Math 160 Discussion Notes
Brian Powers – TA – Fall 2011

2.4 The Inverse of a Matrix

With real numbers we have a conecpt of the "multiplicitive identity" and "multiplicitive inverse". The multiplicitive identity is 1, and for any number n, its multiplicitive inverse is 1/n, so that when the two are multiplied, the product is 1.

With matrices this is not quite so simple. We first have to define an identity for matrices.

The Identity Matrix

The identity matrix is an nxn square matrix with 1s along the main diagonal and 0 everywhere else. We denote it as I_n with n the dimensions, although we will omit the n when the dimension is clear from the context of the problem.

ex) $I_1=\begin{bmatrix}1\end{bmatrix}$, $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

For any square matrix A, IA=AI=A. That is to say, multiplying a matrix by the identity matrix doesn't change its value.

The Inverse of a matrix

For a square nxn matrix A, if there exists a matrix A^{-1} such that $AA^{-1}=I$, then we say A^{-1} is the inverse of A, and we call it "A inverse". Note that A is the inverse of A^{-1} as well. Also, not every matrix has an inverse!

ex) Verify that $\begin{bmatrix}7 & 2\\ 3 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & -2\\ -3 & 7\end{bmatrix}$

We can take the product:

$$\begin{bmatrix}7 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -3 & 7\end{bmatrix}=\begin{bmatrix}7(1)+2(-3) & 7(-2)+2(7)\\ 3(1)+1(-3) & 3(-2)+1(7)\end{bmatrix}=\begin{bmatrix}7-6 & -14+14\\ 3-3 & -6+7\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Finding the Inverse of a 2x2 Matrix

We have an easy way to find the inverse of a 2x2 matrix, if it exists. Consider A= $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

The determinant of A, written detA=ad-bc. If detA≠0, then $A^{-1}=\dfrac{1}{det\,A}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$. If detA=0 then A has no inverse.

ex) Find the inverse of $\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}$

$$\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}^{-1}=\frac{1}{2(7)-3(5)}\begin{bmatrix}7 & -3\\ -5 & 2\end{bmatrix}=-\begin{bmatrix}7 & -3\\ -5 & 2\end{bmatrix}=\begin{bmatrix}-7 & 3\\ 5 & -2\end{bmatrix}$$

ex) Find the inverse of $\begin{bmatrix}3 & 7\\ 5 & -2\end{bmatrix}$

$$\begin{bmatrix}3 & 7\\ 5 & -2\end{bmatrix}^{-1}=\frac{1}{3(-2)-7(5)}\begin{bmatrix}-2 & -7\\ -5 & 7\end{bmatrix}=\frac{1}{-41}\begin{bmatrix}-2 & -7\\ -5 & 7\end{bmatrix}=\begin{bmatrix}2/41 & 7/41\\ 5/41 & -3/41\end{bmatrix}$$

ex) Find the inverse of $\begin{bmatrix} 4 & 2 \\ 10 & 5 \end{bmatrix}$

detA=4(5)-2(10)=20-20=0, so this matrix has no inverse.

Using the Inverse to solve a system of linear equations

Revisiting the system of linear equations

$$\begin{cases} a_{1,1}x_1+a_{1,2}x_2+\cdots+a_{1,m}x_m=b_1 \\ a_{2,1}x_1+a_{2,2}x_2+\cdots+a_{2,m}x_m=b_2 \\ \vdots \\ a_{n,1}x_1+a_{n,2}x_2+\cdots+a_{n,m}x_m=b_n \end{cases}$$

We can represent this as a matrix equation

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ or AX=B.}$$

If A has an inverse A^{-1}, then we can pre-multiply both sides of the equation by A^{-1} to get:

$A^{-1}X=A^{-1}B \rightarrow A^{-1}AX=A^{-1}B \rightarrow IX=A^{-1}B \rightarrow X=A^{-1}B$.

ex) Given that $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, solve $\begin{cases} x+2y+2z=1 \\ x+3y+2z=0 \\ x+2y+3z=-1 \end{cases}$

The system can be expressed as $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5(1)-2(0)-2(-1) \\ -1(1)+1(0)+0(-1) \\ -1(1)+0(0)+1(-1) \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix}$, so x=7, y=-1, and z=-2

ex) In a particular town of 48,000, a disease is causing illness. From one week to the next, ¼ of the well people get sick and 2/3 of the sick people get well. If this week 13,000 people are sick, how many were sick the previous week? What if 14,000 people are sick this week?

If we say x=# people sick this week, y=#people well this week, s=#sick people next week and w=#well people next week we get this system:

$\begin{cases} \frac{1}{3}x+\frac{1}{4}y=s \\ \frac{2}{3}x+\frac{3}{4}y=w \end{cases}$, so in the form of AX=B, we have $\begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 3/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ w \end{bmatrix}$, so detA=

$$\frac{1}{3}\frac{3}{4}-\frac{1}{4}\frac{2}{3}=\frac{1}{12}$$

$$\begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 3/4 \end{bmatrix}^{-1} = 12 \begin{bmatrix} 3/4 & -1/4 \\ -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -8 & 4 \end{bmatrix}$$

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} s \\ w \end{bmatrix}$. x=9s-3w. If s=13,000 then w=35,000 and x=9(13000)-3(35000)=12000

If s=14000 w=34000 and x=9(14000)-3(34000)=24,000.