf{A} \in \mathcal{H}$, each congruence relation θ on $\mathbf{A}$ is uniquely determined by the filter $1/\theta$, and this filter is determined by the minimal elements in it. We use this fact to bound the size of the congruence lattice of a free algebra.

Lemma 6.8. *Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$ for $\mathcal{V}$ a subvariety of $\mathcal{H}$ and $X = \{x_1, \ldots, x_n\}$. Suppose that θ is an arbitrary congruence relation on $\mathbf{F}$. Then for every i the subset $\overline{\mathrm{T}}_i \cap 1/\theta$ has a least element.*

Proof. Let $\bar{p}, \bar{q} \in \overline{\mathrm{T}}_i \cap 1/\theta$ be incomparable. We show that $\bar{r} = (\bar{p} \to (\bar{q} \to \overline{x_i})) \to \overline{x_i}$ is such that $\bar{r} \leq \bar{p}$ and $\bar{r} \leq \bar{q}$. Since $\bar{r} \in \overline{\mathrm{T}}_i \cap 1/\theta$ and $\overline{\mathrm{T}}_i$ is finite, the existence of $\bar{r}$ will justify the claim. Let v be any valuation into a subdirectly irreducible algebra $\mathbf{A} \in \mathcal{H}$. By Lemma 6.1(1) we know that $v(p), v(q), v(r) \in \{v(x_i), 1\}$ An examination of the four possible values that $v(p)$ and $v(q)$ may assume shows that in all cases, $v(p) \leq v(r)$ and $v(q) \leq v(r)$. □

This lemma shows that for each congruence relation θ on $\mathbf{F}$ and every $1 \leq i \leq n$, there exists a least element $m_i \in \overline{\mathrm{T}}_i \cap 1/\theta$. The congruence θ is uniquely determined by the n elements $m_1, \ldots, m_n$. Hence $|\mathsf{Con}\,\mathbf{F}| \leq |\overline{\mathrm{T}}_1|^n = |\overline{\mathrm{T}}_{\leq 1}|^n = 2^{n|E_{\leq 1}|}$. Since the cardinality of $\mathsf{Con}\,\mathbf{F}$ is an upper bound for $\mathrm{G}_{\mathcal{V}}(n)$, we have

$$\mathrm{G}_{\mathcal{V}}(n) \leq 2^{n|E_{\leq 1}|}.$$

For $\mathcal{V} = \mathcal{H}$ we see from (7) that

$$|E_{\leq 1}| = \sum_{r=0}^{n-1} \binom{n-1}{r} q_r 2^{n-1-r}.$$

This sum is bounded above by the expression for $|\mathbf{F}_{\mathcal{J}}(n)|$ in Theorem 5.6. We may therefore invoke the upper bound for $|\mathbf{F}_{\mathcal{J}}(n)|$ given in Theorem 5.8 to conclude that there is a constant b such that for all large n,

$$\mathrm{G}_{\mathcal{H}}(n) \leq 2^{2^{\frac{n^2}{4}+n+b\lg n}}. \tag{8}$$

Next we consider a lower bound for the G-spectrum of $\mathcal{H}$. In [2] there is an argument that provides a lower bound on the G-spectrum of the variety of implication algebras by means of a lower bound on the G-spectrum of the variety of semilattices. We use that argument now to give a lower bound on $\mathrm{G}_{\mathcal{H}}(n)$ in terms of the lower bound on the G-spectrum of the variety $\mathcal{J}$ of join algebras.

If $\mathbf{A} = \langle A, \to \rangle \in \mathcal{H}$, then let $\mathbf{A}_j$ denote the algebra $\langle A, j(x,y) \rangle$ where $j(x,y)$ is the term $(x \to y) \to y$. If $\mathbf{A}$ is a pure Hilbert algebra, then $\mathbf{A}_j$ is a pure join algebra as observed in Lemma 6.1. Moreover, every subdirectly irreducible algebra join algebra is of the form $\mathbf{A}_j$ for some subdirectly irreducible algebra $\mathbf{A} \in \mathcal{H}$. Therefore, in the language of [2] the variety $\mathcal{J}$ of join algebras is a *residual reduct* of $\mathcal{H}$, and so Lemma 4.2 of that paper applies. This means that for $\mathbf{F} = \mathbf{F}_{\mathcal{H}}(X)$, the subalgebra $\mathbf{B}$ of $\mathbf{F}_j$ generated by the set X using only the operation $j^{\mathbf{F}}(x,y)$ has the following properties:

- $\mathbf{B}$ is isomorphic to $\mathbf{F}_{\mathcal{J}}(X)$,
- for every $\theta \in \mathsf{Con}\,\mathbf{B}$ there exists $\overline{\theta} \in \mathsf{Con}\,\mathbf{F}$ for which $\overline{\theta}|_B = \theta$,
- $\overline{\theta}$ is generated as a congruence relation by θ.

Now for each congruence relation θ_S on $\mathbf{F}_{\mathcal{J}}(X)$ that was formed in the argument used to prove Theorem 5.9 we form the congruence relation $\overline{\theta}_S$ on $\mathbf{F}_{\mathcal{H}}(X)$. An argument virtually identical to that given on page 333 of [2] shows that the collection of at least $2^{f(n)/n^2}/n!$ congruences of the form θ_S on $\mathbf{F}_{\mathcal{J}}(n)$ that has the quotients $\mathbf{F}/\theta_S$ all pairwise non-isomorphic extends to a family of congruence relations $\overline{\theta}_S$ on $\mathbf{F}_{\mathcal{H}}(n)$ that also have pairwise non-isomorphic quotients. If we use the lower bound given in Theorem 5.9 and the upper bound in (8) we have the following.

Theorem 6.9. *There is a constant b such that for all sufficiently large n,*

$$2^{2^{\frac{n^2}{4}+n-b\lg n}} \leq \mathrm{G}_{\mathcal{H}}(n) \leq 2^{2^{\frac{n^2}{4}+n+b\lg n}}.$$

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