Stipulations, multivalued logic, and De Morgan algebras

J. Berman and W. J. Blok
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607
U.S.A.

Dedicated to Professor Ivo Rosenberg on the occasion of his sixty-fifth birthday

Abstract

A *stipulation* s is a map from a set Pr of propositional constants into the sentence algebra generated by that set. Given an algebra $\boldsymbol{A}$ of truth values a valuation $f : \mathrm{Pr} \to \boldsymbol{A}$ is *s-consistent* if $f(p) = \bar{f}(s(p))$, for all $p \in \mathrm{Pr}$, where $\bar{f}$ is the natural extension of f to the sentence algebra. If there is a partial order (an "information order") $\leq$ on the domain of the algebra that is preserved by its fundamental operations, then the set of s-consistent valuations into the algebra is also a (nonempty) poset. Natural completeness properties of the partial order transfer to this poset (Theorem 3.7), and if the algebra is in addition order-complete with respect to the partial order the finite posets of s-consistent valuations can be characterized (Theorem 3.8). The results are applied and refined in the case $\boldsymbol{A}$ is the three-element Kleene algebra (Section 4) and the four-element De Morgan algebra (Section 5). The facts about the clones of operations needed for this are known in the Kleene case, and are provided for the De Morgan case: the four-element De Morgan algebra with all four constants added to its type is order-complete with respect to a lattice order (Theorem 5.2), while the clone of operations of the four-element De Morgan algebra consists of all selfdual maps preserving the same lattice order (Theorem 5.8).

1 Introduction

By an *interpretation* of a set Se of sentences in an algebra $\boldsymbol{A}$ of truth values we mean any map $f : \mathrm{Se} \longrightarrow A$. The interpretation is *functional* if the connectives of the underlying language of Se are also the operation symbols for the algebra $\boldsymbol{A}$, and the map f is a (partial) homomorphism with respect to them (considering Se the domain of a (partial) algebra). For example, if Se is the set of sentences built from a set of basic propositions Pr using the familiar connectives $\{\vee, \wedge, \rightarrow, \neg, \mathbb{T}, \mathbb{F}\}$, and $\mathbf{B}_2$ is the two-element Boolean algebra with the same operations, then the classical interpretations are homomorphisms from the word algebra generated by Pr over $\{\vee, \wedge, \rightarrow, \neg, \mathbb{T}, \mathbb{F}\}$ to the algebra $\mathbf{B}_2$. Some of the interpretations will be preferable to others: for example, if a proposition expresses a tautology, one may require the interpretations to assign the value $\mathbb{T}$ to it; we will not always assume this, however. In general, propositions may be true in some interpretation, false in others, and will be assigned $\mathbb{T}$ or $\mathbb{F}$ accordingly.

If we want to take into account the fact the propositions may refer to other propositions in the set Pr, including themselves, we need to worry about the consistency of the interpretation. This phenomenon played an important role in Kripke's "Outline of a theory of truth" [6], and in work following it, such as Visser's "Four-valued semantics and the liar" [11]. For example, if the proposition ℓ (for "liar") expresses its own negation, i.e., ℓ expresses $\neg\ell$, then any *consistent* and functional interpretation $f : \mathrm{Se} \rightarrow \boldsymbol{A}$ must satisfy $f(\neg\ell) = f(\ell)$. This clearly cannot be accomplished if we take for our algebra of truth-values $\boldsymbol{A}$ the algebra $\mathbf{B}_2$. Kripke used the three-valued Kleene algebra instead, which has in addition to the truth values $\mathbb{T}$ and $\mathbb{F}$ a value $\mathbb{N}$ for "undetermined" satisfying $\neg\mathbb{N} = \mathbb{N}$. A consistent and functional interpretation in this algebra does exist: it will assign to the proposition ℓ the value $\mathbb{N}$. In fact, for any set Pr of propositions, however they interact, the map f that sends every sentence built up from propositions in Pr to $\mathbb{N}$ is a consistent and functional interpretation. But it gives little information as it doesn't make any distinctions among the sentences. The three-valued Kleene algebra allows a natural "information ordering", different from the ordering induced by the operations of the algebra, that puts $\mathbb{N}$ at the bottom, and $\mathbb{T}$ and $\mathbb{F}$—themselves incomparable—above it. (See Figure 1.) We will refer to this partially ordered set as $\mathcal{K}$. Its ordering induces a natural ordering on the interpretations, and given a set of sentences, we can now look for consistent, functional interpretations that carry more information, i.e., that are larger than the trivial interpretation mentioned before. The partially ordered set of these interpretations plays an important role in [6].

Visser [11] continued the study of semantics for sentences that allow self-reference, replacing the three-element Kleene algebra by the four-element De Morgan algebra, which has in addition to the three truth-values of the Kleene algebra a value $\mathbb{B}$ for "over-determined". In the information ordering this element sits above both $\mathbb{T}$ and $\mathbb{F}$, giving rise to the information lattice $\mathcal{M}$ (see Figure 1.) The partially ordered set of consistent functional interpretations of any set of sentences in the four-element De Morgan algebra is always a lattice, in contrast with the three-valued Kleene case discussed above.

An important tool in the study of consistent functional interpretations in these two cases has been the observation that they always arise as the fixed points of certain order-preserving operators on the ordered sets of all functional interpretations. In the first part of the paper we investigate this phenomenon in a general setting, with both the three-valued Kleene case and the four-valued De Morgan case as important instances. In particular, we show that if the set of term operations of the algebra $\boldsymbol{A}$ of truth-values consists of maps preserving a certain partial ordering on the algebra—the information ordering in the Kleene and De Morgan cases—then for any set of sentences the consistent interpretations will be precisely the fixed points of a certain order-preserving operator on the poset of all functional interpretations. If the order of the algebra satisfies a certain completeness property this tells us a good deal about the posets of consistent functional interpretations of any given set of sentences in such an algebra (Theorem 3.7). Both the three-element Kleene algebra and the four-element De Morgan algebra satisfy these requirements with respect to their information orders. If the set of term functions of the algebra $\boldsymbol{A}$ of truth values actually coincides with the set of maps preserving the partial order—we say that such an algebra is *order-complete*—then we can characterize the finite partially ordered sets of consistent functional interpretations that can occur, provided the ordering satisfies the completeness property referred to earlier (Theorem 3.8).

It is known that the three-element Kleene algebra with the constant $\mathbb{N}$ added to its type is order-complete. In the last section of the paper we show that the four-element De Morgan algebra with the constants $\mathbb{N}$ and $\mathbb{B}$ added to its type is order-complete as well. The characterization of the finite posets of consistent functional interpretations that can occur obtained in Theorem 3.8 therefore applies in both the Kleene and the De Morgan case if we add the constants to the type of these algebras (Theorems 4.3, 5.3).

If we do not add the constants to the type, the sets of term functions of the three-element Kleene algebra as well as the four-element De Morgan algebra are properly contained in the set of all maps preserving the respective information orders. A good characterization of these sets of term functions is known for the Kleene case (Theorem 4.4), and we obtain a characterization in the De Morgan case (Theorem 5.8). These characterizations allow us to determine the finite partially ordered sets of consistent functional interpretations that can occur in the pure Kleene case (Corollary 4.5) as well as the pure De Morgan case (Corollary 5.9).

The information orders arising from the Kleene and De Morgan algebras have appeared in other contexts. See for example [1] and [2].

2 Conditionally Complete Posets

The information orderings encountered above, viz., the lattice ordering $\mathcal{M}$ of the four-element De Morgan algebra and its subposet $\mathcal{K}$, the information ordering of the three-element Kleene algebra, are representatives of a class of posets that was studied in some depth in [3].

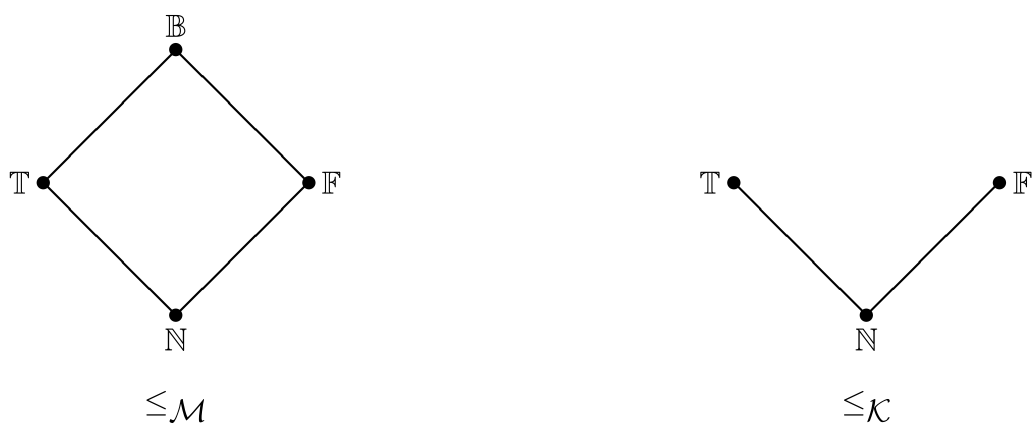

Figure 1

We recall the relevant facts here. A partially ordered set (or *poset* for short) $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ is a non-void set A with a partial order $\leq_{\mathcal{A}}$ (or just $\leq$) on it. For $X \subseteq A$, we say that $a \in A$ is an *upper bound of* X if $x \leq a$, for all $x \in X$, and a *lower bound* if $x \geq a$, for all $x \in X$. The *least upper bound* of X, if it exists, is denoted by $\bigvee^{\mathcal{A}} X$, or $\bigvee X$ for short, if no confusion is likely to arise; similarly, the *greatest lower bound* of X is denoted by $\bigwedge^{\mathcal{A}} X$ or $\bigwedge X$. Recall that a *lattice* is a poset in which every two elements have a least upper bound and a greatest lower bound, and that a lattice is *complete* if every subset X has a least upper bound and a greatest lower bound.

If $\mathcal{A}$ and $\mathcal{B}$ are posets, a function $F : A \to B$ is *order-preserving* if $x \leq_{\mathcal{A}} y$ implies $F(x) \leq_{\mathcal{B}} F(y)$, for all $x, y \in A$; we write then also $F : \mathcal{A} \to \mathcal{B}$. If $\mathcal{A} = \mathcal{B}$ and $a \in A$, then a is a *fixed point* of F if $F(a) = a$; the set of all fixed points of an order-preserving map $F : \mathcal{A} \to \mathcal{A}$ is denoted $\mathrm{Fix}(\mathcal{A}, F)$. Every subset B of A is the universe of the *subposet* $\mathcal{B} = \langle B, \leq_{\mathcal{A}} \cap B^2 \rangle$ of $\mathcal{A}$.

In the next section we will see that the consistent functional interpretations of a set of propositions will emerge as the fixed points of certain order-preserving operators on the poset of all functional interpretations. A well-known theorem by Tarski (see [10]) tells us that if the poset of interpretations is a complete lattice, then the set of fixed points of any order-preserving map is non-empty and forms a complete lattice. This result will apply when we consider interpretations in the four-element De Morgan algebra $\mathbf{M}$, since its information ordering $\mathcal{M}$ is a (complete) lattice, and hence so is the poset of all interpretations in $\mathbf{M}$. It does not apply when we consider interpretations in the three-element Kleene algebra $\mathbf{K}$, since its information ordering is not a lattice. In [3] Tarski's theorem is extended to a wide class of posets that includes the posets of interpretations in $\mathbf{K}$. We need some further definitions.

Definition 2.1. Let $\mu \geq 2$ be a cardinal. A partially ordered set $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ is *μ-conditionally complete* if, for $X \subseteq A$, $\bigvee X$ exists whenever every $Y \subseteq X$ such that $|Y| < \mu$ has an upper bound.

The class of 2-conditionally complete posets coincides with the class of complete lattices. The poset $\mathcal{K}$—the information ordering of the three-element Kleene algebra—is 3-conditionally complete, as is the partially ordered set $\mathcal{K}^I$ for any set I (with coordinate-wise ordering). The 3-conditionally complete posets coincide with the *complete coherent partial orders* (ccpo's) introduced in [11].

A subposet $\mathcal{B}$ of $\mathcal{A}$ is *well-embedded* in $\mathcal{A}$ if every set $X \subseteq B$ that has an upper bound in $\mathcal{A}$ also has an upper bound in $\mathcal{B}$. Observe that if $\mathcal{B}$ has a largest element, then it is well-embedded in every poset $\mathcal{A}$ of which it is a subposet.

A subposet $\mathcal{B}$ of $\mathcal{A}$ is a *retract* of $\mathcal{A}$ if there is an order-preserving map $F : \mathcal{A} \to \mathcal{B}$ such that $F|_B$ is the identity map.

We are now ready to state the generalization of Tarski's theorem referred to above.

Theorem 2.2 ([3]). *Let $\mathcal{A}$ be a μ-conditionally complete poset, $\mu \leq \omega$, and let $\mathcal{B}$ be a subposet of $\mathcal{A}$. The following are equivalent.*

1. *$B = \mathrm{Fix}(\mathcal{A}, F)$, for an order-preserving map $F : \mathcal{A} \to \mathcal{A}$.*

2. *$\mathcal{B}$ is a μ-conditionally complete poset and is well-embedded in $\mathcal{A}$.*

3. *$\mathcal{B}$ is a retract of $\mathcal{A}$.*

Let $\mathcal{C}_\mu$ denote the class of μ-conditionally complete posets, for μ a cardinal. It is shown in [3] that the classes C_μ are *order varieties*, i.e., they are closed under the operations of forming direct products and of forming retracts. A typical example of an $(n+1)$-conditionally complete poset that is not n-conditionally complete, $2 \leq n < \omega$, is

$$\mathcal{T}_n = \langle \{X \subseteq \{0, 1, \ldots, n-1\} : |X| < n\}, \subseteq \rangle.$$

The domain of $\mathcal{T}_n$ is denoted T_n. Note that $\mathcal{T}_2$ is isomorphic to the information ordering $\mathcal{K}$ of the three-element Kleene algebra, introduced earlier. We define $\mathcal{T}_1$ to be the two-element chain.

Theorem 2.3 ([3]). *For $1 \leq n < \omega$ the class $\mathcal{C}_{n+1}$ of $(n+1)$-conditionally complete posets consists of the retracts of direct products of the poset $\mathcal{T}_n$.*

In Sections 3 and 4 we will use these results in our attempt to determine the posets of fixed points of order-preserving maps $F : \mathcal{T}_n^I \to \mathcal{T}_n^I$, I some index set, $2 \leq n < \omega$. The following refinement of Theorem 2.2 will be needed.

If A is a set, $C \subseteq A$, and $F : A \to A$, then we say that F is *C-preserving* if $F(C) \subseteq C$.

Theorem 2.4. *Let $B \subseteq \mathrm{T}_n^I$ be the set of fixed points of an order-preserving map of $\mathcal{T}_n^I$ into itself, for some $2 \leq n < \omega$. Let C denote the set of maximal elements of $\mathcal{T}_n^I$. Then there is an order preserving and C-preserving map $F : \mathcal{T}_n^I \to \mathcal{T}_n^I$ such that B is the set of fixed points of F.*

Proof. By Theorem 2.2 there is a retraction $G : \mathcal{T}_n^I \to \mathcal{B}$; note that B is then the set of fixed points of this retraction. For $b \in B$, let C_b be the set

$\{c \in C : c \geq b\}$. Note that if $b \in C$, then C_b consists of the element b only. If $b \notin C$, then $|C_b| > 1$, and we choose $c_b, c'_b \in C_b$ such that $c_b \neq c'_b$. Now define $F : \mathcal{T}_n^I \to \mathcal{T}_n^I$ as follows:

$$F(x) = \begin{cases} G(x) & \text{if } x \notin C \text{ or } x \in B \\ c_{G(x)} & \text{if } x \in C,\ x \notin B, \text{ and } x \neq c_{G(x)} \\ c'_{G(x)} & \text{otherwise.} \end{cases}$$

Then the fixed points of F are precisely the fixed points of G, i.e., the elements of B, and F is C-preserving. To see that F is order-preserving, let $x, y \in \mathrm{T}_n^I$, $x < y$. Then $x \notin C$, so $F(x) = G(x)$. If $y \notin C$ or $y \in B$ then $F(y) = G(y)$, so $F(x) \leq F(y)$ since G is order-preserving. If $y \in C$ then $F(y) = c_{G(y)} \geq G(y)$ or $F(y) = c'_{G(y)} \geq G(y)$, so in this case too we have

$$F(y) \geq G(y) \geq G(x) = F(x).$$

□

For more on μ-conditionally complete posets see [3].

3 Stipulations

We fix a set $\mathcal{L} = \{\omega_i : i \in I\}$ of connectives (which also serve as operation symbols) each of fixed finite arity, and a set $\mathrm{Pr} \subseteq \{\mathrm{p}_i : i < \omega\}$ of propositional constants. The *$\mathcal{L}$-sentences* are built up in the usual way from Pr using the connectives in $\mathcal{L}$, and we will denote the set of all $\mathcal{L}$-sentences by Se. The *$\mathcal{L}$-sentence algebra* will be denoted by $\mathbf{Se} = (\mathrm{Se}, \mathcal{L})$; it is freely generated by the set Pr.

More generally, an *$\mathcal{L}$-algebra* (or *algebra* for short if the set $\mathcal{L}$ is clear from the context, as it usually will be) is a structure $\boldsymbol{A} = \langle A, \mathcal{L} \rangle$, where A is any non-empty set, and for each operation symbol $\omega_i \in \mathcal{L}$, $i \in I$, say of arity n_i, we have an operation

$$\omega_i^{\boldsymbol{A}} : A^{n_i} \longrightarrow A.$$

The *$\mathcal{L}$-terms* are the syntactic expressions built up from variables $x_0, x_1, \ldots$, using the operation symbols ω_i, $i \in I$ from $\mathcal{L}$ in the usual recursive way. The *$\mathcal{L}$-term functions of $\boldsymbol{A}$* are the operations on $\boldsymbol{A}$ induced by the $\mathcal{L}$-terms. The set of all $\mathcal{L}$-term functions of $\boldsymbol{A}$ forms a *clone* of operations on A, that is, a collection of operations that contains the projection operations and is closed under composition. We denote the set of all term operations of an algebra $\boldsymbol{A}$ by $\mathrm{Clo}\,\boldsymbol{A}$, and the set of all n-ary term operations by $\mathrm{Clo_n}\,\boldsymbol{A}$. By a *valuation of* Pr *in* $\boldsymbol{A}$ we mean a map $f : \mathrm{Pr} \to A$, also denoted $f : \mathrm{Pr} \to \boldsymbol{A}$. Every valuation $f : \mathrm{Pr} \to \boldsymbol{A}$ can be extended in a unique way to a homomorphism $\bar{f} : \mathbf{Se} \to \boldsymbol{A}$; we refer to such a homomorphism as a *functional interpretation* or *interpretation* for short. The set of valuations of Pr in $\boldsymbol{A}$ will be denoted by A^{Pr}. In order to formalize the interrelationships that may exist among the propositional constants we use the following device, introduced in [11].

Definition 3.1. Given a language $\mathcal{L}$ and a set Pr of propositional constants, a *stipulation on* Pr *over* $\mathcal{L}$ or a *stipulation* for short is a map $s : \mathrm{Pr} \to \mathrm{Se}$, where Se is the set of all $\mathcal{L}$-sentences.

The idea here is that for $p \in \mathrm{Pr}$, the proposition p expresses $s(p)$. Thus in the example mentioned in the introduction, if $\neg \in \mathcal{L}$, and $\ell \in \mathrm{Pr}$, where ℓ is the liar sentence, this would be reflected by setting $s(\ell) = \neg\ell$. Observe that since **Se** is freely generated by Pr, every stipulation $s : \mathrm{Pr} \to \mathrm{Se}$ can be extended in a unique way to a homomorphism $\bar{s} : \mathbf{Se} \to \mathbf{Se}$.

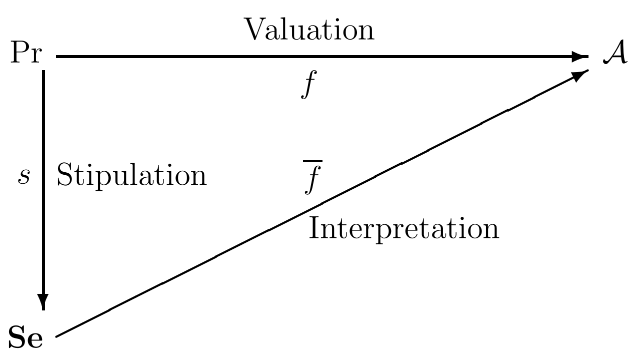

Figure 2

We assume now that we are given a language $\mathcal{L}$, a set Pr of propositional constants, and a stipulation s.

Definition 3.2. A valuation $f : \mathrm{Pr} \to \boldsymbol{A}$ is *s-consistent* if it satisfies

$$f(p) = \bar{f}(s(p)),$$

for all $p \in \mathrm{Pr}$.

Thus the valuation $f : \mathrm{Pr} \to \boldsymbol{A}$ is s-consistent if and only if

$$f = \bar{f} \circ s.$$

Observe that if $f : \mathrm{Pr} \to \mathrm{Se}$ is an s-consistent valuation, then for every $\phi \in \mathrm{Se}$ we have $\bar{f}(\phi) = \bar{f}(\bar{s}(\phi))$, i.e., $\bar{f}$ is an *s-consistent interpretation*.

We associate with the stipulation s an operator $F_s : A^{\mathrm{Pr}} \to A^{\mathrm{Pr}}$ on the set of all valuations of Pr in $\boldsymbol{A}$ by

$$F_s(f) = \bar{f} \circ s.$$

We have then

Lemma 3.3. *The valuation f is s-consistent if and only if it is a fixed point of F_s.*

To decide which operators $F : A^{\mathrm{Pr}} \to A^{\mathrm{Pr}}$ are of the form F_s for some stipulation s, the following lemma will be useful.

Proposition 3.4. *Let $\boldsymbol{A}$ be an algebra, $F : A^{\mathrm{Pr}} \to A^{\mathrm{Pr}}$ a map. Then $F = F_s$ for some stipulation $s : \mathrm{Pr} \to \mathrm{Se}$ if and only if for every $p \in \mathrm{Pr}$ there exists a term $t_p(x_0, \ldots, x_{n-1})$, of arity n, say, and $i_0, \ldots, i_{n-1} < \omega$ such that*

$$\pi_p \circ F(f) = t_p^{\boldsymbol{A}}(f(\mathrm{p}_{i_0}), \ldots, f(\mathrm{p}_{i_{n-1}})),$$

for all $f \in A^{\mathrm{Pr}}$. Here π_p denotes the projection onto the p^{th} coordinate.

Proof. $\Rightarrow$. Let $s : \mathrm{Pr} \to \mathrm{Se}$ be a stipulation and let $F = F_s$. Then $F(f) = \bar{f} \circ s$ for $f \in A^{\mathrm{Pr}}$, and hence for $p \in \mathrm{Pr}$

$$\begin{aligned} \pi_p \circ F(f) &= F(f)(p) \\ &= \bar{f} \circ s(p) \\ &= t_p^{\boldsymbol{A}}(f(\mathrm{p}_{i_0}), \ldots, f(\mathrm{p}_{i_{n-1}})), \end{aligned}$$

where $t_p(x_0, \ldots, x_{n-1})$ is a term such that $s(p) = t_p^{\mathbf{Se}}(\mathrm{p}_{i_0}, \ldots, \mathrm{p}_{i_{n-1}})$.

$\Leftarrow$. For $p \in \mathrm{Pr}$ let s be the stipulation defined by

$$p \mapsto t_p^{\mathbf{Se}}(\mathrm{p}_{i_0}, \ldots, \mathrm{p}_{i_{n-1}}) \in \mathrm{Se}.$$

Then for all $f \in A^{\mathrm{Pr}}$

$$\begin{aligned} F(f)(p) &= \pi_p \circ F(f) \\ &= t_p^{\boldsymbol{A}}(f(\mathrm{p}_{i_0}), \ldots, f(\mathrm{p}_{i_{n-1}})) \\ &= \bar{f}(t_p^{\mathbf{Se}}(\mathrm{p}_{i_0}, \ldots, \mathrm{p}_{i_{n-1}})) = \bar{f} \circ s(p). \end{aligned}$$

Hence $F(f) = \bar{f} \circ s = F_s(f)$, so $F = F_s$. □

If the operator F_s happens to be an order-preserving operator on a partially ordered set, we can say a good deal about its set of fixed points. In order for this to be the case, it suffices to assume that there exists a partial order $\leq$ on the universe of the algebra $\boldsymbol{A}$ with the property that it is preserved by all of the fundamental operations of $\boldsymbol{A}$. More precisely, let $\mathcal{A} = \langle A, \leq \rangle$. Then $\leq$ is *preserved* by the operation $\omega_i^{\boldsymbol{A}}$ for $\omega_i \in \mathcal{L}$ if for all $a_j, b_j \in A$ with $a_j \leq b_j$ for $j < n_i$, we have

$$\omega_i^{\boldsymbol{A}}(a_0, a_1, \ldots, a_{n_i-1}) \leq \omega_i^{\boldsymbol{A}}(b_0, b_1, \ldots, b_{n_i-1}).$$

The set of all (finitary) operations preserved by the partial order $\leq_{\mathcal{A}}$ of a partially ordered set $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ we denote by $\mathrm{Pre}(\leq_{\mathcal{A}})$. Some authors write $\mathrm{Pol}(\leq_{\mathcal{A}})$ for this clone of operations.

The following result is obtained by an easy induction on the complexity of terms:

Lemma 3.5. *Let* $\boldsymbol{A}$ *be an algebra,* $\leq$ *a partial order on* A*,* $\mathcal{A} = \langle A, \leq \rangle$*. If* $\omega_i^{\boldsymbol{A}} \in \mathrm{Pre}(\leq)$*, for all* $i \in I$*, then* $\mathrm{Clo}\, \boldsymbol{A} \subseteq \mathrm{Pre}(\leq)$*.*

Given a partially ordered set $\mathcal{A} = \langle A, \leq \rangle$ the set A^{Pr} of all valuations $f : \mathrm{Pr} \to A$ carries a partial order induced by that of $\mathcal{A}$, by setting for $f, f' \in A^{\mathrm{Pr}}$

$$f \leq f' \quad \text{iff} \quad f(p) \leq f'(p), \text{ for all } p \in \mathrm{Pr}.$$

We will denote this partially ordered set by $\mathcal{A}^{\mathrm{Pr}}$.

Proposition 3.6. *Let* $\boldsymbol{A}$ *be an algebra and* $\leq$ *a partial order on* A *such that* $\mathrm{Clo}\, \boldsymbol{A} \subseteq \mathrm{Pre}(\leq)$*. For any stipulation* $s : \mathrm{Pr} \to \mathrm{Se}$ *the operator* $F_s : \mathcal{A}^{\mathrm{Pr}} \to \mathcal{A}^{\mathrm{Pr}}$ *is order-preserving.*

Proof. Suppose $f, f' \in \mathcal{A}^{\mathrm{Pr}}$ and $f \leq f'$, i.e., $f(p) \leq f'(p)$ for all $p \in \mathrm{Pr}$. Given a sentence $\phi \in \mathrm{Se}$, there is an $\mathcal{L}$-term $t(x_0, \ldots, x_{n-1})$ and there are propositional constants $\mathrm{p}_{i_0}, \ldots, \mathrm{p}_{i_{n-1}} \in \mathrm{Pr}$ such that $\phi = t^{\mathrm{Se}}(\mathrm{p}_{i_0}, \ldots, \mathrm{p}_{i_{n-1}})$. Now

$$\begin{aligned} \bar{f}(\phi) = \quad & t^{\boldsymbol{A}}(f(\mathrm{p}_{i_0}), \ldots, f(\mathrm{p}_{i_{n-1}})) & \leq \\ & t^{\boldsymbol{A}}(f'(\mathrm{p}_{i_0}), \ldots, f'(\mathrm{p}_{i_{n-1}})) & = \bar{f}'(\phi); \end{aligned}$$

here the inequality holds because $\mathrm{Clo}\, \boldsymbol{A} \subseteq \mathrm{Pre}(\leq)$. It follows that for all $p \in \mathrm{Pr}$ we have

$$F_s(f)(p) = \bar{f}(s(p)) \leq \bar{f}'(s(p)) = F_s(f')(p),$$

and hence $F_s(f) \leq F_s(f')$. Thus F_s is order-preserving. ∎

By applying Theorem 2.2 we conclude:

Theorem 3.7. *Let $\boldsymbol{A}$ be an algebra, $\mathcal{A} = \langle A, \leq \rangle$ a μ-conditionally complete partially ordered set for some $2 \leq \mu \leq \omega$ such that* $\mathrm{Clo}\, \boldsymbol{A} \subseteq \mathrm{Pre}(\leq)$. *Then for any stipulation s the set of s-consistent valuations in $\boldsymbol{A}$ is non-empty, and is the domain of a subposet of $\mathcal{A}^{\mathrm{Pr}}$ that is*

(1) μ-conditionally complete,

(2) well-embedded in $\mathcal{A}^{\mathrm{Pr}}$, and

(3) a retract of $\mathcal{A}^{\mathrm{Pr}}$.

If the algebra $\boldsymbol{A}$ of truth values is *order-complete*, i.e., if its set of term functions coincides with $\mathrm{Pre}(\leq)$, we have in the finite case a precise characterization of the posets of consistent valuations that may occur.

Theorem 3.8. *Let* Pr *be a finite set, $\boldsymbol{A}$ a finite algebra, and $\leq$ a partial order on A such that* $\mathrm{Clo}\, \boldsymbol{A} = \mathrm{Pre}(\leq)$. *Let $\mathcal{A} = \langle A, \leq \rangle$, and let $\mathcal{A}^{\mathrm{Pr}}$ denote the partially ordered set of all valuations of* Pr *in $\boldsymbol{A}$. The following are equivalent for a subset B of $\mathcal{A}^{\mathrm{Pr}}$.*

(1) B is the set of s-consistent valuations for some fixed stipulation.

(2) B is the set of fixed points of some order-preserving operator $F : \mathcal{A}^{\mathrm{Pr}} \to \mathcal{A}^{\mathrm{Pr}}$.

In addition, if $\mathcal{A}$ is μ-conditionally complete for some $2 \leq \mu < \omega$, then (1) and (2) are equivalent to

(3) $\mathcal{B}$ is μ-conditionally complete and well-embedded in $\mathcal{A}^{\mathrm{Pr}}$.

(4) $\mathcal{B}$ is a retract of $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (2). Let s be a stipulation. By Proposition 3.6 the associated operator $F_s : \mathcal{A}^{\mathrm{Pr}} \to \mathcal{A}^{\mathrm{Pr}}$ is order-preserving, and by definition of F_s we have $B = \mathrm{Fix}(\mathcal{A}^{\mathrm{Pr}}, F_s)$.

(2) $\Rightarrow$ (1). Let $B = \mathrm{Fix}(\mathcal{A}^{\mathrm{Pr}}, F)$ for some order-preserving operator $F : \mathcal{A}^{\mathrm{Pr}} \to \mathcal{A}^{\mathrm{Pr}}$. For $p \in \mathrm{Pr}$ let π_p denote the projection map of $\mathcal{A}^{\mathrm{Pr}}$ onto the p-th coordinate. Then

$$\pi_p \circ F : \mathcal{A}^{\mathrm{Pr}} \to \mathcal{A},$$

being a composition of order-preserving maps, is order-preserving. If $|\mathrm{Pr}| = n$, then $\pi_p \circ F$ is an n-ary function, and therefore by assumption an n-ary term function of $\boldsymbol{A}$; say, $t_p^{\boldsymbol{A}}(x_0, \ldots, x_{n-1})$ for some term t_p. The conditions of Proposition 3.4 are therefore satisfied (with $i_j = j$, $j = 0, \ldots, n-1$), and hence $F = F_s$ for some stipulation s. Thus (1) follows.

Now assume $\mathcal{A}$ is μ-conditionally complete for some $\mu < \omega$. Then so is $\mathcal{A}^{\mathrm{Pr}}$, and hence Theorem 2.2 is applicable. Thus (2) is equivalent with each of (3) and (4). □

We recall from Section 2 that for all $1 \leq n < \omega$ the poset $\mathcal{T}_n$ is $(n+1)$-conditionally complete, and that every $(n+1)$-conditionally complete poset is a retract of a direct product of copies of $\mathcal{T}_n$. We define $\mathbf{T}_n$ to be the algebra $\langle \mathrm{T_n}, \mathrm{Pre}(\leq)\rangle$ over a language $\mathcal{L}$ containing a symbol for each (finitary) map $f \in \mathrm{Pre}(\leq)$.

Corollary 3.9. *Let $2 \leq n < \omega$. For every finite n-conditionally complete poset $\mathcal{B}$ there exists a stipulation s such that $\mathcal{B}$ is isomorphic to the poset of s-consistent valuations in $\mathbf{T}_n$. In particular, for every $2 \leq n < \omega$, every finite lattice can occur as the poset of s-consistent valuations in $\mathbf{T}_n$.*

Proof. Note that $\mathbf{T}_n$ satisfies the conditions of Theorem 3.8 by definition. It suffices therefore to recall that every n-conditionally complete poset $\mathcal{B}$ is a retract of a direct product of copies of $\mathcal{T}_n$; and it is easy to see that if B is finite, only a finite number m of copies of $\mathcal{T}_n$ are needed. We may thus assume that $\mathcal{B}$ is a retract of $\mathcal{T}_n^{\mathrm{Pr}}$, with $|\mathrm{Pr}| = m$. For the second claim, observe that if $\mathcal{B}$ is a finite lattice, then $\mathcal{B}$ is complete, and hence it is n-conditionally complete for all $2 \leq n < \omega$. □

4 Valuations in the three-element Kleene algebra

In this section we consider what in [5] is called the "strong Kleene scheme"; it is the logical tool of [6].

The three-element Kleene algebra $\mathbf{K}$ has domain $\mathrm{K} = \{\mathbb{T}, \mathbb{F}, \mathbb{N}\}$, and operations $\neg$, $\wedge$, $\vee$ together with the constants $\mathbb{T}$, $\mathbb{F}$. The operations $\wedge$ and $\vee$ are lattice meet and lattice join operations, giving rise to the linear order $\leq_{\mathbf{K}}$ given by $\mathbb{F} \leq_{\mathbf{K}} \mathbb{N} \leq_{\mathbf{K}} \mathbb{T}$. The operation $\neg$ is defined by $\neg\mathbb{T} = \mathbb{F}$, $\neg\mathbb{F} = \mathbb{T}$, and $\neg\mathbb{N} = \mathbb{N}$. The algebra $\mathbf{K}^+$ (the three-element Kleene algebra with constants) has in addition the constant $\mathbb{N}$ among its operations.

By $\mathcal{K}$ we will denote the partially ordered set of which the universe coincides with that of $\mathbf{K}$, i.e., $\{\mathbb{T}, \mathbb{F}, \mathbb{N}\}$, and of which the partial order is the "information order" $\leq_{\mathcal{K}}$ mentioned in the introduction: the reflexive closure of the relation $\{(\mathbb{N}, \mathbb{T}), (\mathbb{N}, \mathbb{F})\}$.

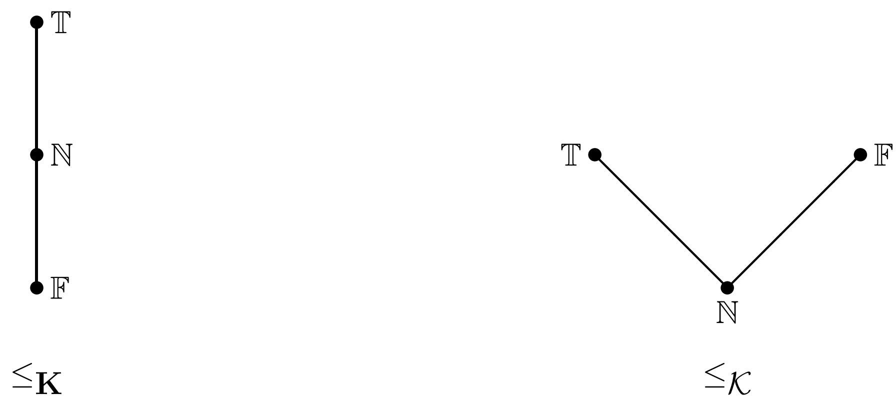

Figure 3

Note that $\mathcal{K}$ is isomorphic to $\mathcal{T}_2$. It is easy to see that all the fundamental operations of $\mathbf{K}$ and $\mathbf{K}^+$ preserve $\leq_{\mathcal{K}}$, and hence, by Lemma 3.5, $\mathrm{Clo}\,\mathbf{K} \subseteq \mathrm{Clo}\,\mathbf{K}^+ \subseteq \mathrm{Pre}(\leq_{\mathcal{K}})$. Theorem 3.7 yields:

Corollary 4.1. *Let s be any stipulation on a set* Pr *of propositional constants over the language of $\mathbf{K}$ or $\mathbf{K}^+$ respectively. The set of s-consistent valuations in $\mathbf{K}$ or $\mathbf{K}^+$ is non-empty, and is the domain of a subposet of $\mathcal{K}^{\mathrm{Pr}}$ that is*
(1) 3-conditionally complete (i.e., a "ccpo"),
(2) well-embedded in $\mathcal{K}^{\mathrm{Pr}}$, and
(3) a retract of $\mathcal{K}^{\mathrm{Pr}}$.

The fact that the set of s-consistent valuations in $\mathbf{K}$ is non-empty was well-known; that it is a 3-conditionally complete poset was observed by Visser [11].

Mukaidono [7] was the first one to prove that every order-preserving function $f : \mathcal{K}^n \to \mathcal{K}$ is actually a term function of $\mathbf{K}^+$:

Theorem 4.2. $\mathrm{Clo}\,\mathbf{K}^+ = \mathrm{Pre}(\leq_{\mathcal{K}})$.

The result was obtained independently by Blamey [4, pp. 36–37]. The theorem tells us that in Corollary 3.9 we can replace the algebra $\mathbf{T}_2$ by $\mathbf{K}^+$ to obtain the following corollary, which was also observed by Visser [12, p. 666].

Corollary 4.3. *For every finite* 3*-conditionally complete poset $\mathcal{B}$ there exists a stipulation s on a finite set* Pr *over the language of $\mathbf{K}^+$ such that $\mathcal{B}$ is* *isomorphic to the poset of s-consistent valuations of* Pr *in* $\mathbf{K}^+$*. In particular, every finite lattice can occur as the poset of s-consistent valuations of* Pr *in* $\mathbf{K}^+$ *for some stipulation s.*

From a logical point of view it may be considered unnatural to allow $\mathbb{N}$ as a logical constant, and the question arises which posets of s-consistent valuations occur if we are allowed to use the logical operations of $\mathbf{K}$ only. Note that if $t(x_0, \ldots, x_{n-1})$ is a term over the language of $\mathbf{K}$, then since $\{\mathbb{T}, \mathbb{F}\}$ is a subuniverse of $\mathbf{K}$ the term function $t^{\mathbf{K}}$ will satisfy

$$t^{\mathbf{K}}(a_0, \ldots, a_{n-1}) \in \{\mathbb{T}, \mathbb{F}\}$$

whenever $a_i \in \{\mathbb{T}, \mathbb{F}\}$ for all $i < n$. It follows from Proposition 3.4 that for any stipulation $s : \mathrm{Pr} \to \mathrm{Se}$ the associated operator $F_s : \mathcal{K}^{\mathrm{Pr}} \to \mathcal{K}^{\mathrm{Pr}}$ defined by $F_s(f) = \bar{f} \circ s$ preserves the set $C = \{\mathbb{T}, \mathbb{F}\}^{\mathrm{Pr}}$. We say that an n-ary operation $t : \mathrm{K}^n \to \mathrm{K}$ preserves the set $\{\mathbb{T}, \mathbb{F}\}$ if $t(a_0, \ldots, a_{n-1}) \in \{\mathbb{T}, \mathbb{F}\}$ whenever $a_i \in \{\mathbb{T}, \mathbb{F}\}$, $i < n$; the set of all finitary operations on K that preserve $\{\mathbb{T}, \mathbb{F}\}$ is denoted by $\mathrm{Pre}(\{\mathbb{T}, \mathbb{F}\})$.

Mukaidono [8] proved

Theorem 4.4. $\mathrm{Clo}\,\mathbf{K} = \mathrm{Pre}(\leq_{\mathcal{K}}) \cap \mathrm{Pre}(\{\mathbb{T}, \mathbb{F}\})$.

We can now show that the conclusion of Corollary 4.3 continues to hold if we restrict ourselves to valuations in $\mathbf{K}$:

Corollary 4.5. *For every finite* 3*-conditionally complete poset* $\mathcal{B}$ *there exists a stipulation s on a finite set* Pr *over the language of* $\mathbf{K}$ *such that* $\mathcal{B}$ *is isomorphic to the poset of s-consistent valuations of* Pr *in* $\mathbf{K}$*. In particular, every finite lattice can occur as the poset of s-consistent valuations of* Pr *in* $\mathbf{K}$ *for some stipulation s.*

Proof. Let $\mathcal{B}$ be a finite 3-conditionally complete poset. By Theorem 2.3 $\mathcal{B}$ is a retract of a direct product of copies of $\mathcal{T}_2$. Since $\mathcal{B}$ is finite, we may assume it is a retract of a finite direct product of copies of $\mathcal{T}_2$, say, of $\mathcal{T}_2^n$. Let C denote the set of maximal elements of $\mathcal{T}_2^n$. By Theorem 2.4 B is the set of fixed points of an order preserving and C-preserving map $F : \mathcal{T}_2^n \to \mathcal{T}_2^n$. Since $\mathcal{T}_2$ and $\mathcal{K}$ are isomorphic we may identify the posets $\mathcal{T}_2^n$ and $\mathcal{K}^n$. The set of maximal elements of $\mathcal{K}^n$ is $\{\mathbb{T}, \mathbb{F}\}^n$, so we may conclude that $\mathcal{B}$ is isomorphic to the poset of fixed points of an order-preserving map $F : \mathcal{K}^n \to \mathcal{K}^n$ that also preserves the set $\{\mathbb{T}, \mathbb{F}\}^n$. Now for all $p \in \mathrm{Pr}$ the map $\pi_p \circ F : \mathcal{K}^n \to \mathcal{K}$ is also order preserving, and preserves the set $\{\mathbb{T}, \mathbb{F}\}$, and hence, by Theorem 4.4 is a term function of $\mathbf{K}$, say, $t_p(x_0, \ldots, x_{n-1})$. Then for $f \in \mathcal{K}^n$ we have $\pi_p \circ F(f) = t_p^{\mathbf{K}}(f(\mathrm{p}_0), \ldots, f(\mathrm{p}_{n-1}))$. It follows from Proposition 3.4 that $F = F_s$ for some stipulation $s : \mathrm{Pr} \to \mathrm{Se}$, and hence that $\mathcal{B}$ is the poset of s-consistent valuations in $\mathbf{K}$.

The second claim follows since every finite lattice is 3-conditionally complete. □

5 Valuations in the four-element De Morgan algebra

We now consider the scheme proposed and studied in [11], where the algebra of truth values is the four-element De Morgan algebra $\mathbf{M}$.

The domain of $\mathbf{M}$ is the set $\{\mathbb{T}, \mathbb{F}, \mathbb{N}, \mathbb{B}\}$, and its operations are $\neg, \wedge, \vee$ together with the constants $\mathbb{T}, \mathbb{F}$. The operations $\wedge$ and $\vee$ are lattice meet and lattice join, giving rise to the Boolean order $\leq_{\mathbf{M}}$ given by $\mathbb{F} \leq_{\mathbf{M}} \mathbb{N}, \mathbb{B} \leq_{\mathbf{M}} \mathbb{T}$. The operation $\neg$ is given by $\neg\mathbb{T} = \mathbb{F}$, $\neg\mathbb{F} = \mathbb{T}$, $\neg\mathbb{N} = \mathbb{N}$, $\neg\mathbb{B} = \mathbb{B}$. The algebra $\mathbf{M}^+$ (the four-element De Morgan algebra with constants) has in addition the constants $\mathbb{N}$ and $\mathbb{B}$ among its operations.

By $\mathcal{M}$ we will denote the partially ordered set of which the universe coincides with that of $\mathbf{M}$, i.e., with universe $\{\mathbb{T}, \mathbb{F}, \mathbb{N}, \mathbb{B}\}$, and of which the partial order is the "information order" $\leq_{\mathcal{M}}$ mentioned in the introduction: the Boolean lattice order with $\mathbb{N} \leq_{\mathcal{M}} \mathbb{T}, \mathbb{F} \leq_{\mathcal{M}} \mathbb{B}$.

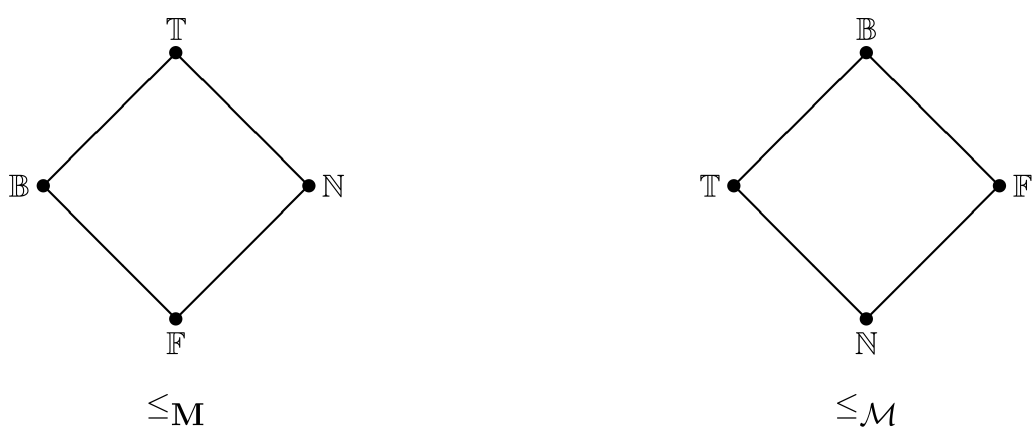

Figure 4

Note that $\mathcal{K}$ is a subposet of $\mathcal{M}$. It is easy to see that the fundamental operations of $\mathbf{M}$ and $\mathbf{M}^+$ preserve $\leq_{\mathcal{M}}$, and hence, by Lemma 3.5, $\mathrm{Clo}\,\mathbf{M} \subseteq \mathrm{Clo}\,\mathbf{M}^+ \subseteq \mathrm{Pre}(\leq_{\mathcal{M}})$. Since $\leq_{\mathcal{M}}$ is a complete lattice, so is the poset $\mathcal{M}^{\mathrm{Pr}}$ of all valuations of Pr in $\mathbf{M}$ or $\mathbf{M}^+$, for any set Pr of propositional constants. We can thus apply Tarski's theorem (Theorem 3.7 in the case $n = 2$):

Corollary 5.1. *Let s be any stipulation on a set* Pr *of propositional constants over the language of* $\mathbf{M}$ *or* $\mathbf{M}^+$ *respectively. The set of s-consistent valuations in* $\mathbf{M}$ *or* $\mathbf{M}^+$ *is non-empty, and is the domain of a subposet of* $\mathcal{M}^{\mathrm{Pr}}$ *that is*

(1) a complete lattice, and

(2) a retract of $\mathcal{M}^{\mathrm{Pr}}$*.*

It is well-known that every complete lattice is a retract of a product of copies of the two-element chain (this is also the case $n = 1$ of Theorem 2.3), and it follows easily that hence it is a retract of a direct product of copies of $\mathcal{M}$ as well. The following result—the proof of which is postponed to the next section—allows us to show that therefore every finite lattice can occur as the lattice of s-consistent valuations in $\mathbf{M}^+$ for some stipulation s.

Theorem 5.2. $\mathrm{Clo}\,\mathbf{M}^+ = \mathrm{Pre}(\leq_{\mathcal{M}})$.

Theorem 3.8 now yields:

Corollary 5.3. *Let $\mathcal{B}$ be a finite lattice. There is a stipulation s on a finite set* Pr *such that $\mathcal{B}$ is isomorphic (as a poset) to the poset of s-consistent valuations in* $\mathbf{M}^+$*.*

In contrast with the situation in the Kleene case, the last corollary does not continue to hold if we drop $\mathbb{N}$ and $\mathbb{B}$ as operations, i.e., if we replace $\mathbf{M}^+$ by $\mathbf{M}$. Let $d : \mathrm{M} \to \mathrm{M}$ be the map defined by $d(\mathbb{T}) = \mathbb{T}$, $d(\mathbb{F}) = \mathbb{F}$, $d(\mathbb{N}) = \mathbb{B}$, and $d(\mathbb{B}) = \mathbb{N}$. For any set I, the map d induces a map

$$\bar{d} : \mathrm{M}^I \to \mathrm{M}^I$$

by setting $\bar{d}(f) = d \circ f$, for $f \in \mathrm{M}^I$. The map $\bar{d}$ is an example of an order-reversing involution of $\mathcal{M}^I$. Here a map $d : \mathcal{B} \to \mathcal{B}$ from a poset $\mathcal{B}$ into itself is an *order-reversing involution* if it satisfies

(1) $x \leq y$ iff $d(x) \geq d(y)$, and

(2) $d(d(x)) = x$,

for all $x, y \in B$.

Proposition 5.4. *Let s be a stipulation on the set* Pr *of propositional constants over the language of* $\mathbf{M}$*. A valuation f in* $\mathbf{M}$ *is s-consistent if and only if the valuation $d \circ f$ is s-consistent.*

Proof. For any valuation f of Pr in $\mathbf{M}$, f is s-consistent if and only if $\bar{f} \circ s = f$. The map $d : \mathbf{M} \to \mathbf{M}$ is an automorphism of $\mathbf{M}$, and so

$$\begin{aligned} \bar{f} \circ s = f \qquad & \text{iff} \qquad d \circ \bar{f} \circ s = d \circ f \\ & \text{iff} \qquad \overline{d \circ f} \circ s = d \circ f. \end{aligned}$$

Thus f is s-consistent if and only if $d \circ f$ is s-consistent. □

In order to get closer to a characterization of the posets of s-consistent valuations in $\mathbf{M}$ we now introduce a class of richer structures defined and studied by Visser [11] for this purpose.

Definition 5.5. A structure $\langle \mathcal{B}, d \rangle$ is a *complete selfdual lattice* if the partially ordered set $\mathcal{B}$ is a complete lattice, and if $d : \mathcal{B} \to \mathcal{B}$ is an order-reversing involution.

Corollary 5.1 together with Proposition 5.4 yield:

Corollary 5.6. *Given any stipulation s on a set* Pr *of propositional constants over the language of* $\mathbf{M}$*, the set of s-consistent valuations in* $\mathbf{M}$ *is the domain of a complete selfdual lattice.*

This was first shown by Visser [11]. He also proved the following converse (see [11, Fact 2.2.1]).

We say that a map $F : \mathrm{M}^I \to \mathrm{M}^I$ is *selfdual* if F commutes with the map $\bar{d}$ defined earlier: $\bar{d} \circ F = F \circ \bar{d}$.

Theorem 5.7 ([11]). *Let $\langle \mathcal{B}, d \rangle$ be a complete selfdual lattice. Then $\langle \mathcal{B}, d \rangle$ is isomorphic to the complete selfdual lattice of fixed points of an order-preserving and selfdual map $F : \mathcal{M}^I \to \mathcal{M}^I$, for a suitable index set I.*

An n-ary operation $f : \mathrm{M}^n \to \mathrm{M}$ is called *selfdual* if $d \circ f = f \circ \bar{d}$. Note that if $|\mathrm{Pr}| = n$, then a map $F : \mathrm{M}^{\mathrm{Pr}} \to \mathrm{M}^{\mathrm{Pr}}$ is self-dual if and only if all of the n-ary operations $\pi_p \circ F : \mathrm{M}^{\mathrm{Pr}} \to \mathrm{M}$ are selfdual. In the next section we will show

Theorem 5.8. Clo $\mathbf{M}$ *consists of all selfdual operations in* $\mathrm{Pre}(\leq_{\mathcal{M}})$.

Corollary 5.9. *For every finite selfdual lattice* $\langle \mathcal{B}, d\rangle$ *there exists a stipulation* s *on a finite set* Pr *over the language of* $\mathbf{M}$ *such that* $\langle \mathcal{B}, d\rangle$ *is isomorphic to the poset of* s*-consistent valuations of* Pr *in* $\mathbf{M}$*.*

Proof. Let $\langle \mathcal{B}, d\rangle$ be a finite selfdual lattice. By Theorem 5.7 we may assume it is the poset of fixed points of some order-preserving and selfdual map $F : \mathcal{M}^I \to \mathcal{M}^I$, for some set I. Since B is finite we may assume I is finite as well, and in fact take I to be a finite set Pr of propositional constants. Now for any $p \in \mathrm{Pr}$ the map $\pi_p \circ F : B^{\mathrm{Pr}} \to B$ is a finitary operation that preserves $\leq_{\mathcal{M}}$ and that is selfdual, and hence is a term function of $\mathbf{M}$ by Theorem 5.8. Proposition 3.4 is now applicable, and yields that $F = F_s$ for some stipulation s on Pr over the language of $\mathbf{M}$. □

6 The term functions of the four-element De Morgan algebra

In this section we provide proofs of Theorems 5.2 and 5.8. Our proofs are essentially elaborations on the standard proof that every operation on the two-element set $\mathrm{B} = \{\mathbb{T}, \mathbb{F}\}$ is a term operation of the two-element Boolean algebra $\mathbf{B}_2$. We review this proof in order to outline the strategy of our two proofs and to establish some notation.

The partial order on the algebra $\mathbf{B}_2$ is denoted $\leq_{\mathbf{B}_2}$, and is given by $\mathbb{F} \leq_{\mathbf{B}_2} \mathbb{T}$. Consider an information ordering $\leq_{\mathcal{B}}$ on B in which $\mathbb{T}$ and $\mathbb{F}$ are incomparable. It is easily seen that $\mathrm{Pre}(\leq_{\mathcal{B}})$ consists of all finitary operations on B. So if we can argue that $\mathrm{Clo}\,\mathbf{B}_2$ also consists of all finitary operations on B, then we will have proven that $\mathrm{Clo}\,\mathbf{B}_2 = \mathrm{Pre}(\leq_{\mathcal{B}})$, a result analogous to Theorems 5.2 and 5.8.

We wish to show that if $f(x_1, \ldots, x_n) \in \mathrm{Pre}(\leq_{\mathcal{B}})$, then $f(x_1, \ldots, x_n) \in \mathrm{Clo}\,\mathbf{B}_2$. For each $w \in \mathrm{B}^n$ define

$$X_w^{\mathbb{T}} = \{x_i \ : \ w_i = \mathbb{T}\} \cup \{\neg x_i \ : \ w_i = \mathbb{F}\}.$$

Form the Boolean term $p_w = \bigwedge X_w^{\mathbb{T}}$. Thus each p_w may be viewed as an atom in the free Boolean algebra on n free generators. The p_w are sometimes called *minterms.* Note that $p_w(w) = \mathbb{T}$ and p_w is the least member of $\mathrm{Clo_n}\,\mathbf{B}_2$ whose value on w is $\mathbb{T}$. Our proofs for $\mathbf{M}$ and $\mathbf{M}^+$ involve constructing analogous terms for each of the three truth values $\mathbb{T}, \mathbb{B}$, and $\mathbb{N}$. If we let

$$g = \bigvee_{\substack{w \in \mathrm{B}^n \\ f(w) = \mathbb{T}}} p_w,$$

then $g \in \operatorname{Clo} \mathbf{B}_2$ and a standard argument using the partial order $\leq_{\mathbf{B}_2}$ shows that $g = f$.

Our proofs of Theorems 5.2 and 5.8 are similar but necessarily more complicated since instead of two truth values there are four, and instead of the linear order of $\leq_{\mathbf{B}_2}$ we have the partial order $\leq_{\mathbf{M}}$ in which $\mathbb{B} \vee \mathbb{N} = \mathbb{T}$, a fact that substantially complicates the argument.

Proof of Theorem 5.2. As already observed each of the basic operations of $\mathbf{M}^+$ is in $\operatorname{Pre}(\leq_{\mathcal{M}})$, so by Lemma 3.5 we have $\operatorname{Clo} \mathbf{M}^+ \subseteq \operatorname{Pre}(\leq_{\mathcal{M}})$.

Let $n > 0$ be arbitrary. We wish to show that every $f(x_1, \ldots, x_n) \in \operatorname{Pre}(\leq_{\mathcal{M}})$ is in $\operatorname{Clo_n} \mathbf{M}^+$. Let $X = \{x_1, \ldots, x_n\}$ and denote by $\neg X$ the set $\{\neg x_1, \ldots, \neg x_n\}$. For $w = (w_1, \ldots, w_n) \in \mathrm{M}^n$ define $X_w^{\mathbb{B}}$ and $X_w^{\mathbb{N}} \subseteq X \cup \neg X$ by

$$X_w^{\mathbb{B}} = \{x_i : w_i = \mathbb{T} \text{ or } \mathbb{B}\} \cup \{\neg x_i : w_i = \mathbb{F} \text{ or } \mathbb{B}\}$$

and

$$X_w^{\mathbb{N}} = \{x_i : w_i = \mathbb{T} \text{ or } \mathbb{N}\} \cup \{\neg x_i : w_i = \mathbb{F} \text{ or } \mathbb{N}\}.$$

Form $q_w = \mathbb{B} \wedge (\bigwedge X_w^{\mathbb{B}})$ and $r_w = \mathbb{N} \wedge (\bigwedge X_w^{\mathbb{N}})$. In the event that $X_w^{\mathbb{B}}$ or $X_w^{\mathbb{N}}$ are empty we note that $\bigwedge \emptyset = \mathbb{T}$, so $q_w = \mathbb{B}$ and $r_w = \mathbb{N}$ in this situation.

Define

$$g = \Big(\bigvee_{\substack{w \in \mathrm{M}^n \\ f(w) = \mathbb{B}}} q_w\Big) \vee \Big(\bigvee_{\substack{w \in \mathrm{M}^n \\ f(w) = \mathbb{N}}} r_w\Big) \vee \Big(\bigvee_{\substack{w \in \mathrm{M}^n \\ f(w) = \mathbb{T}}} q_w \vee r_w\Big).$$

Then $g \in \operatorname{Clo} \mathbf{M}^+$. We show that $f = g$ by arguing that for all $y \in \mathrm{M}^n$ the inequality $g(y) \leq_{\mathbf{M}} f(y)$ holds and for each y, depending on the value of $f(y)$, there is a summand in g that when evaluated at y has value $f(y)$.

The term operation q_w has range $\{\mathbb{B}, \mathbb{F}\}$ since $q_w(y) \leq_{\mathbf{M}} \mathbb{B}$. Moreover, q_w preserves $\leq_{\mathcal{M}}$ as do all term operations of $\mathbf{M}$. Note that $q_w(w) = \mathbb{B}$ since $\mathbb{B} \wedge \mathbb{T} = \mathbb{B}$ and $\mathbb{B} \wedge \mathbb{B} = \mathbb{B}$.

Let $y \in \mathrm{M}^n$ be arbitrary. The order $\leq_{\mathcal{M}}$ is extended to the product M^n in the natural way. If $w \leq_{\mathcal{M}} y$, then $q_w(y) = \mathbb{B}$ since q_w preserves $\leq_{\mathcal{M}}$. We claim that if $w \not\leq_{\mathcal{M}} y$, then $q_w(y) = \mathbb{F}$. For if $w \not\leq_{\mathcal{M}} y$, then there exists an i for which $w_i \not\leq_{\mathcal{M}} y_i$. If $w_i = \mathbb{B}$, then both x_i and $\neg x_i$ are in $X_w^{\mathbb{B}}$ and

$y_i \in \{\mathbb{T}, \mathbb{F}, \mathbb{N}\}$, and so $q_w(y) = \mathbb{F}$. If $w_i = \mathbb{T}$, then x_i but not $\neg x_i$ is in $X_w^{\mathbb{B}}$ and $y_i \in \{\mathbb{N}, \mathbb{F}\}$, so $q_w(y) = \mathbb{F}$. A similar argument applies if $w_i = \mathbb{F}$. The fourth case that $w_i = \mathbb{N}$ is not possible if $w_i \not\leq_{\mathcal{M}} y_i$.

Thus, if $f(w) = \mathbb{B}$, then $q_w(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$ since the range of q_w is $\{\mathbb{B}, \mathbb{F}\}$, and if $q_w(y) = \mathbb{B}$, then $w \leq_{\mathcal{M}} y$ and so $f(w) \leq_{\mathcal{M}} f(y)$, which gives $f(y) = \mathbb{B}$.

Similar arguments show that $r_w(w) = \mathbb{N}$; for all $y \in \mathrm{M}^n$ if $y \leq_{\mathcal{M}} w$, then $r_w(y) = \mathbb{N}$ and if $y \not\leq_{\mathcal{M}} w$, then $r_w(y) = \mathbb{F}$; and if $f(w) = \mathbb{N}$, then $r_w(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$.

Finally, if $f(w) = \mathbb{T}$, then $q_w(w) \vee r_w(w) = f(w)$ since $\mathbb{B} \vee \mathbb{N} = \mathbb{T}$. For all $y \in \mathrm{M}^n$ the inequality $q_w(y) \vee r_w(y) \leq_{\mathbf{M}} f(y)$ holds. To verify this inequality we consider three cases.

1. If $w <_{\mathcal{M}} y$, then $q_w(y) \vee r_w(y) = \mathbb{B} \vee \mathbb{F} = \mathbb{B}$, but $\mathbb{T} \leq_{\mathcal{M}} f(y)$ so $f(y) = \mathbb{T}$ or $\mathbb{B}$. Thus $q_w(y) \vee r_w(y) = \mathbb{B} \leq_{\mathbf{M}} f(y)$.

2. If $y <_{\mathcal{M}} w$, then $q_w(y) \vee r_w(y) = \mathbb{F} \vee \mathbb{N} = \mathbb{N}$, but $f(y) \leq_{\mathcal{M}} f(w) = \mathbb{T}$ so $f(y) = \mathbb{T}$ or $\mathbb{N}$. Thus $q_w(y) \vee r_w(y) = \mathbb{N} \leq_{\mathbf{M}} f(y)$.

3. If y and w are $\leq_{\mathcal{M}}$ incomparable, then $q_w(y) \vee r_w(y) = \mathbb{F} \vee \mathbb{F} = \mathbb{F}$, and so $q_w(y) \vee r_w(y) \leq_{\mathbf{M}} f(y)$ regardless of the value of $f(y)$.

So each summand in g is below f in the order $\leq_{\mathbf{M}}$ and for each $y \in \mathrm{M}^n$ there is a summand that when evaluated on y has value $f(y)$. This completes the proof. □

We next prove Theorem 5.8, which states that $\mathrm{Clo}\,\mathbf{M}$ consists of all selfdual operations in $\mathrm{Pre}(\leq_{\mathcal{M}})$. In this argument we work with the algebra $\mathbf{M}$ so we do not have access to the constants $\mathbb{B}$ and $\mathbb{N}$ as we did in the previous proof. Thus there are fewer term operations in $\mathrm{Clo}\,\mathbf{M}$ but the restriction to selfdual operations is precisely what is needed to compensate for this loss.

Proof of Theorem 5.8. As in the previous proof $\mathrm{Clo}\,\mathbf{M} \subseteq \mathrm{Pre}(\leq_{\mathcal{M}})$. A computation shows that each of the operations $\vee, \wedge, \neg, \mathbb{T}, \mathbb{F}$ is selfdual. The property of being selfdual is preserved under composition of operations so every operation in $\mathrm{Clo}\,\mathbf{M}$ is selfdual and in $\mathrm{Pre}(\leq_{\mathcal{M}})$.

We argue the reverse inclusion. For every $w \in \mathrm{M}^n$ let $X_w^{\mathbb{B}}$ and $X_w^{\mathbb{N}}$ be as defined in the proof of Theorem 5.2, that is,

$$X_w^{\mathbb{B}} = \{x_i : w_i = \mathbb{T} \text{ or } \mathbb{B}\} \cup \{\neg x_i : w_i = \mathbb{F} \text{ or } \mathbb{B}\}$$

and

$$X_w^{\mathbb{N}} = \{x_i : w_i = \mathbb{T} \text{ or } \mathbb{N}\} \cup \{\neg x_i : w_i = \mathbb{F} \text{ or } \mathbb{N}\}.$$

Define

$$X_w^{\mathbb{T}} = \{x_i \ : \ w_i = \mathbb{T}\} \cup \{\neg x_i \ : \ w_i = \mathbb{F}\}.$$

Form $q_w = \bigwedge X_w^{\mathbb{B}}$, $r_w = \bigwedge X_w^{\mathbb{N}}$ and $t_w = \bigwedge X_w^{\mathbb{T}}$. In the event that any of the sets $X_w^{\mathbb{B}}, X_w^{\mathbb{N}}$, or $X_w^{\mathbb{T}}$ are empty the corresponding term operations q_w, r_w, or t_w are the constant function $\mathbb{T}$ since $\bigwedge \emptyset = \mathbb{T}$.

Let $f(x_1, \ldots, x_n)$ be an arbitrary selfdual member of $\mathrm{Pre}(\leq_{\mathcal{M}})$. That f enjoys the following properties is easily verified.

1. $f(\{\mathbb{T}, \mathbb{F}\}^n) \subseteq \{\mathbb{T}, \mathbb{F}\}$.
2. $f(\{\mathbb{T}, \mathbb{F}, \mathbb{N}\}^n) \subseteq \{\mathbb{T}, \mathbb{F}, \mathbb{N}\}$.
3. $f(\{\mathbb{T}, \mathbb{F}, \mathbb{B}\}^n) \subseteq \{\mathbb{T}, \mathbb{F}, \mathbb{B}\}$.

Let $\mathrm{S}_- = \{\mathbb{T}, \mathbb{F}, \mathbb{N}\}^n$, $\mathrm{S}_+ = \{\mathbb{T}, \mathbb{F}, \mathbb{B}\}^n$, and $\mathrm{S}_0 = \mathrm{M}^n - (\mathrm{S}_+ \cup \mathrm{S}_-)$. Note that $\mathrm{S}_+ \cap \mathrm{S}_- = \{\mathbb{T}, \mathbb{F}\}^n$ and that both $\mathbb{B}$ and $\mathbb{N}$ appear in each element of S_0.

Define

$$g = \Big(\bigvee_{\substack{w \in \mathrm{M}^n \\ f(w) = \mathbb{B}}} q_w\Big) \vee \Big(\bigvee_{\substack{w \in \mathrm{M}^n \\ f(w) = \mathbb{N}}} r_w\Big) \vee \Big(\bigvee_{\substack{w \in \mathrm{S}_+ \cup \mathrm{S}_- \\ f(w) = \mathbb{T}}} t_w\Big) \vee \Big(\bigvee_{\substack{w \in \mathrm{S}_0 \\ f(w) = \mathbb{T}}} (q_w \vee r_w)\Big).$$

We prove that $g = f$ by showing that for all $y \in \mathrm{M}^n$ the inequalities $f(y) \leq_{\mathbf{M}} g(y)$ and $g(y) \leq_{\mathbf{M}} f(y)$ obtain.

Let $w = (w_1, \ldots, w_n) \in \mathrm{M}^n$ be arbitrary subject only to the constraint that there is at least one i for which $w_i = \mathbb{B}$. Then both x_i and $\neg x_i$ are in $X_w^{\mathbb{B}}$. So $q_w(w) \leq_{\mathbf{M}} \mathbb{B}$. All other x_j or $\neg x_j$ in $X_w^{\mathbb{B}}$ contribute either $\mathbb{B}$ or $\mathbb{T}$ to the value of $q_w(w)$. Since $\mathbb{B} \wedge \mathbb{T} = \mathbb{B}$ it follows that for all such w we have $q_w(w) = \mathbb{B}$.

Likewise, if there is at least one i for which $w_i = \mathbb{N}$, then $r_w(w) = \mathbb{N}$. If $w \in \mathrm{S}_+ \cup \mathrm{S}_-$, then $t_w(w) = \mathbb{T}$ while if $w \in \mathrm{S}_0$, then $q_w(w) \vee r_w(w) = \mathbb{B} \vee \mathbb{N} = \mathbb{T}$.

We show that $f(y) \leq_{\mathbf{M}} g(y)$ for all $y \in M^n$ by considering the possible values for $f(y)$. If $f(y) = \mathbb{B}$, then from (1), (2) and (3) we see that $y_i = \mathbb{B}$ for at least one i. Thus $q_y(y) = \mathbb{B} = f(y)$ so $f(y) \leq_{\mathbf{M}} g(y)$. Similarly, if $f(y) = \mathbb{N}$, then $r_y(y) = \mathbb{N} = f(y)$ and $f(y) \leq_{\mathbf{M}} g(y)$. If $f(y) = \mathbb{T}$ and $y \in \mathrm{S}_+ \cup \mathrm{S}_-$, then $t_y(y) = \mathbb{T} = f(y)$ and if $y \in \mathrm{S}_0$, then $(q_y \vee r_y)(y) = \mathbb{T} = f(y)$. So again $f(y) \leq_{\mathbf{M}} g(y)$ in these cases. Finally, if $f(y) = \mathbb{F}$, then $f(y) \leq_{\mathbf{M}} g(y)$ since $\mathbb{F}$ is the least element of the partial order $\leq_{\mathbf{M}}$.

It remains to show that $g(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$. We will do this by showing that each of the four types of summands in g, when evaluated at y, has value $\leq_{\mathbf{M}} f(y)$. We first prove some facts about q_w and r_w.

Let $w \in \mathrm{M}^n$ be such that $w_i = \mathbb{B}$ for at least one i. So both x_i and $\neg x_i$ are in $X_w^{\mathbb{B}}$. We have already seen that $q_w(w) = \mathbb{B}$. Consider $y \in \mathrm{M}^n$. If $w \leq_{\mathcal{M}} y$, then $q_w(y) = \mathbb{B}$ since q_w preserves $\leq_{\mathcal{M}}$. On the other hand, if $q_w(y) = \mathbb{B}$, then wherever $w_j = \mathbb{B}$ the value of y_j must also be $\mathbb{B}$, wherever $w_j = \mathbb{T}$ the value of y_j must be $\mathbb{B}$ or $\mathbb{T}$, and wherever $w_j = \mathbb{N}$, the value of y_j must be $\mathbb{B}$ or $\mathbb{F}$. Hence if $q_w(y) = \mathbb{B}$, then $w \leq_{\mathcal{M}} y$. Since q_w is selfdual, we have $q_w(\bar{d}(w)) = d(\mathbb{B}) = \mathbb{N}$. Moreover, if $y \leq \bar{d}(w)$, then $q_w(y) = \mathbb{N}$ since q_w preserves $\leq_{\mathcal{M}}$. An argument similar to the one just given shows that if $q_w(y) = \mathbb{N}$, then $y \leq \bar{d}(w)$. For no y does $q_w(y) = \mathbb{T}$ since $q_w \leq_{\mathbf{M}} x_i \wedge \neg x_i$. Thus we have shown for every $w \in \mathrm{M}^{\mathrm{n}}$ that has at least one $w_i = \mathbb{B}$,

4. $w \leq_{\mathcal{M}} y$ if and only if $q_w(y) = \mathbb{B}$,
5. $y \leq_{\mathcal{M}} \bar{d}(w)$ if and only if $q_w(y) = \mathbb{N}$,
6. $q_w(y) = \mathbb{F}$ in all other cases.

Now suppose $f \in \mathrm{Pre}(\leq_{\mathcal{M}})$ is selfdual with $f(w) = \mathbb{B}$. We have already seen that this implies that there is at least one i for which $w_i = \mathbb{B}$. So q_w satisfies (4), (5), and (6) and it follows that for all $y \in \mathrm{M}^n$ the inequality $q_w(y) \leq_{\mathbf{M}} f(y)$ holds.

A similar argument for the summands of g corresponding to those w where $f(w) = \mathbb{N}$ shows that $r_w(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$.

Next consider the summands of g of the form t_w for $w \in \mathrm{S}_+ \cup \mathrm{S}_-$ with $f(w) = \mathbb{T}$. We claim that $t_w(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$. If y is such that $t_w(y) = \mathbb{F}$, then the claim is immediate. If $t_w(y) = \mathbb{N}$, then y is obtained from w by changing some $\mathbb{T}$'s or $\mathbb{F}$'s to $\mathbb{N}$, and changing $\mathbb{N}$'s and $\mathbb{B}$'s arbitrarily. In the case that $w \in \mathrm{S}_+$ we have $y \leq_{\mathcal{M}} w$ since $\mathbb{B}$ is maximal in the partial order $\leq_{\mathcal{M}}$ and no w_i is $\mathbb{N}$. If $w \in \mathrm{S}_-$, then $y \leq_{\mathcal{M}} \bar{d}(w)$ since $\bar{d}(w) \in \mathrm{S}_+$ and every $\mathbb{N}$ that is in w becomes a $\mathbb{B}$ in $\bar{d}(w)$, and $\mathbb{B}$ is maximal in $\leq_{\mathcal{M}}$. Thus, in either case $f(y) \leq_{\mathcal{M}} \mathbb{T}$ since $f(w) = f(\bar{d}(w)) = \mathbb{T}$. This shows $f(y) \in \{\mathbb{T}, \mathbb{N}\}$, which implies $t_w(y) \leq_{\mathbf{M}} f(y)$. A similar argument in the case that $t_w(y) = \mathbb{B}$ gives $w \leq_{\mathcal{M}} y$ or $\bar{d}(w) \leq_{\mathcal{M}} y$ and $f(y) \in \{\mathbb{T}, \mathbb{B}\}$, and thus $t_w(y) \leq_{\mathbf{M}} f(y)$. Finally, if $t_w(y) = \mathbb{T}$, then either $w \leq_{\mathcal{M}} y \leq_{\mathcal{M}} \bar{d}(w)$ or $\bar{d}(w) \leq_{\mathcal{M}} y \leq_{\mathcal{M}} w$. Therefore $f(y) = \mathbb{T}$ since $f(w) = f(\bar{d}(w)) = \mathbb{T}$ and $f \in \mathrm{Pre}(\leq_{\mathcal{M}})$.

The final summands we need to consider are those arising from those w for which $f(w) = \mathbb{T}$ and $w \in \mathrm{S}_0$. In this case w has at least one $w_i = \mathbb{B}$ and at least one $w_j = \mathbb{N}$. Thus the displayed conditions (4), (5), and (6) hold for w and q_w since $w_i = \mathbb{B}$ and three dual conditions hold for w and r_w since $w_j = \mathbb{N}$. We claim that $(q_w \vee r_w)(y) \leq_{\mathbf{M}} f(y)$ for all $y \in \mathrm{M}^n$. If $(q_w \vee r_w)(y) = \mathbb{F}$, then the claim is immediate. If $(q_w \vee r_w)(y) = \mathbb{B}$, then since $\mathbb{B}$ is join-irreducible in the lattice structure of $\mathbf{M}^+$, $q_w(y) = \mathbb{B}$ or $r_w(y) = \mathbb{B}$. So $w \leq_{\mathcal{M}} y$ or $\bar{d}(w) \leq_{\mathcal{M}} y$ must hold because of the properties of r_w and q_w. This gives $\mathbb{T} = f(w) \leq_{\mathcal{M}} f(y)$ or $\mathbb{T} = d(\mathbb{T}) = f(\bar{d}(w)) \leq_{\mathcal{M}} f(y)$ since $f \in \mathrm{Pre}(\leq_{\mathcal{M}})$. Thus $\mathbb{T} \leq_{\mathcal{M}} f(y)$, so $f(y) = \mathbb{T}$ or $\mathbb{B}$ as desired. If instead $(q_w \vee r_w)(y) = \mathbb{N}$, then we can argue as we just did to show $f(y) \leq_{\mathcal{M}} \mathbb{T}$, and thus $f(y) = \mathbb{T}$ or $\mathbb{N}$. Finally, if $(q_w \vee r_w)(y) = \mathbb{T}$, then $\{q_w(y), r_w(y)\} = \{\mathbb{B}, \mathbb{N}\}$ since $\mathbb{T}$ is not in the range of q_w or of r_w. If $q_w(y) = \mathbb{B}$ and $r_w(y) = \mathbb{N}$, then $y \leq_{\mathcal{M}} w$ and $w \leq_{\mathcal{M}} y$. So $y = w$ and $f(y) = f(w) = \mathbb{T}$. If instead $q_w(y) = \mathbb{N}$ and $r_w(y) = \mathbb{B}$, then we can argue that $y = \bar{d}(w)$. So $f(y) = f(\bar{d}(w)) = d(\mathbb{T}) = \mathbb{T}$. □

7 Open Problems

We conclude the paper by drawing attention to some obvious questions that present themselves but that we have left unanswered.

Firstly, wherever we obtained a characterization of the posets of s-consistent valuations into some algebra of truth values, we restricted ourselves to *finite* posets, and hence to stipulations and valuations on finite sets Pr of propositional constants.

Problem 7.1. Find generalizations of Theorem 3.8 and Corollaries 3.9, 4.3, 4.5, 5.3, 5.9 in which the finiteness assumption is dropped.

It seems likely that some topological considerations will have to come into play in such a generalization.

In Section 3 we obtained satisfying characterizations of the posets of s-consistent valuations into the algebra $\mathbf{T}_n$. Let $\mathbf{T}_n'$ be the algebra with domain T_n and operations the set of all finitary and order-preserving operations on $\mathcal{T}_n$ that in addition preserve the set C of maximal elements of $\mathcal{T}_n$. Just as in the Kleene case (Corollary 4.5), we can appeal to Theorem 2.4 to show that in Corollary 3.9 we can replace the algebra $\mathbf{T}_n$ by $\mathbf{T}_n'$.

In the case $n = 2$ $\mathbf{T}_2'$ is term equivalent to the three-element Kleene algebra $\mathbf{K}$, and thus is an honest algebra of truth values. In the information ordering $\mathcal{T}_2$, the maximal elements correspond to the truth values $\mathbb{T}, \mathbb{F}$, while the bottom element corresponds with the value $\mathbb{N}$ that represents a state of insufficient information. This algebra may be viewed as an "information algebra" for the two-element Boolean algebra $\mathbf{B}_2$. Similarly, the maximal elements of $\mathcal{T}_n$ form the domain of a subalgebra of $\mathbf{T}'_n$. This subalgebra is the n-element simple Post algebra, and thus an n-valued analog of $\mathbf{B}_2$. The algebra $\mathbf{T}'_n$ can then be viewed as an "information algebra" for the n-element simple Post algebra. The question is whether for $n > 2$ the algebra $\mathbf{T}'_n$ is term equivalent to a "natural" algebra of truth values, just as $\mathbf{T}'_2$ is term equivalent to $\mathbf{K}$. It is known that for every n the clone of operations of $\mathbf{T}'_n$ is finitely generated since it contains a near-unanimity term.

Problem 7.2. Is there a "natural" choice for the generating operations?

Here by a "natural" choice we think of logically relevant operations such as those reflecting disjunction, conjunction, negation and possibly others. The family of all finitary order-preserving operations on $\mathcal{T}_n$ is investigated by Mukaidono and Rosenberg in [9] and a similar problem is posed there.

References

[1] N.D. Belnap Jr., A useful four-valued logic, in: *Modern Uses of Multiple-Valued Logic*, edited by J. Dunn and G. Epstein, 8–37, 1977. Reidel, Dordrecht.

[2] J.A. Bergstra, I. Bethke and P.H. Rodenburg, A propositional logic with 4 values: true, false, divergent and meaningless, *J. Appl. Non-Classical Logics* 5, 199–217 (1995).

[3] J. Berman and W.J. Blok, Generalizations of Tarski's fixed point theorem for order varieties of complete meet semilattices, *Order* 5, 381–392 (1989).

[4] S. Blamey, Partial logic, in: *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenther, vol. 3, 1–70, 1986. Reidel.

[5] A. Gupta and N. Belnap, *The Revision Theory of Truth*, 1993. MIT Press.

[6] S. Kripke, Outline of a theory of truth, *Journal of Philosophy* 72, 690–716 (1975).

[7] M. Mukaidono, On the B-ternary logical functions — A ternary logic considering ambiguity, *Systems, Computers, Controls* 3, 27–36 (1972).

[8] M. Mukaidono, Regular ternary logic functions — Ternary logic functions suitable for treating ambiguity —, *IEEE Proceedings of the Thirteenth Symposium on Multiple-Valued Logic*, 286–291 (1983).

[9] M. Mukaidono and I. Rosenberg, Clones of isotone operations for half-bound orders and regular functions, *IEEE Proceedings of the Seventeenth Symposium on Multiple-valued Logic*, 349–355 (1987).

[10] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, *Pacific J. Math.* 5, 285–309 (1955).

[11] A. Visser, Four-valued semantics and the liar, *Journal of Philosophical Logic* 13, 181–212 (1984).

[12] A. Visser, Semantics and the liar paradox, in: *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenthner, vol. 4, 617–706, 1989. Reidel.